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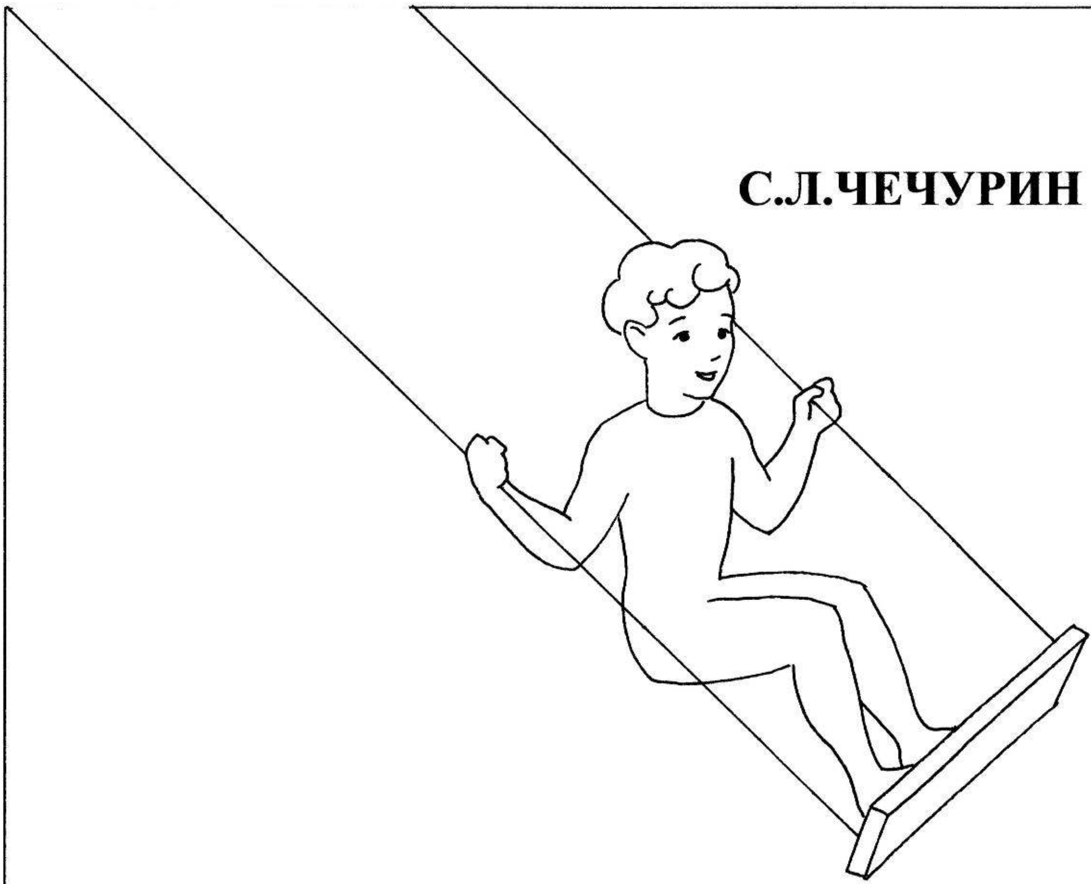
Parametric Resonance

Pain and Joy

St. Petersburg State Polytechnic University Press
2014

НАУЧНО-ПОПУЛЯРНАЯ БИБЛИОТЕКА

С.Л.ЧЕЧУРИН



ПАРАМЕТРИЧЕСКИЙ РЕЗОНАНС-БОЛЬ и РАДОСТЬ

ORIGINAL COVER

ABSTRACT

Of multifarious processes studied forced oscillation, parametric oscillation and self-excited oscillation are specified in the modern classical oscillation theory. The conditions of oscillation occurrence/excitation and existence/stability are of interest. A principal role in solving that problem is pertained to parametric oscillations because both forced and self-excited oscillations lose their instability and become beyond physical existence under parametric resonance excitation conditions. Moreover there is a reason to believe that self-excited oscillations are sustained parametric oscillations. An influence area of parametric oscillations and resonance is under fast broadening. Parametric oscillations in economic models, a parametric direction in biology ("water-on-life" activation), and a parametric nature of field and vibration influences on living organisms in medicine were recently revealed. And wherein is occasionally that elusive magical force of parametric oscillations and resonance? The matter will concern about that.

This paper is a popular scientific publication. Firstly it will serve for engineers and designers whose specialization is far from a theory of oscillations. It will be useful for physics-and-mathematics-oriented young people, and the knowledge of parametric oscillation properties and features given in Part I, which is not difficult, can be useful for untechnical experts in the cause of integration of physics, chemistry, biology, medicine and other sciences on the basis of mathematical modeling.

Editor and Translator Elen Kutueva

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BACKGROUNDS

The existence of parametric oscillations is coupled with the existence of the universe, and their history counting billions of years derives from the history of foundation and development of worlds. In all the instances, watching a monkey flying from one branch to another by means of parametric oscillations, it becomes clear that the use of parametric oscillations is related to a pre-human period. But the human being's study of parametric oscillations and parametric resonance was started quit recently or some hundred add years ago. And it was in spite of the fact that parametric resonance made a lot of mess over its history.

The beginning of parametric oscillation study is put down to the investigations of Mathieu and Hill equations. In 1868 Mathieu researched membrane oscillations and came to a simple second-order differential equation with a harmonically variable absolute term containing no derivative. Even the numerical equation solution was surprising: it included an infinite number of alternate parametric oscillation excitation areas. Subsequently the oscillations were presented in the form of Ayns-Strett diagrams to be used up to now. Solving the problem of moon orbit determination Hill also considered the second-order differential equation in which an absolute term was arbitrarily varied. He obtained the solution in the form of an infinite determinant called Hill's determinant later on. Having been confronted with complexity of the derived solutions researchers temporarily relaxed their activity in that direction.

The excellent service in pointing an important role of studying the equations with periodical coefficients belongs to the distinguished Russian mathematician A.M. Lyapunov. At the beginning of the last century he proved that equilibrium stability of dynamic systems is governed by stability of linear stationary systems or incremental equations formed to the first linear approximation. Motion stability of a dynamic system including its oscillation stability is governed by equilibrium stability of a linear nonstationary system with time-varying parameters and also incremental equations formed to the first linear approximation. In other words the motion stability problem was reduced to the problem of equilibrium stability of a linear nonstationary/parametric system.

An interest in studying linear nonstationary systems was steeply grown. At the first half of the last century both rigorous and rough research methods of high-order linear nonstationary/parametric systems including those with distributed parameters were emerged.

In this paper the main attention is concentrated on a physical aspect of parametric oscillations and resonance. The choice of the frequency analysis method characterized by pictorial physical and geometrical interpretations shows the correlation with that aspect. In common case this single frequency harmonic approach method is known as Described Function Method. As to nonstationary systems it can be named harmonic stationaryzation [1,2].

The main difficulty in writing was making a hard choice between the simplicity of stating and the complexity of reasoning. It did not succeed in supposing only one of the two ways. That is why the paper consists of two parts. Part I is simple and clear and Part 2 is relatively complex. The subject of Part 2 is mainly based on senior school courses in physics and mathematics. The simple differential equation are used for transfer to frequency region only.

Thus, the basic goal of this paper is initially to introduce the reader to parametric oscillations and the phenomena of parametric resonance. According to the objective all the illustrations are accompanied with elemental examples of single-frequency harmonic oscillations and several results of second part are printed in a small type. A careful reader will be able to find many interesting things in the parametric resonance phenomena, which are quite often strict judges in respect of assessments of completeness and carefulness of constructions and engineering solutions

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PART 1. PHENOMENA OF PARAMETRIC RESONANCE

Several Words about Oscillations

Oscillations and oscillating processes have been in existence regardless of a human will in gravitational, electrical and magnet fields, liquid and gaseous media, and the combinations of fields and media. For a long time oscillations were only interpreted as periodic motions of bodies, fields and media under the effects of applied/external forces, moments, extraneous oscillating processes and other just now so-called disturbances including periodic ones.

This point of view is also prevailing among those of our contemporaries whose professional activities do not cover physical and mathematical concepts. This does not allow explaining a number of such mysterious phenomena as self-excitation and sudden amplitude jumps of oscillations, the reasons of excitation and stability of oscillations, etc. Just not long ago so-called parametric oscillations were distinguished among multiform oscillatory processes to attract a close attention of researchers.

Quite a lot books have been written about oscillations. Neither complete nor even popular statement of the problems existing in the theory and practice of oscillations is covered in this paper. It is highly desirable for the uninformed reader to see the popular science book by the American scientist R. Bishop [3]. The paper includes an extremely limited amount of the definitions needed hereinafter.

At present free oscillations, forced oscillations, parametric oscillations and self-excited oscillations as a special case of sustained free oscillations are differentiated. The free oscillations shown in Figure 1.1. are excited by a shock disturbance such as, for example, the damped air oscillation following striking on a bell to produce the sound.

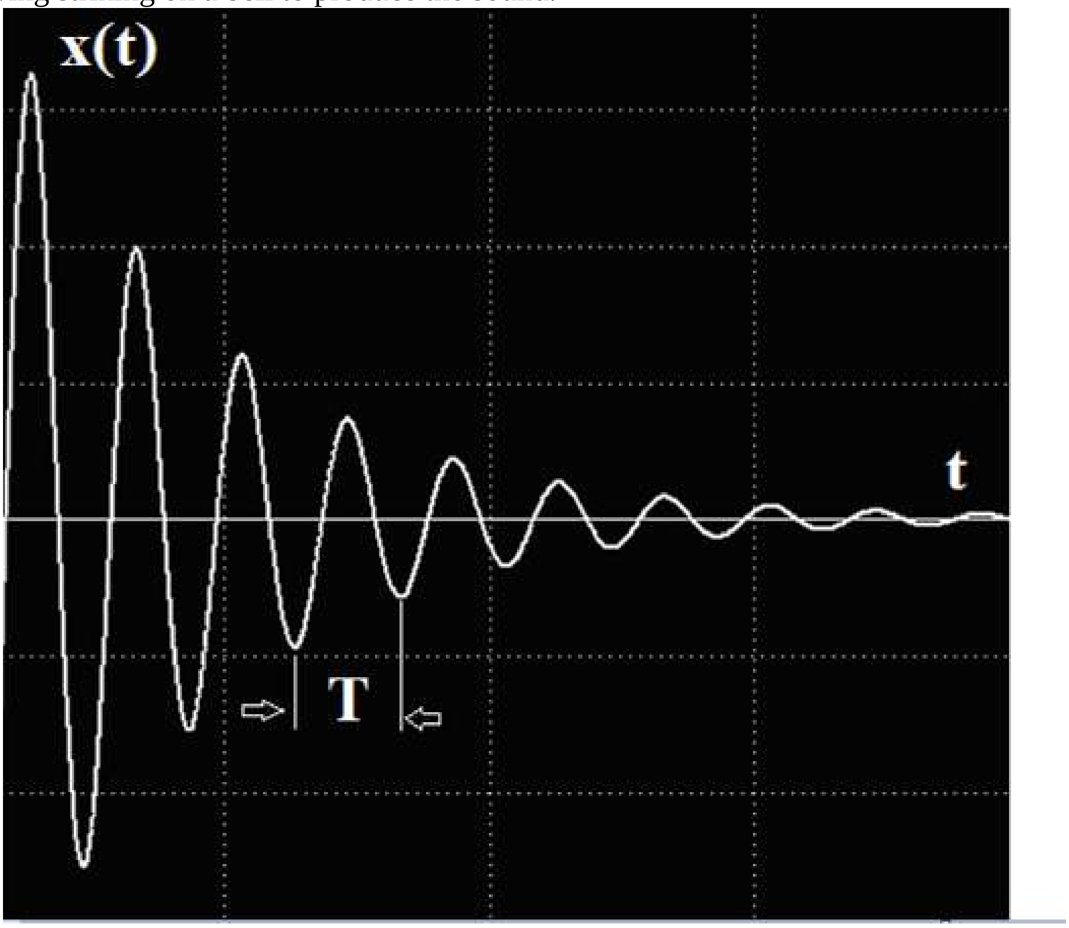


Figure 1.1. Free oscillation

The free oscillation frequency with a period, T, is called the natural oscillatory object frequency $\omega_0 = 2\pi / T$. When the forcing periodic disturbance frequency coincides with the natural frequency of the object, the sustained forced oscillations of the same frequency have the maximum amplitude, A, proportional to the amplitude of the forcing oscillations (see Figure 1.2.).

In addition to the amplitude, A, the sustained oscillations shown in Figure 1.2. are characterized by the circular frequency $\omega=2\pi/T$ (rad/s) and linear frequency $f=1/T$ (Hz), where T (s) oscillation period. The running phase $\psi=\omega t$ (rad) is often used along the x-axis instead of the running time, t. The relative phase or phase of oscillations $\varphi=\omega\tau$ (rad) is an important parameter of an oscillating process. φ defines the shift, τ, of oscillations relative to a certain reference harmonic wave of the same frequency. The forcing/external oscillation is usually taken as a reference signal given in Fig. 1.2. in a thin line.

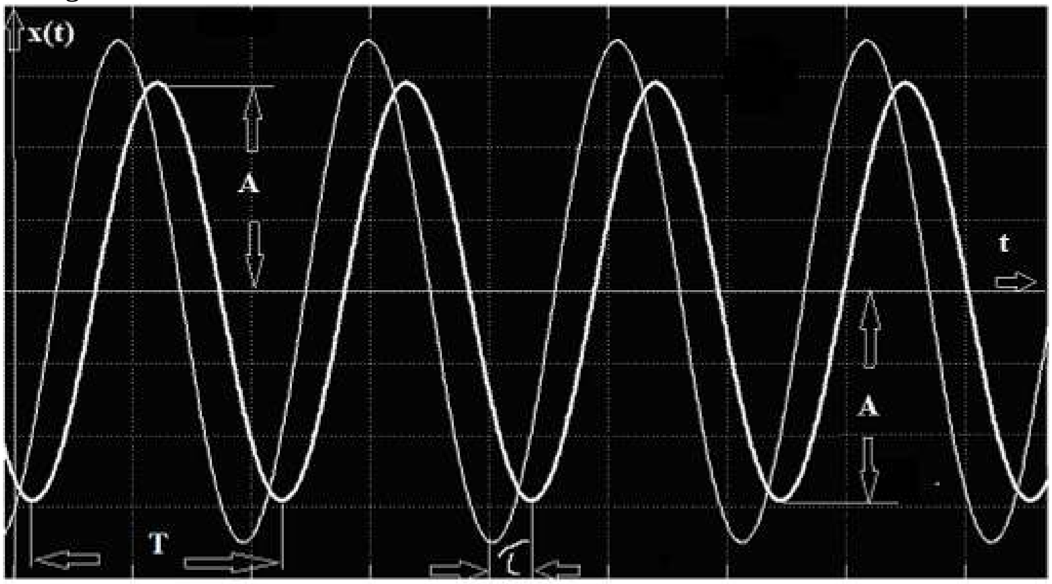


Figure 1.2. Forced oscillations

An oscillatory object is characterized by coordinates and parameters. The coordinate time variations generally form the motion of a dynamic object and particularly define the periodic motion of the oscillatory object. The oscillatory object complexity and describing equation order are assessed by an amount of coordinates to describe oscillating process conditions. As soon as either free or forced oscillations are exited dependent coordinates also oscillate to a different degree. The oscillation degree and interconnections of the coordinates are governed by the oscillatory object parameters. As a rule the object parameters are constants forming the equation coefficients according to the operating principles of the object. Thus, for instance, a circulatory system of a living organism includes a heart as a blood pressure converter, vessels (arteries, veins and capillaries), a liver as a blood-forming organ and a brain and nerve fibers as a control system. The system coordinates are the blood pressures at the different points of the organism, rates of pressure variations, the blood velocities and volume flow rates. The system parameters are blood viscosity, vascular system drag/dimensions, a hart volume, nervous system status, etc. The heart operates in periodical pulse oscillation modes. That’s why the blood pressure varies in a periodical way too. The external disturbances with respect to the circulatory system are atmospheric pressure variations, environment temperature changes and the variations of physical activities and moral stresses.

Following the disturbance oscillations, at first the control subsystem changes a vessel state (by compressing/releasing) and then it alters the hart beet frequency and filling to rise or decrease of the blood pressure. In such a way the circulatory system parameters are periodically varied. Of course, that is only the simplified description of such much more complex dynamic system like the circulatory system. The objective of this paper is popularization of scientific knowledge. So, complex dynamic systems are not considered hereinafter.

If object parameters are constant, the object/system is called stationary. When object parameters are changed in time, such object/system is called nonstationary. When the nonstationary object parameters are periodically changed, such object/system is called periodic nonstationary or parametric. The both stationary and nonstationary systems may be linear and nonlinear by a kind of their description.

Thus, the main attention will be hereinafter focused on periodically nonstationary oscillatory objects and systems, both linear and nonlinear. The parametric systems became widespread in scientific-technical activities such as mechanics, electrical engineering, radio engineering, automatic control, instrument engineering, hydrodynamics, aerodynamics, etc. In all signs, the priority in studying parametric systems belongs to mechanics. So, we start our story from that subject.

Parametric Resonance in Mechanics

This section is aimed to introduce the phenomenon of parametric oscillations by simple and clear examples from mechanics and get the reader ready for some acquaintance with more complex parametric phenomena. It should be noted, there are not really many examples of the oscillatory objects and systems in which parametric resonance openly appears in its true form. It more often "prefers" to hide mysteriously behind complex oscillating processes.

Common pendulum. Figure 1.3. shows an ordinary pendulum. Assume that environmental/air resistance and the frictional force at the suspension point/support, O, are minor, i.e. the pendulum is near ideal. If firstly the pendulum is deflected from its vertical position to the right at the angle +α or to the left at the angle –α and released then, the free pendulum oscillations will decay (see Figure 1.1.). Under the vertical suspension point vibration Δx(t)= a sinΩt (see Figure 1.3.) the complementary acceleration $\varepsilon(t) = a\Omega^2 \sin \Omega t$ acts on the pendulum. Its min/max value is $\pm a\Omega^2$. So, the pendulum weight becomes alternating as $P = m(g - a\Omega^2 \sin \Omega t)$. The damped pendulum oscillations begin to grow on at the vibration frequency, Ω, close to the doubled natural frequency, ω_0, as soon as the vibration amplitude exceeds a certain threshold value $a \succ a_{\text{th}}$. ***Those are divergent parametric oscillations excited and there comes parametric resonance.***

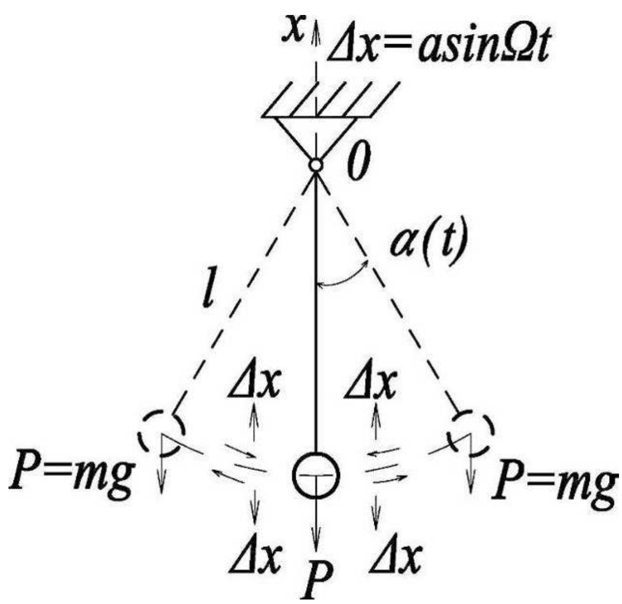


Figure 1.3. Pendulum with suspension point vibration

Figure 1.4. shows the pendulum parametric oscillations and the suspension point oscillations versus time.

It is important to note on the basis of the above simple example that the parametric resonance excitation occurs at a certain shift, τ, between the suspension/parameter oscillations Δx(t) (see

Figure 1.4. in a thin line) and the pendulum oscillations α(t) (see Figure 1.4. in a heavy line). The shift is about a half of the pendulum oscillation period, which corresponds to a quarter

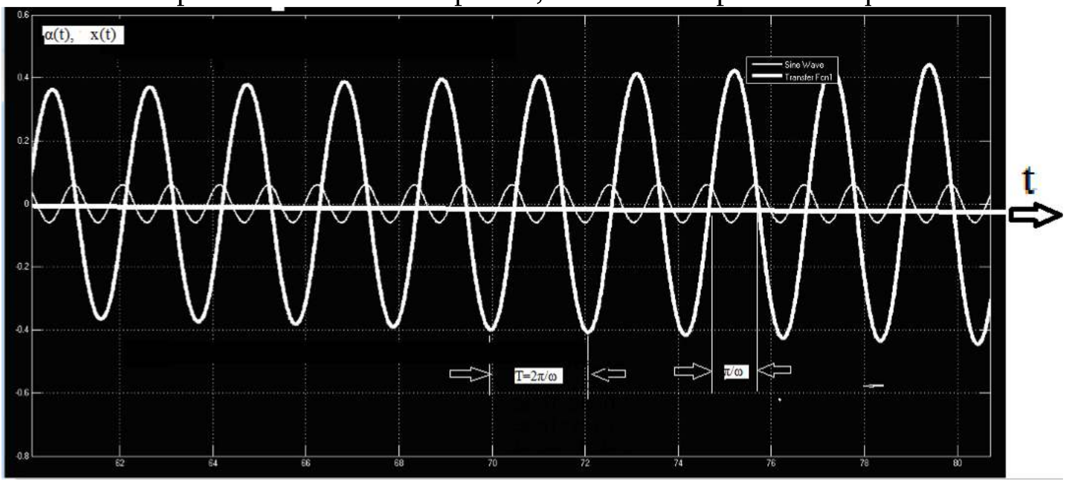


Figure 1.4. Ramp parametric oscillation of pendulum

of the pendulum oscillation period or phase shift ψ=ωτ=2πτ/T (rad) equal to 90°. It is easy to explain the necessity in such shift in terms of physical principles. The pendulum oscillation period includes four quarters. The first quarter I (see Figure 1.3. and Figure 1.19.) located between the vertical line and the maximum +α and the third quarter III located between the vertical line and the minimum –α are the deceleration quarters because the gravity moment (the product of the pendulum length and the projection of gravity to a moving direction) is oriented against the pendulum moving direction. The second quarter II and the fourth quarter IY are the acceleration quarters because the gravity moment coincides with the pendulum moving direction. Thus, to swing the pendulum effectively there is a need to diminish the gravity moment within the deceleration segments by decreasing gravity and to increase that within the acceleration segments. Hence, to provide the needed vibration/parameter behavior the pendulum support, O, has to be moved downward in the deceleration segments where the pendulum weight reduces because a vibration acceleration value is subtracted from a gravitational acceleration value, and it has to move upward in the acceleration segments where the pendulum weight enlarges as a result of the summation of the above accelerations. ***Parametric resonance is excited in that way.***

There is an essential difference between parametric oscillations and forced oscillations. The forced pendulum oscillations occur when an external periodic moment is applied to the pendulum, for example, while you swing the latter and the pendulum oscillation frequency coincides with the external action variation frequency whereas the parametric oscillation frequency is two times less than the parameter/oscillation variation frequency. The parametric oscillation phase is fixed whereas the forced oscillation phase depends on the oscillatory object behavior and the forcing disturbance frequency.

Spring pendulum. We analyze the following example. It is a spring pendulum inside a gravitational field (see Figure 1.5.). The pendulum has a stable equilibrium state at the point $x = \ell_0$, where the weight, P, is balanced by the opposite force of expanded spring. If the load is slightly pulled downward or raised and then released, the damped free oscillations of the load Δx(t) are observed along the vertical x-axis in the condition when the resistance forces are minor.

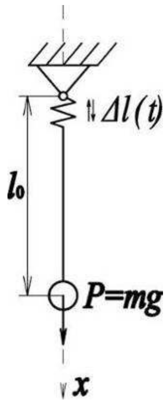


Figure 1.5. Spring pendulum

At a certain peak-to-peak increment of the periodic vertical oscillations the new pendulum oscillations α(t) about a center of rotation arises in the plane of Figure 1.5. These are parametric oscillations. That phenomenon is not difficult to explain. During the vertical oscillations the pendulum length $\ell(t) = \ell_0 + \Delta\ell(t)$ changes periodically together with the periodic spring divergence and compression. In turn, the periodic increment in pendulum length results in the periodic decrease in gravity moment within the deceleration quarters, and it increases within the acceleration quarters. Therefore the parametric oscillations α(t) similar to those presented in the previous example, occur (see Figure 1.6.), and the oscillation time diagram is also similar to that in Figure 1.4. where the oscillations with a length of $\Delta\ell(t)$ take place instead of the vibrations Δx(t).

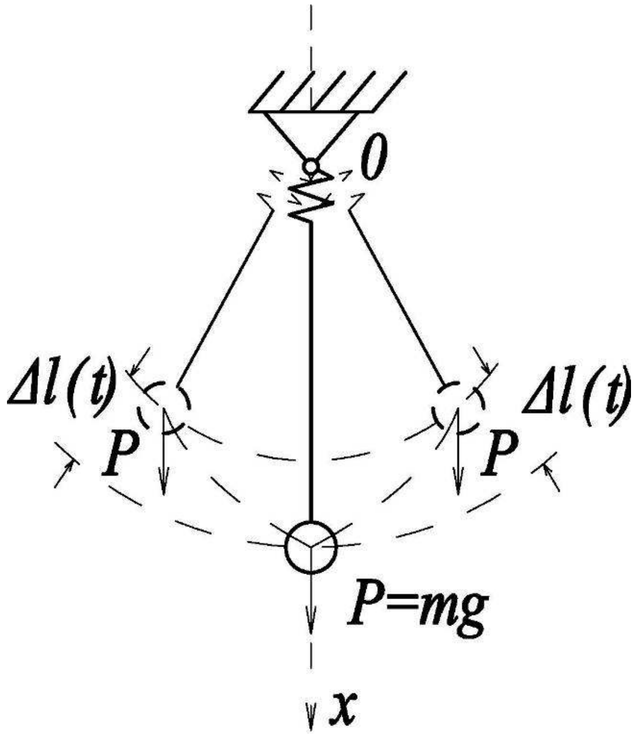


Figure 1.6. Spring pendulum oscillation

Of course, as the vertical length oscillations decay, parametric oscillations are eliminated too. But if the vertical load oscillations are supported by a periodic forcing disturbance, thereby sustained parametric oscillations are sustained. The interesting fact is that while the forced oscillations of pendulum deviation angle are kept sustained, the vertical load oscillations with a double angle oscillation frequency also begin. Those oscillations are also forced because they are not excited by a variable parameter (e.g. a variable spring rate) but related to the spring extensions at α=0 at the expense of the maximum vertical gravity projections and the centrifugal pendulum rotation force.

The parametric oscillations considered here were discovered long ago at the down of train operations in the course of the study of swing carriage couplers. At that time the coupler springs had low stiffness. The transverse parametric oscillations threatening with accidents appeared under the oscillations of springs and integral trains in the horizontal plane along a railroad way.

Elastic shaft rotation. Figure 1.7. shows an elastic rectangular shaft. It has both horizontal, c_x, and vertical, c_y, flexural stiffnesses differing from each other, and the constant torsional stiffness, c_z. One of the shaft ends is supported as a cantilever in a spindle and the other one is free. While the spindle rotates about the z-axis, the free end of the shaft sags along the y-axis under the action of gravity. A sag value is scaled inversely with the alternate stiffness values c_x and c_y. The flexural homogeneous shaft has uniformly distributed mass, and its stiffness depends on the distance to gravity point. In other words, the elastic shaft is characterized by length-distributed mass and stiffness as distinct from the lumped pendulum parameters.

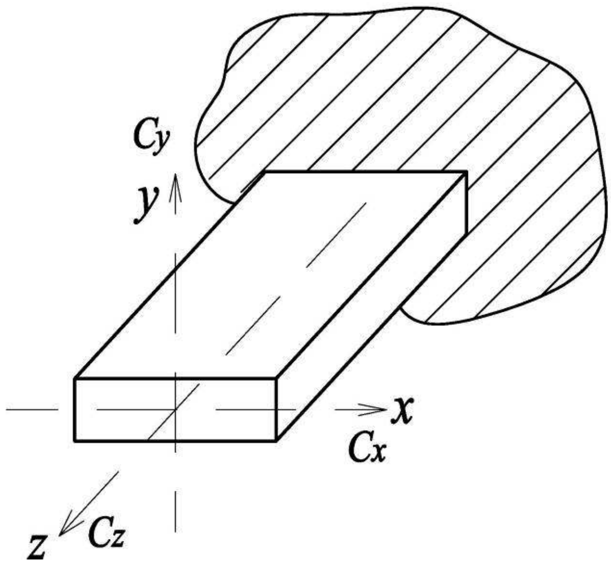


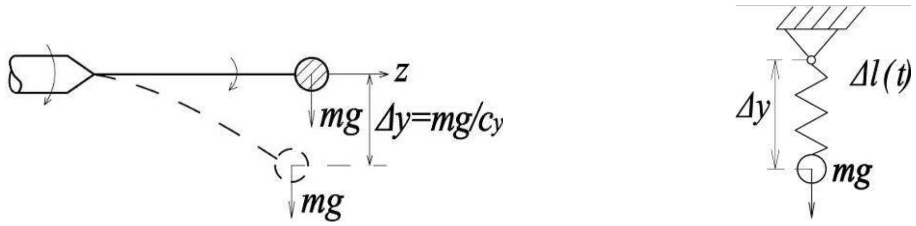
Figure 1.7. Elastic shaft rotation

It is known that distributed parameter objects definable in mathematical physics by partial differential equations can have an infinite set of natural frequencies and complex combination oscillations. During approximate calculations realized in many kinds of engineering software, including FEA, the descriptions of distributed parameter systems are substituted for their discrete analogs of ordinary differential equations, i.e. difference equations or finite-difference equations. The systems described by approximate difference equations also have periodic frequency characteristics and an infinite series of natural frequencies

$$\omega_n = n\omega_0,$$

where n=1,2,3 and ω_0 is the first/fundamental frequency.

For illustration we substitute the distributed mass for the lumped mass located in the middle of the shaft and suspended from an instantaneous plate. The plate possesses three kinds of stiffness as it is shown in Figure 1.7. For simplicity, we assume stiffness along the y-axis is mach less than that along the x-axis. In such approximation the rotating shaft model is given in Figure 1.8. The lumped load with weight P=mg periodically sags two times in a rotation period at the distance $\Delta y = mg / c_y$, i.e. doubled-period continuous forced flexural oscillations of the plate occur. The simple model of such oscillations in the form of the spring pendulum can be seen in Figure 1.8.b) equivalent to Figure 1.5.



a) b)

Figure 1.8. Simplified model of elastic shaft oscillation

As it was previously shown the parametric oscillations α(t) about the z-axis arise at the shaft rotation frequency under parameter/pendulum length oscillations (see Figure 1.6.). Thus, the flexural oscillations Δy(t) activate the torsional parametric oscillations α(t).

This kind of commonly destructive parametric resonance is called flexure-torsion flutter. For instance, this takes place when a shaft bar is under HS turn processing having even not large eccentricity.

As it was mentioned above, the parameter-distributed objects are attributed by a lot of natural oscillation frequencies. As a rotation frequency increases, primarily the first most dangerous resonance more often appears at the frequency ω_0. Just that very case is shown in Figure 1.8. Its danger lays in a low oscillation frequency and, as a rule, a high oscillation amplitude at a low oscillation damping rate. To avoid breaking the spindle revolutions are needed to be diminished or enlarged. As soon as the revolutions are increased to the next natural frequency $n\omega_0$ the parametric resonance of lower amplitude and in a more complex form/mode is exited again, and in that case one or several quarters of the oscillatory wave are fitted over the shaft length.

In the investigated example the shaft revolution leads to forced periodic oscillations, and as a result parametric oscillations are excited. To launch flexure-torsion flutter rotating is not necessary. The flutter arises when a liquid/gas (air stream) flows around oscillatory objects. This often causes the accidents of structures, bridges, aircraft, etc. The descriptions of some of them are presented hereinafter in the sections on hydrodynamics and aerodynamics.

At the end of the section we consider one of the examples in which parametric resonance plays not a destructive but constructive part.

Inversed pendulum. It was presented in the section Ordinary pendulum (see Figure 1.3.) that whiles the suspension point vibrates in a vertical direction the nascent parametric oscillations result in the loss of equilibrium stability. The inversed vertical pendulum is shown in Figure 1.9. Its vertical position is not stable if there is no any supporting external forcing. Objects like that are called structurally unstable because they cannot be brought to a stable state at all the values of constant parameters (e.g. length and weight).

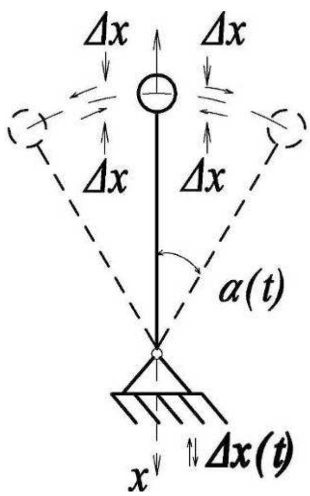


Figure 1.9. Inversed pendulum oscillations

Just as in the case of an ordinary pendulum, in the presence of the suspension point vertical vibration $\Delta x = a \sin \Omega t$ the alternating vibration acceleration $-a\Omega^2 \sin \Omega t$ is periodically summarized/subtracted from the gravitational acceleration, g, at the frequency, Ω. As a result of the cycle parameter/pendulum weight variations resonance and parametric oscillations originated in an anti-phased manner stabilize the unstable pendulum position. The following obvious inequality is the necessary stabilization condition:

$$a\Omega^2 > g .$$

The resonance was detected by P.L. Kapitza for the first time in the middle of the last century. He was the first who showed that an inversed pendulum becomes stable at its vertical position when

$$a\Omega > \sqrt{2g\ell} ,$$

where a is the vibration amplitude, ℓ is the pendulum length. The last condition means that the maximum linear velocity of the support movement has to exceed the free fall velocity of the pendulum from the height equal to the pendulum length. In literature the stable-under-vibration inversed pendulum is called Kapitza's pendulum.

Since time immemorial the visual demonstration of Kapitsa's pendulum has been shown during the children's contests in which a vertical stick is kept on a palm oscillating in a vertical/horizontal plane for making the longest time.

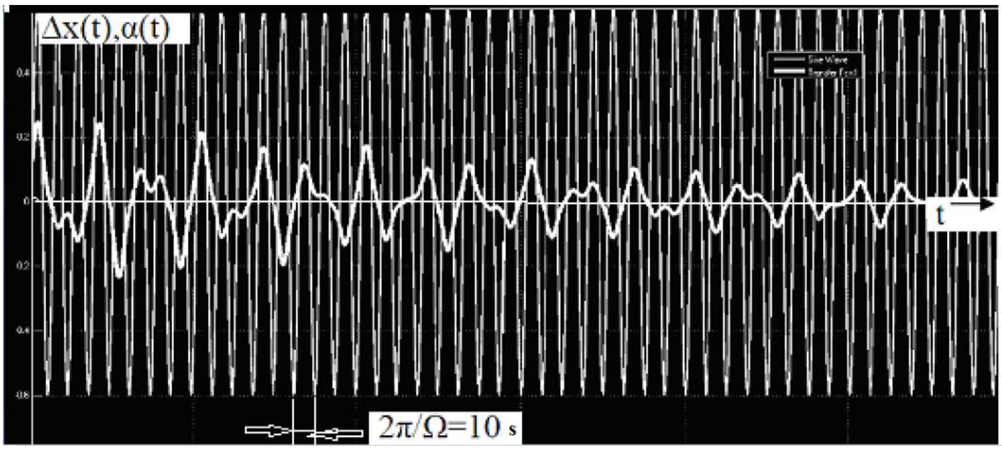


Figure 1.10. Transient of inversed pendulum stabilization

The transient of inversed pendulum stabilization is obtained by numerical simulations of Kapitza's pendulum with a one meter length (see Part 2, Oscillation process numerical simulation). One can see in Figure 1.10. that stability is reached at the 0.6 m vibration amplitude and the 60 m/s^2 acceleration.

Parametric resonance in electrical engineering

Parametric oscillatory circuit. At the end of 1930th Academicians L.I. Mandelshtam and N.D. Papalekcy proposed to apply parametric resonance in making oscillators. The RLC-oscillatory circuit with periodically changed capacitor capacitance, C, was used to generate the oscillations (see Figure 1.11.). The capacitor capacitance was alternated following the cyclic variations of the capacitor plate gap. However the conductive disk provided with slots and rotated between the plates by an electromotor was suitable to a greater degree to modulate the electric field of the capacitor. The parameter/capacitor capacity oscillation frequency was varied by changing the electromotor speed, and the parametric oscillations of electric current and voltage at the circuit elements were excited within the circuit natural frequency. At the same time the parametric oscillation frequency close to the circuit natural frequency was two times less than the parameter oscillation frequency.

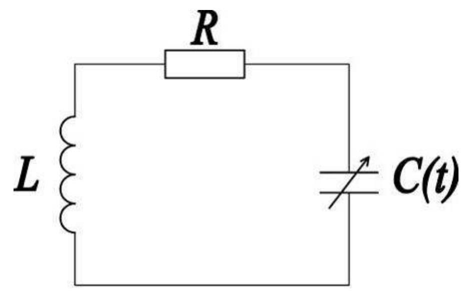


Figure 1.11. Electric oscillatory circuit

Exciting the parametric oscillations is accompanied with the accumulation of oscillation energy in the circuit. If one slides the capacitor plates apart, thereby diminishing the capacitance, as soon as the extreme voltage is reached, then the capacitor voltage, V, is increased because the capacitor charge q=CU cannot be rapidly changed owing to a capacitance lag. As the voltage grows circuit energy grows as square of the voltage. If the capacitor plates are moved closer to each other as soon as the zero capacitor voltage is reached, the circuit energy is not decreased. Thus the accumulation of circuit energy occurs provided that energy growth exceeds energy loss at the active resistance, R. It should be noted, that the energy of parametric oscillations in the circuit grows at the expense of energy loss for moving the capacitor plates or rotating the disk to modulate the capacitor field. The example considered is also related to the infrequent cases in which parametric oscillations are mentioned to arise in their pure forms.

At present the energy of parametrically excited oscillations is not often utilized because of an emergence of powerful semiconductor key/thyristor technologies and alternating-voltage generators/inverters based on those technologies. The development and wide-spread usage of parametrically excited oscillations have not gone towards the energy but generation and oscillation amplification of super high frequencies/SHF direction in radiocommunications and radiolocation. To generate and amplify parametric SHF, highly small-sized, noise-eliminating resonators/circuits are used.

Ferroresonance – hidden parametric resonance. The phenomenon of ferroresonance has been known in electrical engineering for a long time. This happens in respect of forced oscillations in the RLC-circuit consisting of the inductor, L, with a steel core. This phenomenon appears under certain conditions as amplitude jumps of forced oscillations while the forcing oscillation amplitude/frequency is smoothly varied.

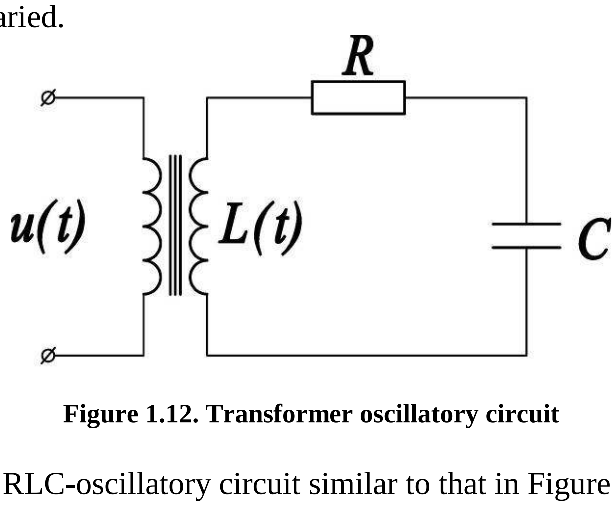


Figure 1.12. Transformer oscillatory circuit

Figure 1.12. gives the RLC-oscillatory circuit similar to that in Figure 1.11. but equipped with the transformer input of alternating voltage u(t)=usinωt. The secondary coil of the transformer is based on a steel core and serves as a circuit inductor. As the amplitude and the voltage increase the secondary voltage amplitude at the inductor, L, grows to a certain moment, following which increasing the secondary voltage slows down. The further increase in input voltage brings to the circuit voltage jump, i.e. stepwise resonance or ferroresonance takes place. The specific characteristic of those jumps is the binding presence of steel core in the coil, where from the name ‘ferroreso-

nance' comes. When the core is missed the jumps are not observed. It turned out that a true cause of the jumps was the parametric resonance excitation in the circuit. In Figure 1.12. the inductance, C, is a variable. Permeability of the core decreases to the low value equal to air permeability as long as the amplitude of the current magnetizing the core increases. In such a case they say about saturation of the core magnetic. Thus, there take place maximum inductance at the linear section of magnetization while there comes no saturation. Inductance is zero under saturation. The frequency of inductance oscillations is $\Omega=2\omega$ and the frequency of new growing parametric oscillations is equal to that of forced oscillations because saturation occurs two times in the period $T=2\pi/\omega$ when the maximum amplitude of magnetizing current is realized. The forced oscillations and the first parametric resonance oscillations are composed and the amplitude jump is observed. And it is not possible to distinguish the parametric oscillations from the forced oscillations, which explains mysterious effects of that resonance.

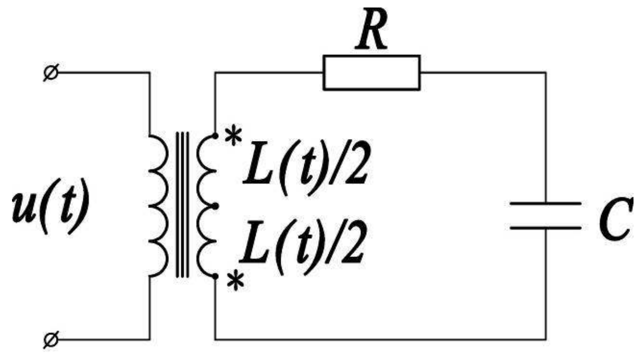


Figure 1.13. Inductive parametric resonance

It is not possible to distinguish parametric oscillations so indeed but they can be separated out in their true form. For that purpose the conductor coil is made in the form of two opposing identical coils, either coil inductance is L/2 (see Figure 1.13.). In that case the primary forcing voltage is antiphasely transformed to either of the two circuit coils, and the final voltage at the secondary transformer terminals is zero. In other words, there are no forced oscillations in the circuit. Nevertheless, in the circuit (see Figure 1.13.) considerable oscillations arise at the jump/ferroresonance frequency presented in Figure 1.12. These are the parametric oscillations hidden before and they are extracted in their pure form, i.e. we deal with the true reason of ferroresonance.

The effect of parametric oscillation extraction from forced oscillations can be explained by the fact that in spite of missing the forced oscillations in the circuit (see Figure 1.13.) magnetizing the steel core is kept on by the forcing voltage of primary circuit, and the circuit inductance equal to the sum of two coil inductance goes on changing from its maximum to about zero.

Flexure-torsion resonance known from mechanics causes a number of problems in electromechanics, e.g. in production and service of powerful power plant turbogenerators. Unbalancing under manufacturing and mounting of large-tonnage rotors can lead, for example, to flexure-torsion resonance in use of turbogenerators, which happened after putting into operation Krasnoyarsk hydroelectric power station.

Parametric resonance in fluid dynamics

In numerous complex problems of fluid dynamics either a motion of medium (liquid, gas, air) in which an object is situated or an object (aircraft, a rocket, a ship) motion inside an environment and also medium/object interference are considered. As a rule those problems cannot be analytically solved. They are accomplished using numerical methods or physical modeling of a medium and an object. The analogies between elementary fluid-dynamic problems touched on below and the simple tasks from mechanics and electrical engineering previously examined are sighted.

From time immemorial projecting and servicing water transports (river boats, sea craft, etc) developers' efforts have been focused on the impact of liquid/water flow and wave generation on oscillatory objects. For instance it is connected with navigation safety. Navigation hazard has been confirmed by numerous accidents in the history of navigation.

To a first approximation water-craft in water is a stable pendulum thanks to the fact that its center of gravity is located under the water below water-craft waterline. The pendulum is capable of swinging at the angles which can cause flipping water-craft provided that the frequency of water surface fluctuations (rough water) is equal to the water-craft natural frequency or divisible by the latter. Such a situation is mostly dangerous if badly-fixed cargo is transported or bilge tanks are partially loaded. So about 40 years ago the information appeared in press that the dry cargo ship Komsomolets Uzbekistana was a total wreck in the conditions of not great choppy sea in consequence of cargo displacement. Fortunately, the ship's crew was rescued by the US coast guard. Probably, for the reason stated above, the information which was often concealed was published in the press.

In accidents like that cyclic rolling-induced cargo/parameter displacements periodically change the position of water-craft center of gravity and amplify capsizing moment. That is why the transportation with badly fixed cargo and partially loaded liquid tanks is forbidden according to navigation instructions. By the instructions changing a water-craft course is also recommended to mitigate rocking by making an angle with a wave front. Though a wavelength remains as it was before, rocking both period and amplitude are shifted from the water-craft natural frequency. Not so long ago parametric oscillations of water-craft moving down wind and wave were exhibited. At that moment first the water-craft stern was coming up and down at a wave crest, and the bow was following of doing the same. Thus the longitudinal oscillations of the water-craft were run two times in a wave period, and the transverse oscillations similar to those in respect of a spring pendulum were occurring.

Not only wave generation can be at the bottom of both forced and parametric oscillations. Another nature of the oscillations is water flowing around an oscillatory object. Vortex rotations of the liquid are formed during water flowing around a body (see e.g. the cylindrical solid in Figure 1.14.). While the vortexes detach from the body surface periodical forces applied to the body are generated. Those forces are capable of arising forced transverse oscillations. These oscillations generated together with both parametric and self-excited oscillations are called stall flutter.

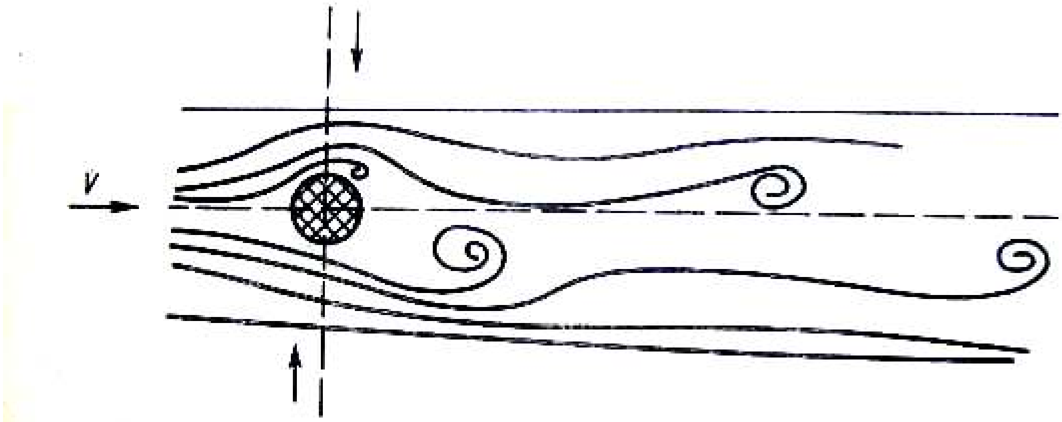


Figure 1.14. Vortex detachment

In whole, it should be noted that oscillation processes in liquids have a complex nature and most often combine forced parametric oscillations and nonlinear self-excited oscillations.

Let us consider several simple examples.

During the Great Patriotic war special mines were used to provide security of northern ports and naval bases. They were located at cable-anchored buoys submerged superficially. Some mines were soon found to detonate whereas no water-craft was thereabout. In the issue of close study of the mines it was turned out that the false bursts were connected with a small currents flowing around the buoys to result in stall flutter. The generated oscillations initiate the detonations of the mines.

Another rather comical example is proposed for digress the reader from war subjects. Fishery-lovers are well known with stall flutter. It exhibits as slow reverse oscillations of the boat bow-anchored at a stream. Such movements are too dangerous during bottom-rod angling. In such situations a short rod curves gradually brining in an arch form and an inexperienced fisher catches at a landing net in looking forward to catch a fish. Alas! Big fish takes the bait in another way!

As an example the following occurrence from the author's experience is given.

"Once in early spring I drove my boat from Lake Ladoga to the Gulf of Finland. My boat came to the Neva by the evening. It was getting dark and my boat approached to the Ivanovsky rapids. The current is always strong in that region of the river and it becomes swift at the time of a spring water flood. At dusk I suddenly saw a strange phenomenon ahead: a certain 'being' was going from under the water. Having gone up to its 2-meter full height the being started to lower and disappeared under the water then. A little later the strange phenomenon happened again. By that time I had gained considerable experience in sailing: I rowed across Lake Ladoga from Valaam to Novaya Ladoga at a stormy night, astonished at over-water mirages in fair weathers, and one day I went down as a result of the collision with a log (fortunately I could put a tarpaulin patch to the breach), etc. But at that moment I was ill at ease. What this could be? Boating close with 'the monster' I saw a big submerged river buoy going from under the water."

The approximate analysis of the buoy oscillations is given in the section 'Buoy' resonance (see Part 2). It turned out that forced oscillations of the buoy submerged to be initiated, for example, by stall flutter are able to cause parametric resonance which, in turn, gives a rise to the amplitude jump of initial forced oscillations. The initiation of hidden parametric oscillations is connected with the fact that the moment, M, pushing out the buoy nonlinearly depends on the deviation angle as follows:

$$M = M_0\alpha + M_1\alpha^3 .$$

So under the forced oscillations of the angle $\alpha(t) = A\sin\omega t$ the parameter/rate of moment change varies periodically:

$$dM/d\alpha\big|_{\alpha(t)} = M_0 + 3M_1A^2\sin^2\omega t = M_0 + 1{,}5M_1A^2(1+\cos 2\omega t) .$$

It seems that the first 'hidden' parametric resonance with a forced oscillation frequency, ω, can arise at the double parameter variation frequency.

Parametric resonance in aerodynamics

In hydrodynamics the effect of a moving medium/liquid on oscillatory objects is mostly investigated. In aerodynamics the influence of a stagnant environment/air upon a moving oscillation object or a separate part of the object/ aircraft is usually considered. In essence the tasks on oscillations and stability of buildings/structures under the action of wind loadings are also related to aerodynamics. First of all among the structures are bridges and high-rise towers and buildings. Although wind loadings are not comparable with the action of the atmosphere on aircraft it is worth a lot to ignore them.

The story of Tacoma Narrows Bridge is widely known. That bridge is one in twelve the masterpieces of the American bridge engineering. In 1940 swinging in the directional breeze the bridge crashed down. In the old amateur oscillation images one could watch the growing divergent torsional and flexural oscillations of the structure. The flexural oscillations arose on the leeward side of the bridge during the airflow. They were due to eddy formations and the difference of upper and lower pressures to initiate the parametric excitation of torsional oscillations. The latter in turn increased the flexural oscillation amplitude - The bridge crashed down owing to flexure-torsion flatter. It was a lot to happen in last century. What about our days?

Have a look at the lower Figure. One can see the dangerous oscillations of the bridge across the Volga near Volgograd. In the October of 2009 it was put into service in a solemn atmosphere and just in the May of 2010 the traffic over the bridge was stopped up. Shipping was also forbidden in that zone of the Volga.

30 m/s-velocity flaws were observed in the May of 2010. Peak-to peak oscillations reached two meters. It seems that great wind loads weakened the bolts attaching the separate spans to the abutments. There is a video recording in Internet (see the website http://pro-volgograd.ru) in which one can see the flexural-torsion oscillations of the spans at the wind velocity that is much less than 30 m/s, judging by wave making, (see the upper image in Figure 1.15.) and the same kind of oscillations between the separate bridge towers (see the lower image in Figure 1.15). Luckily it cost without wreck.

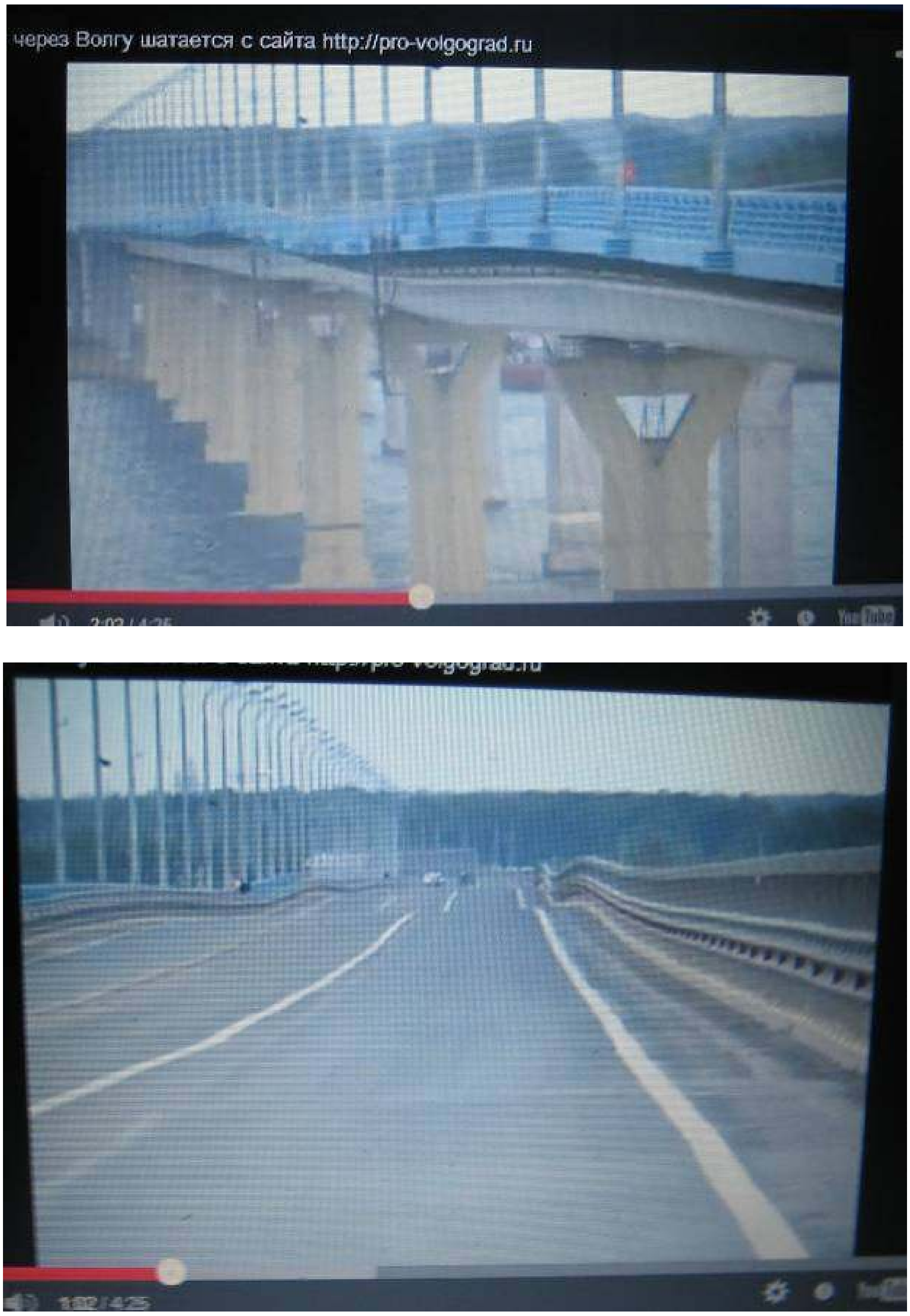


Figure 1.15. Views of new bridge over Volga (from vebsite http://pro-volgograd.ru)

The new 7-kilomiter bridge was built for about 10 years for nearly 25 billion roubles. At present it is locked. Pressing and sizeable repair lies ahead. Just so parametric resonance may penalize for errors and “economy”.

Aircraft is to overcome air flows at the speeds which are several orders higher than a wind velocity. Hence, air quality is assumed to be motionless relative to aircraft. The second conclusion is wind tunnel tests of structural/element models are obligatory to be performed in connection with the high flight speeds of aircraft and possible oscillation excitation.

As a rule an aircraft wing has modest flexural stiffness and is able to make minor oscillations in the vertical plane. Sometimes the oscillations can be watched through a side aircraft window. The torsional wing stiffness about the longitudinal axis is too greater, so torsional oscillations can be only recorded by devices.

An aircraft wing can be simplistically conceived as a beam with overhang end to be fixed in the aircraft case. The beam stiffness is obtained to correspond to the wing-averaged torsional/flexural stiffness.

By virtue of the fact that the vertical cross sections of the top and lower wing surfaces are provided with the mutually unsymmetrical convexities, i.e. different flow conditions are realized at the surfaces, minor both flexible and torsional oscillations can be produced as a result of the different surface pressures.

Again assuming the similarity with a spring pendulum, one can conclude that flexure-torsion flutter is feasible: under certain conditions forced flexural oscillations can initiate parametric torsional oscillations. The flexure-torsion flutter is illustrated in Figure 1.16. where the vertical flexural oscillations are laid on the torsional/rotary oscillations, i.e. forced and parametric oscillations are interrelated. Thus, it is not so easy to understand “who” is guilty of the oscillations.

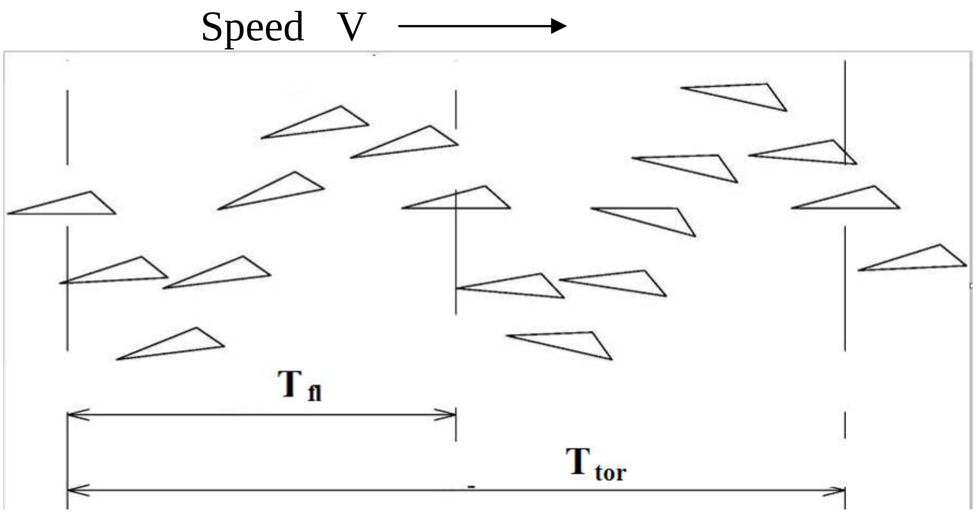


Figure 1.16. Flexure-torsion oscillations

In 1950s having reached sound velocities, jet aircraft faced the problem of overcoming a sound barrier. The problem was that as the aircraft velocity approached to the 1200 km/h sound velocity threatening vibrations of structural elements and a case started. So-called sound flatter was initiated. The point is that while the subsonic velocity increases the frequency of sound waves effecting on aircraft diminishes. After takeoff the sound wave frequencies are within the acoustic frequency spectrum of operating engine in the range from hundreds to thousand hertz. As the velocity grows the frequency of sound wave influence diminishes in proportion to the difference between the velocities of sound and aircraft. Following the diminishing and affecting the sound waves alternately concur with all the natural frequencies of flying spacecraft.

The situation is close to the considered rotation of the elastic rectangular shaft when the shaft rotational velocity changes from zero to maximum. The high frequency vibrations exert no serious influence on the shaft and the aircraft. And yet in advancing low natural frequencies the vibration amplitude is increased to its maximum value corresponding to the first natural frequency.

Because the first natural frequency of aircraft is around several hertz depending on the aircraft structure the most dangerous sound flatter is realized as soon as the difference in velocities of sound and aircraft is reached of about several scores of meters per a second, i.e. while approaching to a sound barrier. Just here the problem arises if the aircraft velocity is needed to be increased or decreased. In both cases the dangerous vibration is shut. At the development outset of jet aircraft there was no sufficient power for engines to be hopped, and flight velocity was to be decreased without any alternative option. As engines were improved their velocities grew as high as 3,000 km/h and sound flatter was overcome by abruptly increasing the aircraft velocity up to the value when the safe flatter duration is not over 1 second.

With regard to sound flatter, it little differs from flexure-torsion flatter. A 3-D sound wave periodically changes the aircraft pressure to vary, for example, vertical stiffness of the wings and initiate the parametric both flexural and further torsional oscillations.

Such kinds of oscillations can be excited in respect of not only the wings but the steering system and the aircraft sternpost. Moreover other kinds of oscillations are possible depending on flight conditions and structural philosophy. In whole, flatter is a challenging examination for aircraft and a test pilot. In conclusion it should be noted that there are a lot of oscillation control methods at designers' disposal, and the choice of natural structure frequencies beyond a frequency disturbance spectrum takes a priority place among them.

Self-excited oscillation or parametric oscillation?

Well, both steady and parametric linear dynamic systems can have convergent free oscillations under their stable conditions and increasing oscillations when those systems are unstable. The increasing parametric oscillations are called parametric resonance. Steady (nonincreasing and sustained) free oscillations of autonomous nonlinear systems are called self-excited oscillations. The term "self-excited oscillations" was introduced by Academician A.A. Andronov at the beginning of the last century and became customary in the theory of oscillations. At the same time the introduced term does not allow for understanding the reasons of oscillation self-excitation and stability.

Subsequently it turned out that self-excitation oscillation conditions in a nonlinear system are at the spectrum limit of the first parametric resonance of a linear system. Several explanations on the matter can be found hereafter (see Part 2, Forced oscillations). This can be explained by the fact that the slope of nonlinear characteristic/parameter F(x) oscillates following the x-coordinate oscillations. At that moment the parametric resonance oscillations corresponding to so-called soft excitation are initiated in a nonlinear system and "put into orbit" the steady/self-excited oscillations. These are steady oscillations if their "orbit" is stable. In turn the self-excited oscillation orbit will be stable if there are no any kinds of parametric resonance but the first. The essential requirement for self-excited oscillation stability is the shift of those oscillations from the excitation limit to the region where parametric resonance lacks/presents while increasing/decreasing the self-excited oscillation amplitude.

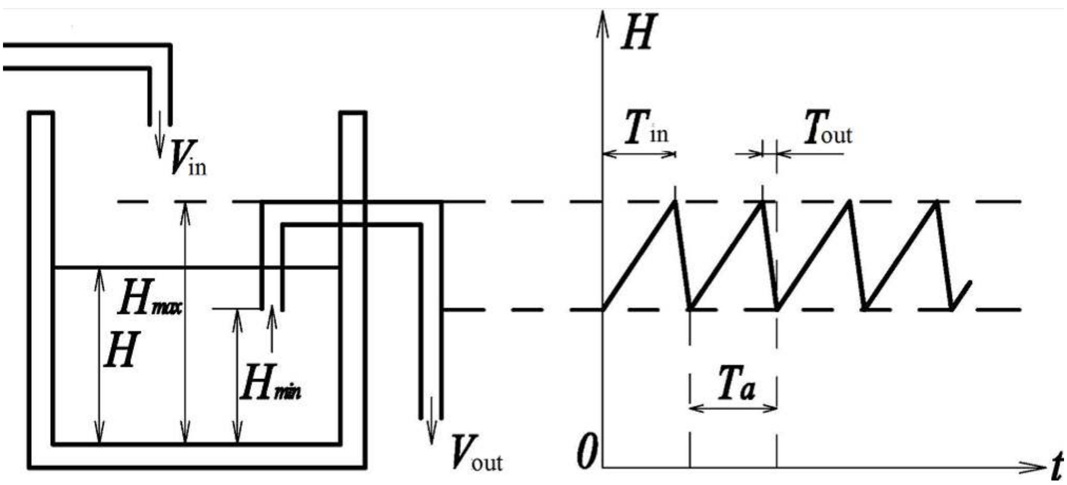


Figure 1.17. Self-excited oscillations of liquid level in Tantalus vessel

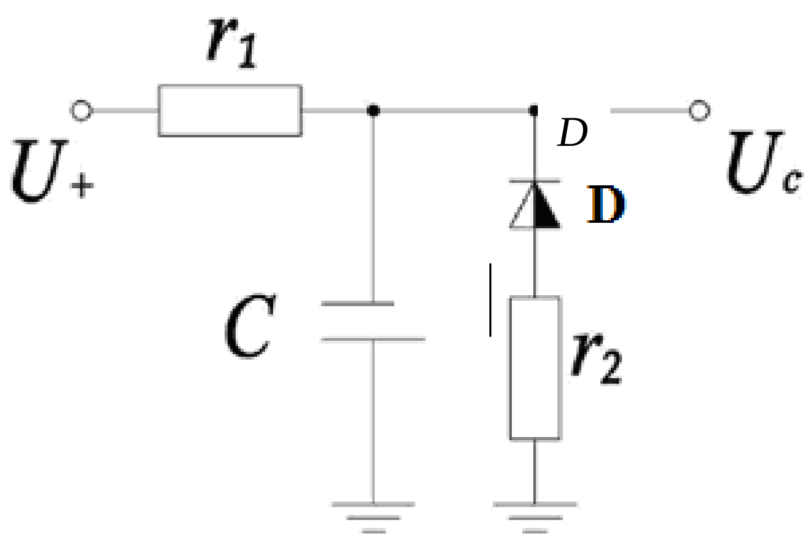
Thus, there are quit a lot reasons for self-excited oscillations to be considered steady parametric oscillations of nonlinear systems. It is the third kind of parametric oscillations taking place during the spring pendulum oscillations (see Part 2, Oscillation process numerical simulations). The simplest examples of numerous self-oscillatory systems are given hereinafter.

Let us consider the example of the hydraulic self-oscillatory system (see Figure 1.17.) which is called Tantalus's vessel. The vessel is filled up with water at the constant velocity V_{in} .

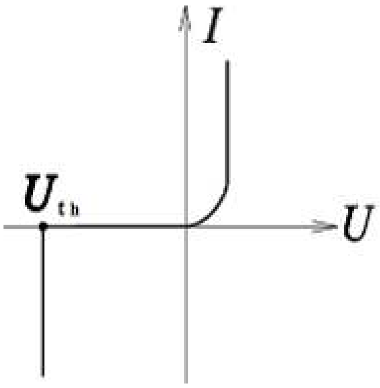
Once the maximum water level, $H_{\max}$, is reached the water starts to drain at the velocity $V_{out} > V_{in}$ as long as the minimum water level, $H_{\min}$, is reached. Hereupon water draining is stopped and filling up the vessel up to its maximum level is started again. The time diagram of water level self-excited oscillations is also given in Figure 1.17 on the right.

One can vary the drain speed by either changing the drain port diameter or lengthening the exhaust vertical pipe section. The presented hydraulic system is simultaneously both nonlinear and parametric because periodically turning on/off the drain port with a period T_a can be considered as a key nonlinear delay link or periodically step-wise varied parameter.

The electronic analog of Tantalus's vessel is given in Figure 1.18a. The capacitor C, makes the function of the vessel. Charging the capacitor is performed from the source of voltage U, through the resistor, r_1. As soon as the threshold voltage U_{th}, is reached at the diode/dinistor or controlled thyristor D, the capacitor is discharged through the resistor r_2 .Because $r_2 << r_1$ the capacitor will become discharged and the great back diode resistance will be recovered. Low charging the capacitor is begun again. The voltage-current characteristic of the diode is given in Figure 1.18c, where $T_1 = r_1 C$ and $T_2 = r_2 C$ are time constants.



a)



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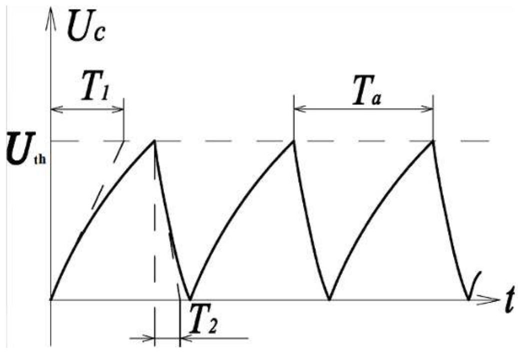


c)

Figure 1.18. Self-excited capacitor voltage oscillations

There are many other mechanical, electrical and electronic self-oscillatory systems, such as impact machines, lever clockworks, generators escapement mechanism and here it is - Stop! Attention! A rap was heard in the kitchen: the water has been boiled in the saucepan and that is its cap knock - one can watch parametric resonance. Self-excited oscillations set in after one or two minutes, those are steady parametric oscillations.

Ordinary swing paradoxes

Would a human been use parametric oscillations and parametric resonance? At present there may be both “yes” and “no”.

Certainly, visiting gymnastic competitions many people watched performances of master gymnasts and amazed by their excellent sporting mastery of trained bodies. The masters exploit parametric oscillations in an extremely accurate manner in making exercises at gymnastic apparatus, such as rings, parallel bars and a horizontal bar. So, for example, jumping up to the bar and rallying the sportsmen throw his body forward, makes one or two controlled forward swings/oscillations and just turns to handstand by an inversed pendulum. Just a little keeping the last position the sportsman makes several circulations, and breaking away from the bar he flies headlong forward, makes one/double somersault and lands on his legs. Bravo! The variable parameter/inertia moment and parametric oscillations were neatly used to reach top mastery!

The inertia moment varies following the pendulum length changes to arise parametric oscillations, which lose their stability and proceed to revolutions as soon as the oscillation amplitude becomes higher than $3\pi/4$. The revolutions are sustained and developed by means of the minor periodical variations of inertia moment too.

The most popular and favorite apparatus for children and young people is an ordinary swing. Just what kinds of emotions one can watch at a swing: ringing laugher and crying, fright and joy, rapture and brave daring! And how! Is there another place on the Earth where one can also undergo zero-gravity in gravity conditions? Justly swing services are invaluable in training of cosmonauts/astronauts of all generations.

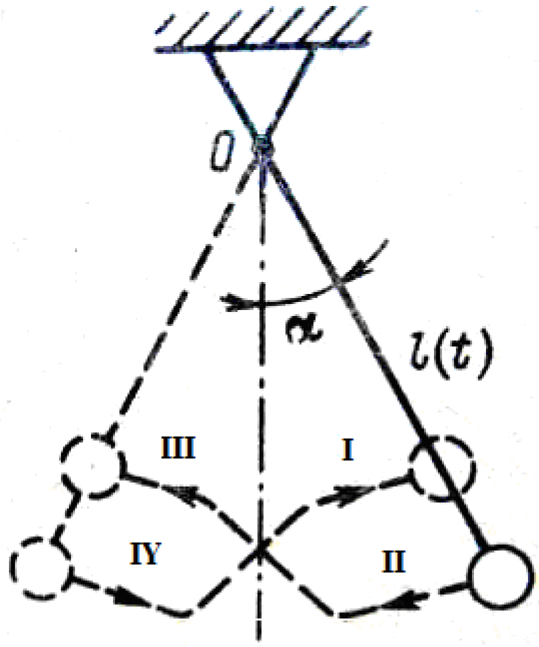


Figure 1.19. Swing oscillations

The swing parametric oscillations are excited by periodically moving a swinging man in the vertical plane (see Figure 1.19), i.e. by periodically changing the pendulum inertia moment and effective length. The swing is easiest to be rocked at the parametric oscillation frequency equal/close to the free oscillation natural frequency of the swing with a motionless load.

Moreover the frequency of parameter variation or vertical load/person movement has to be two times higher than the oscillation frequency of the swing. (Note: if two persons swing, they squat by turns to provide a two-fold parameter change in the swing oscillation period). The arising parametric resonance is called the first/main parametric resonance. But the above mentioned parametric oscillation excitation conditions are not still sufficient. Another important parametric excitation condition is the parameter oscillation phase relative to the swing oscillations. Load/person down movement (by squatting) has to be started as soon as the swing reaches its maximum swing deflection in either direction at the zero angular velocity of oscillations. In turn the load/person upward movements or the reductions of pendulum effective length have to be at the zero swing deflection from vertical position, i.e. at the maximum angular velocity. This requirement of effective swinging is apparent, for example, from the pendulum description in the case when the suspension point vibrations (see Figure 1.3.). The effective length just has to be increased in the second quarter II and in the fourth quarter IY (the acceleration quarters) of the pendulum oscillation period and it has to be decreased in the first quarter I and in the third quarter III (the deceleration quarters). The question is absolutely appropriate just now if what you should do to come the swing to a stop urgently in the situation when have swinging you want to jump down, and of course it is dangerous for you to leap down from the swing and you cannot reach neither hand nor foot of yours to the ground surface.

In whole it should be noted that such a simple apparatus as an ordinary swing is surprising in respect of an amount of effects and paradoxes occurring in the process of parametric oscillations. First of all, swing oscillations and parametric oscillations on the whole, have the lower parametric excitation threshold. So, in the section Ordinary swing (see Part 2) the threshold excitation condition is given as follows:

$$\lambda \geq \frac{\pi}{6}\xi \approx 0.5\xi ,$$

where $\ell = \ell_0 + \Delta\ell(t)$ is the pendulum suspension length, $\lambda = \Delta\ell / \ell_0$ is the relative parameter oscillation amplitude and ξ is a damping coefficient. Because of this threshold condition it is useful for young parents to bear in mind the following long-standing event happened to the author, and it reminds him that at times parents are unfair to their children without taking notice /

"Once swinging my three-year-old son at a wooden swing made by my grandfather I made up my mind that it is high time for the son to swing himself. Having explained the methods of winging I afforded the opportunity for him and imparted an initial deviation to the swings. My boy tried with all his strength but parametric resonance was not excited. After his several unsuccessful attempts I removed him from the swing in an emphatic manner. We got perfectly angry with each other - And a strong childish crying exploded the silence of Shuvalovsky Lake. My wife appeared immediately to the howl. On occasions like that she was interested in scientific propositions less than my pedagogic principle, and she was not interested in the latter at all! The son was placed at the swing, and I was given the exact instructions on the allowable oscillation amplitude - And I had to take up the forced oscillations again." [1].

Surely, I was not right. And just now I give a simplified principle for practical testing the threshold excitation condition h>4ℓ_0/N. It means that the swing suspension length, ℓ_0, to be divided by the child's stature are to be less than 1/4N, where N is an amount of the free oscillations of the swing with a motionless load, and the oscillations are attained at the minor initial swing deviation. The practic principle is derived under the following assumptions: the h/3 squats (h is the child's stature) are allowed, this corresponds to the h/6 swing suspension length variation; the swing frame weight is not taken into account.

The other feature of the swing parametric oscillations differing from those of forced oscillations is also interesting. There is a maximum parametric excitation frequency, and parametric resonance is not feasible beyond one. Your author gives several of the other interesting parametric oscillation paradoxes. To this purpose let us mentally cover the swing frame together with a swinging person by the sphere non-transparent for a wingside spectator. The spectator can ask the first paradoxical question, "Why does the sphere swing and the oscillations do not decay in the conditions when external forces luck?"

The following paradox involves the frequency/period of parametric oscillations. The equality $1-\gamma^2 = \pm 2\lambda(1+2\gamma^2)/\pi$ follows the swing parametric oscillation excitation condition when ξ=0 and α is minor (see the section Ordinary swing). According to the above equality the two values of the relative oscillation frequency $\gamma = \Omega / 2\omega_0$ correspond to each value of the relative variation of the effective length $\lambda = \Delta\ell / \ell_0$. So $\gamma_1^2 = 0.682$ and $\gamma_2^2 = 1.52$ when $\lambda = 0.2$. The absolute oscillation frequencies $\omega_1^2 = 0.682 g / \ell_0$ and $\omega_2^2 = 1.52 g / \ell_0$ agree with them. Measuring the pendulum length, ℓ_0, the wingside spectator can calculate the pendulum frequency $\omega_0^2 = g / \ell_0$ by Thompson's formula taken from the well-known school curriculum. And the last calculated value coincides with none of both possible and real frequencies of swing oscillations. Of course the last paradox can be simply explained by the fact that the spectator cannot watch the parameter oscillations and so he considers the pendulum oscillations as free just when the parametric oscillations differ from those free.

The author gives the next paradox of parametric oscillations. In case if swaying the swing, i.e. its suspension length variation, occurs once over the period, the second parametric resonance is excited, and the parametric oscillations become unsymmetrical relative to the vertical. The wingside spectator can note the following paradox: the pendulum symmetrically swings relative to the vertical-unmatched axes while external forces lack. Of course the paradox can be explained by the fact that parametric oscillations are not symmetrical forced ones.

Certainly, the swing merit is great in learning parametric oscillations. Why, a notorious swing is an excellent teaching aid indispensable in learning parametric resonance.

Parametric oscillations and human being

As yet a human being continues only to cognize the enlarged area of parametric oscillation influence. Thus, in recently studying the specifics of macroeconomic modeling parametric oscillations were found in the multicommodity balance model and in juncture/market cycles of Gudvin's model (See Chechurin L.S. Applied Economic Problems in[2]).

Just lately scientists came to the conclusion based on numerous measurements on the parametric nature of the impact of different oscillation fields, such as radiation field, vibration field, gravitation field, electric field, magnet field, etc, on living organisms [4]. Presumably the impact can be explained by the "life" water effect [5]. Under the impact of an external oscillatory field the water dipole becomes ordered in the cells of the living organism. This is resulted in the stable dipole pair to be formed in the water at the frequency equal to half an external action frequency, and the dipole group/cluster or region of synchronously-oscillating, identical-orientated dipole pairs entails persistent immunity and therapeutic actions conformably to many diseases.

It should be noted that long before the given explanations therapeutic actions were discovered from various devices, such as the VIATON generating vibrations, the ALMAG, generating magnetic oscillations, the MAVIT, generating both vibrations and oscillations, ultrasonic devices, etc. It seems that an ordinary and simultaneously cheap swin*g* could fully occupy its fitting place among health-improving devices like those because a swing is the unique parametric training device in which the subsonic oscillations of gravitational field are generated. The training device might be useful for the elderly to a greater extent than the young.

It would be not right to complete this section at that point and not to mention how parametric resonance effects on human character and life. So, considerable will-power and resolution are required from a test-pilot to overcome the aircraft sound flatter. The pilot's prize is a joy in flying at the silence of air space. In other life collisions strong-willed personal qualities are required from a human been in getting over "life flatter". All is much complex in live and the complexity is often beyond all mathematical formulations and physico-mathematical modeling. In order to get out somewhat and to avoid the flatter in some difficult life circumstances the author gives the following common example.

A cyclist goes by bicycle. A large puddle emerges in front of him, and it is not clear for him whether to stop or go on. He carries out neither of the two - He applies a brake, and his bicycle drives slowly in the puddle - the puddle gets deeper and deeper, and the bicycle speed gets slower and slower - Stop! The cyclist becomes an unstable inversed pendulum at the zero bicycle speed. His violent efforts on handling handlebars cannot help because the oscillations occur in another plane. And the hard result takes place: the rider and his bicycle lay in the puddle. Of course, those are not "a life flatter", but how look at!

PART 2. BRIEF GUIDE TO PARAMETRIC OSCILLATION RIDDLES

As appears from the previous description parametric resonance can both harm and help. At the same time in spite of the presence of parameter oscillations parametric resonance can both be excited and not. This part is aimed at understanding the conditions of presence/absence its excitation. In that case a conscious chance of the purposeful stimulation of parametric resonance excitation/absence appears.

Forced oscillations

Ordinary pendulum description. The pendulum oscillations are schematically illustrated in Figure 2.1. where the actuating forces are presented as follows: $F_p = P\sin\alpha$ is the P weight component perpendicular to the arm, ℓ; F_r is the environmental resistance force (air/liquid, bearing friction forces) proportionate to the angular velocity of pendulum turn, ω; F_a is the inertia force proportionate to the angular acceleration, ε.

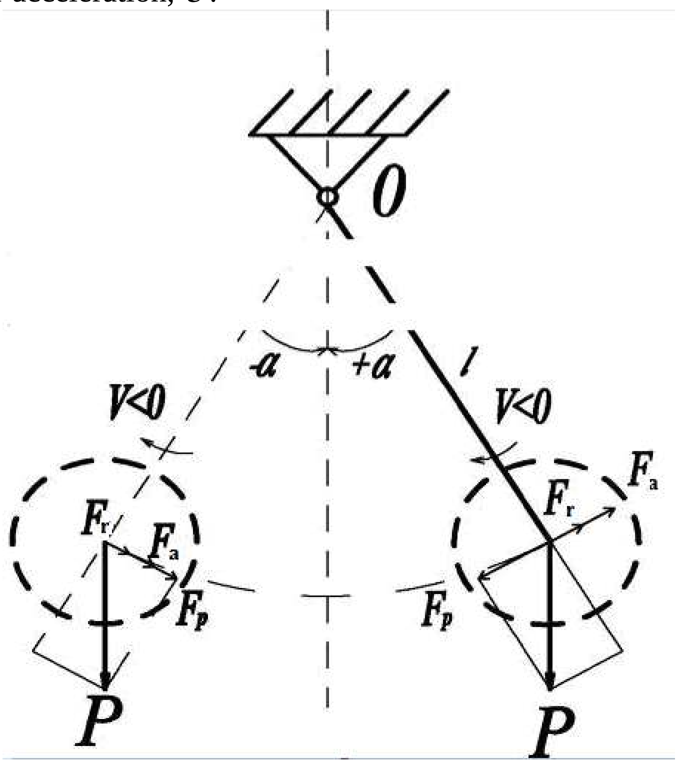


Figure 2.1. Ordinary pendulum oscillation

The above forces moments as the products of the forces by the arm, ℓ, relative to the pendulum rotation axis, O, are denoted by M_p, M_r, M_a. According to the main mechanics law for a rigid rotation the sum of the moments is zero:

$$M_a + M_r + M_p = 0.$$

Assume $\sin\alpha \approx \alpha$ for slight angular deviations. Denote the specific moments $c = M_p/\alpha$, $b = M_r/\omega$, $a = M_a/\varepsilon$, where α, ω, and ε are angle, velocity and acceleration, correspondingly. Using the denotations we rewrite the equation in the form

$$a\varepsilon(t) + b\omega(t) + c\alpha(t) = 0,$$

where $\varepsilon, \omega, \alpha$ are the coordinates/variables/time functions; $a, b,$ c are constant parameters. The linear differential equation for free pendulum oscillations is obtained because an angular acceleration is the second derivative of the angle with respect to time and the angular velocity is the first derivative of the angle with respect to time. The corresponding operator equation is

$$(ap^2 + bp + c)\alpha(t) = 0 \qquad (1)$$

where the operator, p, is a first-order derivative and p^2 is that of second-order.

The algebraic equation (1) in which the quadratic polynomial in the brackets is called a characteristic polynomial and the equality

$$ap^2 + bp + c = 0$$

is called a characteristic equation. The characteristic equation (3) and its roots completely determine the free oscillations at a specified initial deviation.

We can observe the free oscillations after deflecting the pendulum to either of the sides and releasing it (see Figure 1.1.). The oscillation period, T, is found at b=0 by the well-known Thompson's formula:

$$T = 2\pi\sqrt{\frac{a}{c}}\,.$$

While the parameters $a, b,$ c are positive, the pendulum free oscillations have a decayed oscillatory manner if the characteristic equation roots are complex conjugate (see Figure 1.1.), and the decayed manner is monotone if the characteristic equation roots are real. The roots are purely imaginary at b=0 and continuous oscillations occur. The continuous oscillation amplitude is equal to an initial deflection. Lastly even if one of the characteristic equation coefficients/parameters is negative, the unstable process of free pendulum motion is monotonous divergent at the real roots and it is oscillatory divergent at the complex conjugate roots.

The problem is high-difficult or not feasible at all to define the roots of high (higher than four) order polynomials concerning complex objects in an analytic way. Numerical calculations of the roots and the transients present no difficulties using modern computing technique, but the obtained results are greatly limited to be applied. The point is that not so much a transient itself is important for an innovative technology developer as its qualitative characteristics, such as stability, a stability factor, nature (oscillatory or monotonic), an oscillation damping factor, a maximum amplitude, transient time, etc. Lastly the connection of the transient factors with parameters is extraordinary important for a correct parameters selection. And the story is not complete. Very likely, the main thing is that the definition of mathematical formulation adequate to a complex object, such as aircraft, a ship, a bridge, an architectural structure, etc, which was constructed on the basis of various physical operating principles of mechanics, electromechanics, aerodynamic, hydro- and thermodynamics demands great professionalism unattainable at present because of considerable differentiation of sciences.

Frequency characteristics. Designers and engineers try to find a break in the deadlock through full-scale and physical model testing of constructed objects in different media and air/liquid flows and also by exciting and measuring object oscillations using oscillators, vibrators, shakers, and special instrumentation. In particular available either calculated/experimental or experiment-calculated data (the latter is better) on frequency characteristics of constructed objects could be much useful.

Experimental frequency characteristics are recorded in a forced oscillation mode. A disturbing harmonic action (input action/signal) from a generator/vibrator is applied to one of the chosen points/input of an electrical circuit/structure. The fixed input oscillation frequency and amplitude are set. The steady harmonic oscillation amplitude and phase are measured at the different object points/outputs; the their phase is more infrequent to be measured. The output-to-set input oscillation amplitude ratio is the modulus, A, of the frequency characteristic. The phase difference between output oscillations and input ones is the frequency characteristic phase, φ. A the input oscillation frequency is changed and the measurements are performed again. The amplitude-phase-frequency characteristic is obtained in respect of the chosen input and output as follows:

$$A(\omega) = \frac{A_{out}}{A_{in}}, \quad \varphi(\omega) = \varphi_{out} - \varphi_{in}.$$

The more complex the object, the more input and output points, the greater is the family of the obtained frequency characteristics. Usually the input signal phase is assumed to be zero, i.e. the relative shift between output and input signals is counted. If that is so, the measured output signal phase is concurrently the frequency characteristic phase $\varphi(\omega) = \varphi_{out}(\omega)$.

The frequency characteristics can be calculated by the steady forced oscillation equation which results from the free oscillation operator equation (1) by adding the forcing harmonic oscillations $F_{\text{in}}(t)$ to its left side:

$$(ap^2 + bp + c)\alpha(t) = F_{in}(t).$$

The forcing oscillations are assumed to be $F_{\text{in}}(t) = A_{in} \sin \omega t$. Because the equation is linear the steady forced pendulum oscillations are of the same form and frequency, ω, but differ in an amplitude and a phase, i.e. $\alpha(t) = A_{\text{out}} \sin(\omega t + \varphi)$. Hence, the steady forced oscillation equation is as follows:

$$(ap^2 + bp + c)A_{out} \sin(\omega t + \varphi) = A_{in} \sin \omega t.$$

For clearness the symbolic method well-known from theoretical electrical engineering is applied here. According to that method the single-component harmonic signal $A\sin(\omega t + \varphi)$ is represented in the complex plane by the vector, A, located at the angle φ to the positive real line (see Figure 2.2.).

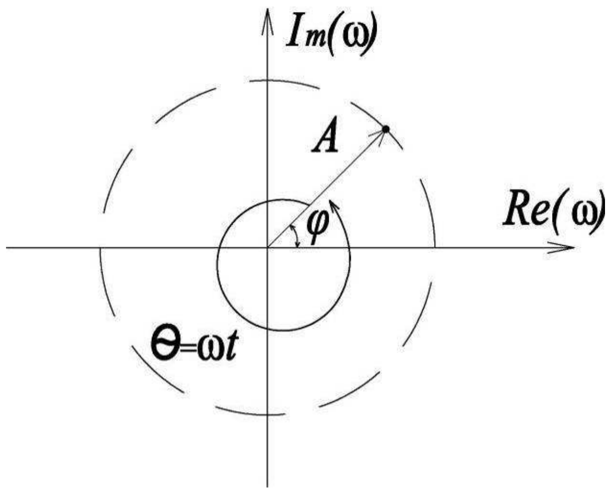


Figure 2.2. Complex plane of frequency characteristic

In Figure 2.2. the point A having the initial phase, φ, rotates round the origin of coordinates to change the phase, θ, at the angular velocity ω because $\theta = \omega t$. And the vector, A, circumscribes the circle $Ae^{j(\omega t+\varphi)} = Ae^{j\omega t}e^{j\varphi}$ in the complex plane. After replacing the sine functions by the complex exponentials function the equation of steady forced oscillation takes the form when p=jω

$$\left[ap^2 + bp + c\right]_{p=j\omega} A_{\text{out}} e^{j(\omega t+\varphi)} = A_{in} e^{j\omega t}.$$

After canceling both sides by exp(jωt) the equation comes to the equality form

$$A_{out} e^{j\varphi} = \left[\frac{1}{ap^2 + bp + c}\right]_{p=j\omega} A_{\text{in}} \qquad (2)$$

The fractionally rational function $W(p) = (ap^2 + bp + c)^{-1}$ is called the transfer pendulum function. Because the transfer function denominator is a characteristic polynomial the function defines all the oscillation object/pendulum features in terms of input and output coordinates chosen. It is important for us that the transfer function, if the operator p is substituted for jω, forms the calculated amplitude-phase-frequency characteristic W(jω) in the complex plane:

$$W(j\omega) = \frac{1}{a(j\omega)^2 + b(j\omega) + c}.$$

As any complex function, the amplitude-phase-frequency characteristic can be written in the two forms - complex and exponential. The complex form includes real and imaginary parts as follows:

$$W(j\omega) = \mathrm{Re}(\omega) + j\,\mathrm{Im}(\omega),$$

where Re(ω)=ReW(jω) is a real frequency characteristic and Im(ω)=ImW(jω) is called an imaginary one. It was mentioned above the exponential form in view of the description of recording the experimental frequency characteristics:

$$W(j\omega) = A(\omega)e^{j\varphi(\omega)} = |W(j\omega)| arctg \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)},$$

where A(ω) and φ(ω) are both amplitude and phase frequency characteristics. Thus, it is resulted from the equality (2) that the output oscillation phase dependence on the frequency is the phase-frequency object characteristic, and the output oscillation amplitude is equal to the amplitude-phase characteristic modulus multiplied by the input oscillation amplitude.

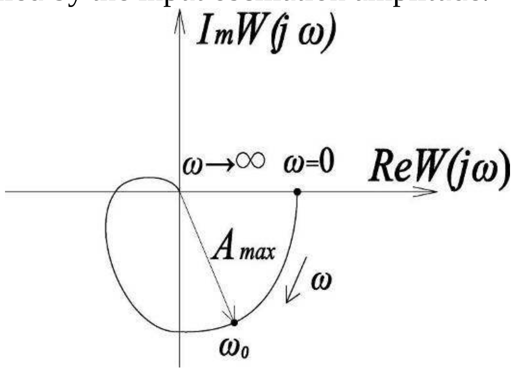


Figure 2.3. Amplitude-phase-frequency characteristic of pendulum

The amplitude-phase-frequency characteristic of the pendulum is given in Figure 2.3. The characteristic begins at the point a^{-1} (ω=0) of the positive real semiaxis, and it ends $(\omega \to \infty)$ at the origin after passing through the both fourth and third quadrants. The characteristic has its maximum distance to the origin where the modulus is maximum at the point of resonance frequency, ω_0, within the fourth quadrant:

$$\omega_0^2 = \frac{c}{a} - \frac{b^2}{2a^2}.$$

The phase characteristic changes from zero to -180°, i.e. the vector, ω, rotates clockwise at the $-\pi$ rad angle as the frequency varies from zero to infinity.

For an ideal pendulum b=0, i.e. there is no oscillation damping, and the amplitude-frequency characteristic modulus goes to infinity and the phase-frequency characteristic jumps from zero to $-\pi$ at the resonance frequency $\omega_0 = \sqrt{c/a}$. For the purpose of lumping oscillatory objects at a small b value it is useful to estimate the resonance frequency as $\omega_0 \approx \sqrt{c/a}$ and the maximum modulus as $A_{\max} \approx 1/b\omega_0$ with phase $\varphi(\omega_0) \approx -\pi/2$ at the imaginary negative semiaxis point, ω_0.

We, hereinafter, shall not calculate and plot amplitude-phase-frequency characteristics, and go beyond their qualitative characteristic demonstrations. Their accurately plotting is easy by either experimental data or a transfer function using the Matlab computing medium.

We return to the forced oscillations. Substituting in the equation (2) p of jω we get the frequency equation

$$A_{out}(\omega)e^{j\varphi(\omega)} = \frac{1}{a(j\omega)^2 + b(j\omega) + c}A_{\text{in}} = W(j\omega)A_{\text{in}}$$

or

$$\frac{A_{out}(\omega)}{A_{in}}e^{j\varphi(\omega)} = W(j\omega).$$

So the amplitude-phase-frequency characteristic modulus is the ratio of the forced oscillation amplitude to the disturbing force amplitude. The forced oscillation phase is equal to the frequency characteristic phase because the input disturbance phase is assumed to be zero. In essence, the physical meaning of frequency characteristic lies in this. If you know the amplitude-phase-frequency characteristic, you can define the forced oscillation amplitude and phase for each frequency, ω, under the rated input oscillation amplitude, and also find the necessary input oscillation amplitude on the basis of the required output one.

In conclusion it should be made one important remark. Dealing with forced oscillations in a linear stationary system we had in view that the system is stable, i.e. its zero equilibrium is stable. Otherwise forced oscillations practically lack and cannot be observed. In this section we could do along without the above remark since the stable pendulum was considered. Moreover all the systems described by a characteristic quadratic with positive coefficients are stable too. The stated definitions are also true for systems of higher orders. In such case the remark is witty. Of numerous well-known stability criterions of linear stationary systems Nyquist's amplitude-phase-frequency criterion (Nyquist's criterion) is needed in the further explanations. According to the Nyquist's criterion a system is stable if its complex frequency characteristic W(jω) does not cover the (-1,j0) negative real semiaxis point.

We briefly sum up at the end of the section. The frequency characteristic completely defines the forced oscillation mode in a linear stationary dynamic system. In a system like that forced oscillations occur at any input oscillation amplitude and have no an excitation threshold. The forced symmetrical oscillation frequency coincides with the symmetrical input one.

Parametric oscillations

Amplitude-phase characteristic of periodic parameter. Let some variable parameter, ν, includes the constant component, v_0, and the variable component, ν(t)>0, and so $v(t) = v_0 + v_1(t)$. Meanwhile, our interest is only the variable component. So the constant component is not taken into account and as a whole ν(t) is considered to be positive (ν(t)>0). Therefore the constant component is positive too ($v_0 > |v_1(t)|$). Assume a harmonic parameter variation rule at the frequency, Ω, i.e.

$$v_1(t) = v_1 \sin\Omega(t-\tau),$$

where τ is the phase shift indicating a certain parameter variation phase $\psi = \Omega\tau$ relative to the periodic input coordinate change $x_{\text{in}}(t)$ at the frequency, ω, and the amplitude, A:

$$x_{in}(t) = A\sin\omega t.$$

Determine a transfer coefficient for the first harmonic parameter.

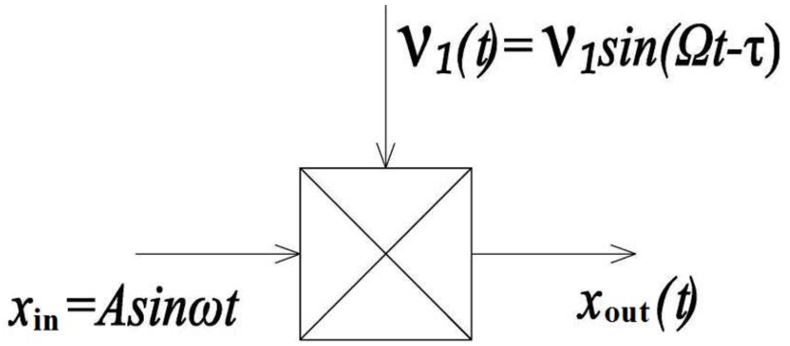


Figure 2.4. Periodically nonstationary parameter

Because the parameter is a proportionality factor for the input coordinate, the output coordinate is the product of the two as seen in Figure 2.4:

$$x_{\text{out}}(t) = Av_1 \sin(\Omega t - \psi)\sin\omega t\,.$$

Express the product of the sines in the form:

$$x_{\text{out}}(t) = Av_1 \sin(\Omega t - \psi)\sin\omega t = Av_1 \sin(\Omega t - \psi)\cos(\omega t - \pi/2) =$$
$$= 0{,}5\mathrm{A}\,v_1\{\sin[(\Omega - \omega)\mathrm{t} - \psi + \pi/2] + \sin[(\Omega + \omega)t - \psi - \pi/2]\}\,.$$

From here it follows that there are two harmonic constituents at such modulator/product output and those are a sum and a difference of the frequencies. There is an only case, while the parameter variation frequency is twice as high as the coordinate oscillation frequency (Ω=2ω), the oscillations agree with the input coordinate frequency arise at the modulator output. Our interest is just the case when the output coordinate includes the first ω-frequency harmonic input signal constituent, see the first summand, and the third 3ω-frequency harmonic input signal constituent, see the second summand.

Applying the symbolic harmonic signal notation again we determine the complex input coordinate transfer coefficient by the first harmonic modulator without taking into account the third harmonic which is minor as a rule. Write down as follows:

$$X_{\text{in}}(j\omega) = Ae^{j\omega t}\,,$$
$$X_{\text{out}}(j\omega) = 0.5\mathrm{A}\,v_1 \mathrm{e}^{\mathrm{j}(\omega t - \psi + \pi/2)} = 0.5Av_1 e^{j\omega t}e^{-j\psi}e^{j\pi/2}\,.$$

Their ratio is the amplitude-phase characteristic of the harmonically-varied parameter/modulator with respect to the first harmonic. Because exp(jπ/2)=j in the complex plane we obtain

$$W(j\psi) = \frac{X_{\text{out}}(j\omega)}{X_{\text{in}}(j\omega)} = 0.5\,jv_1 e^{-j\psi}\,. \qquad (3)$$

Thus, the transfer coefficient of double frequency-varied parameter is equal to $v_1/2$ with respect to the modulus, where v_1 is the first harmonic parameter variation amplitude. It is significant that the transfer coefficient does not depend on the input amplitude, A, and the frequency, ω, but the parameter oscillation phase and the coordinate. The fundamental difference between the parameters is that the time-varied parameter shifts an input oscillation phase at the output at the arbitrary angle, ψ, depending on the phase shift between the oscillations of parameter and input coordinate. In the complex plane the amplitude-phase characteristic W(jψ) at the arbitrary phase shift, ψ, is a $v_1/2$ circle centered at the origin.

Parametric resonance. We consider the illustration of parametric resonance excitation by the elementary LRC-oscillatory circuit (see Figure 1.11.).

In a free-flowing mode without external sources of power and oscillations the identical electric current flows through all the three circuit elements, and the total three-element voltage is zero. The L-inductor voltage is proportional to the current rate of change. The C-capacitor voltage is accumulated as the current flows in time in proportion to the total current. The R-resistor voltage is proportionate to the current. The zero three-element voltage condition can be expressed in the form

$$L\frac{di}{dt}+C^{-1}\int idt+Ri=0\,.$$

In an operator form dividing by the operator, p, corresponds to integrating. So

$$Lpi+\frac{i}{Cp}+Ri=0$$

or

$$Lp^2 i+Rpi+C^{-1}i=0\,.$$

The equation matches with the pendulum equation (1) by type and it can be expressed in the same general view:

$$(ap^2+bp+c)x(t)=0\,, \qquad (4)$$

where x(t) = α(t) for a pendulum and x(t) = i(t) for a RLC-circuit. It is important to note in terms of the two models presented in the above examples that the initial parameters ℓ, m, R, L, C can be parts of the combined parameters a, b, c in their various combinations. Therefore either one or several the combined parameters can be temporary variables.

For the purpose of significant beginning we assume that the parameter, c, is harmonically variable as

$$c(t)=c_0+c_1\sin\Omega t\,,$$

where $c_0 > c_1$ as it was mentioned above. According to (3) the complex parameter transfer coefficient c(t) takes the form

$$W(j\psi)=c_0+0.5jc_1e^{-j\psi}\,.$$

Removing the time coordinate x(t) and replacing the parameter by the coefficient W(jψ) we convert (16) to the complex frequency plane by the substitution p=jω:

$$a(j\omega)^2+bj\omega+c_0=-0.5jc_1e^{-j\psi}\,.$$

Inverting the left and right sides we get the equation in its final form:

$$W(j\omega)=\frac{1}{a(j\omega)^2+bj\omega+c_0}=-\frac{2}{jc_1}e^{j\psi}\,.$$

This is the parametric resonance excitation condition at the parameter oscillation frequency Ω=2ω. It is not difficult to define the excitation condition in its analytic form if one equates the moduluses in the left and right sides with each other:

$$\left[(c_0-a\omega^2)^2+(b\omega)^2\right]^{-1/2}=2/c_1\,.$$

Nevertheless a graphic illustration of the excitation condition in the complex plane of the object amplitude-phase-frequency characteristic is important, namely, parametric resonance is excited in the case when the amplitude-phase-frequency characteristic W(jω) leaves the $2/c_1$ central circle at the frequencies ω=Ω/2.

The last condition illustration is given in Figure 2.5. concerning the frequency characteristic W(jω) of the oscillatory both pendulum and electric circuit. Here the contracted notations are used to denote the real part R=ReW(jω) and the imaginary part I=ImW(jω) of the axes.

As a rule the frequency characteristic leaves the parametric resonance circle at the resonance frequencies corresponding to the maximum characteristic modulus. The critical circle radius and the allowable parameter oscillation amplitude are to be determined by those resonance frequencies.

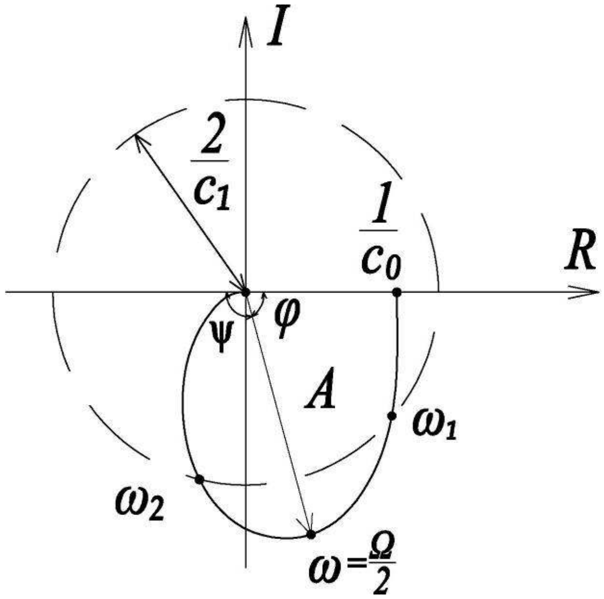


Figure 2.5. Parametric resonance excitation condition

If the parametric resonance excitation condition is met, a linear dynamic system becomes unstable and its oscillations grow up unlimitedly. The parametric resonance excitation condition obtained specifies the frequency domain (ω_1, ω_2) where the stability factor with respect to the modulus is zero/negative.

And what about the stability factor with respect to a phase? Where does it disappear? It is not difficult to answer the last question. Since there is no oscillation synchronization of the parameter itself and its input coordinate in a periodically nonstationary dynamic system, any phase, ψ, between the parameter and the parameter input coordinate can be settled within the range from 0 to 360 degrees. In this case as it follows from the amplitude-frequency characteristic of variable parameter the same phase, ψ, is settled between the input and output parameter oscillations. It is interesting that in the periodically nonstationary dynamic system self-synchronization occurs and the phase, ψ, equal or exceeding the object stability factor by a phase is settled. In other words, if modul margin of stability is negative, periodical parameter insets negative phase shift equal to stability phase margin. So parametric resonance is excited.

The special case is when the frequency characteristic originates outside the circle. In that case the parameter variation amplitude at low frequencies is higher than the constant parameter component, and instead of parametric resonance the loss of free oscillation stability occurs in the system in the segments where c(t)<0.

Parametric oscillation types. Up to here we considered periodically nonstationary systems consisting a single-frequency parameter. The latter was harmonically changed at the frequency, Ω. Simultaneously the parametric resonance excitation condition at the frequency ω=Ω/2 was deduced. The sustained coordinate parametric oscillations with a doubled variation parameter frequency are called the main/first parametric resonance presented in Figure 1.4. By the way, sometimes this kind of resonance is called a fist subharmonic resonance. As a rule the first parametric resonance oscillations have an excitation threshold, which is governed by an inverse circle radius of the first parametric resonance. Because $W^{-1}(j\psi) = \dfrac{2}{jc_1} e^{j\psi}$ the excitation threshold follows the equality

$|W(j\omega)| = 2/c_1$.

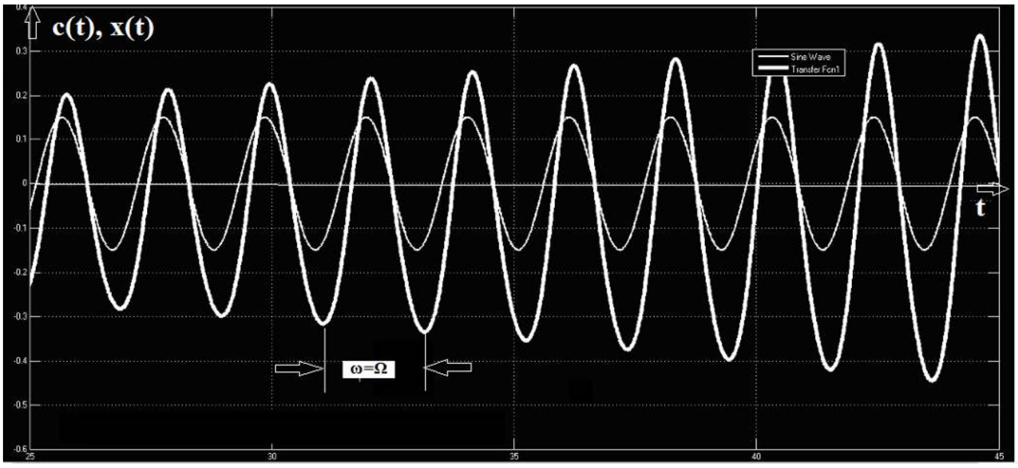


Figure 2.6. Second parametric resonance oscillation

The undamped coordinate oscillations can be observed at the frequency equal to the parameter variation frequency ω=Ω. These are the second parametric resonance oscillations and their frequency coincides with a forced oscillation frequency. The second parametric resonance oscillations have an excitation threshold, which most often exceeds that of the first resonance, and they are unsymmetrical because they include the constant constituent (see Figure 2.6.). The inverse circle radius of the second parametric resonance is more than that of the first resonance. Which of the two parametric resonances is excited in the system with one natural frequency, ω_0, depends on the proximity of parameter oscillation frequency to one of the following approximations: $\Omega \approx 2\omega_0$ and $\Omega \approx \omega_0$.

In case of the multifrequency parameter variation when there are several harmonics, either first or second parametric resonance can be excited at the frequency one of the harmonics depending on the object resonance frequency. These are resonances of higher orders, n, (third, fourth, etc.) at the frequency ω=nΩ/2 (n is a number of a higher harmonic). Essentially these are either first or second parametric resonances at a nth harmonic frequency.

In complex systems characterized by several natural frequencies and particularly in distributed ones the parametric oscillations can be excited at several frequencies. Those are so-called combination parametric oscillations. The subsequent statement does not go beyond the first parametric resonance.

Examples

Let us consider, first, two oppositely dual problems with respect of parametric resonance: 1) parametric equilibrium instability of an ideal pendulum and 2) parametric stabilization of unstable pendulum equilibrium.

Example 1. The aim is to swing the ordinary pendulum (see Figure 2.1.) using its suspension point oscillations. While the suspension point oscillates at the amplitude, a, and the frequency, Ω, in the vertical plane a periodically-changed acceleration and the corresponding force $ma\Omega^2 \sin\Omega t$ acting along the mg-gravity line are periodically imparted (added/subtracted) to the pendulum. In the ideal case, when environmental resistance is missed and the inertia moment $J = ml^2$, the pendulum oscillation operator equation takes the form

$$ml^2 p^2\alpha(t) + (mg - ma\Omega^2 \sin\Omega t)l\alpha(t) = 0$$

or

$$p^2\alpha(t) + \frac{g - a\Omega^2 \sin\Omega t}{l}\alpha(t) = 0.$$

Eliminate time, t, from the equation by substituting the harmonically-changed parameter for its complex amplitude-phase circle equation and turn to the frequency plane by replacing p=jΩ/2 as applied to the first parametric resonance:

$$(j\frac{\Omega}{2})^2 + \frac{g}{l} = j\frac{a\Omega^2}{2l}e^{-j\psi}.$$

The obtained equality (21) is just suitable for analyzing in the plane of reverse frequency characteristic. Nevertheless turn to the plane of ordinary object frequency characteristic by reversing the left and the right sides of this equality as follows:

$$\frac{1}{(j\Omega/2)^2 + g/l} = \frac{2l}{ja\Omega^2}e^{j\psi}. \qquad (5)$$

The left side of (5) is the amplitude-phase-frequency characteristic of the ideal pendulum. This real characteristic originates (Ω=0) at the point $(g/l)^{-1}$ of the positive real semiaxis. It moves to infinity ($+\infty$) along the positive line at the resonance frequency $\Omega/2 = \omega_0 = \sqrt{g/l}$ as Ω is increased. There is a characteristic discontinuity from $+\infty$ to $-\infty$ at that resonance frequency, and then it reverts to the origin along the negative real semiaxis. This implies that there are two intersections ($\Omega_1/2, \Omega_2/2$) of the frequency characteristic with the circle of reverse amplitude-phase parameter characteristic (see Figure 2.7).

As opposed to complex objects, in this simple example the analytical solution is simpler than the graphic one. The first parametric resonance excitation/parametric equilibrium pendulum instability condition can be deduced by equating the moduluss in the sides of the direct (5)/ reverse equality with each other:

$$a \geq \left|(2g/\Omega^2) - l/2\right|.$$

This implies the following condition

$$a \geq \frac{l}{2}\left|1 - 4\frac{\omega_0^2}{\Omega^2}\right|$$

and the boundary (upper/low) resonance excitation frequencies

$$\frac{\Omega_{1,2}}{2} = \omega_0(1 \pm 2a/l)^{-0,5}, a < l/2.$$

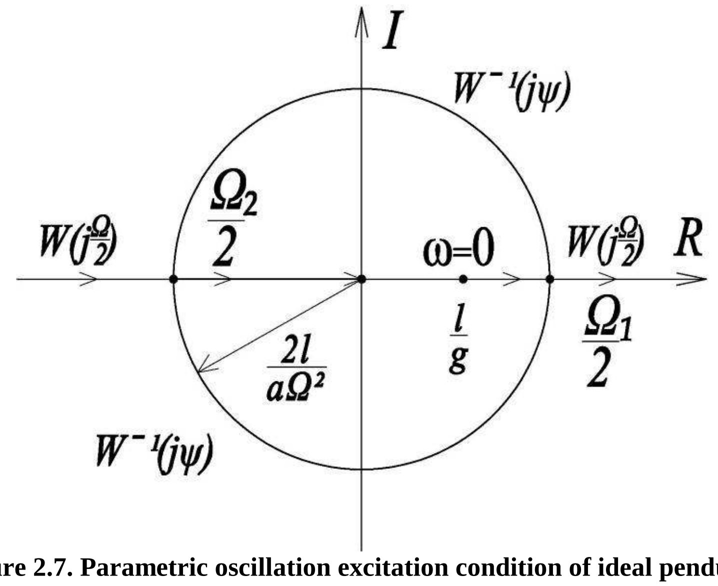


Figure 2.7. Parametric oscillation excitation condition of ideal pendulum

Example 2. Consider the inverse problem of unstable equilibrium stabilizing of an ideal inversed pendulum in respect of Kapitza’s pendulum with the help of its rotation axis vibrations.

The inversed pendulum equations for the case of vibrating its rotation axis differ from those of an ordinary sustained pendulum just in a g-acceleration sign. Therefore we write the frequency parametric resonance excitation condition(5) as follows:

$$\frac{1}{(j\Omega/2)^2 - g/l} = \frac{2l}{ja\Omega^2} e^{j\psi} .$$

From here implies the first parametric resonance excitation condition in the form

$$a\Omega < \frac{l\Omega}{2} + \frac{2g}{\Omega} .$$

It is the first stabilization condition of the inversed pendulum.

The second condition also results from the same operator equation for a reversed pendulum

$$p^2 - \frac{g - a\Omega^2 \sin\Omega t}{l} = 0$$

For the case of the second parametric resonance at p=jΩ=jω frequency equation follows:

$$(j\Omega)^2 - g/l = W_2(j\psi) ,$$

where $W_2(j\psi)$ is the circle-shaped amplitude-phase characteristic of the same variable parameter $(a\Omega^2/l)\sin\Omega t$ with respect to the second parametric resonance oscillations. The rough assessment of the circle radius is

$$\left|W_2(j\psi)\right| \approx 2(a\Omega^2/l)^2 .$$

Substituting rough assessment into the frequency equation from modulus equality we define the reduced-second stabilization condition

$$a\Omega > \sqrt{2gl} .$$

Combining the both conditions we obtain the well-known stability conditions in respect of the unstable inversed pendulum equilibrium:

$$\sqrt{2gl} < a\Omega < \frac{l\Omega}{2} + \frac{2g}{\Omega} .$$

The two examples given illustrate the wonderful ability of parametric resonance to add directly opposite properties to an oscillation object.

The strict reader has the right to ask, “What are these trite-known results for? In reply one can give the two reasons unrelated with each other. First, both examples show that it is easy to get the results. Secondly, why would not you design, for example, a new wall electromechanical clock instead of various electronic quartz clocks in terms of both new parametric and forgotten old pendulum principles? A pendulum provides a uniform rate of a clock and parametric resonance excites oscillations. Such clock is able of ornamenting any housing and, above all, the considerable virtual therapeutic result: rhythmic oscillations of a heavy pendulum inspire its holders with both regular and tranquil life.

In the following examples the qualitative analysis of parametric resonance excitation ability is given in a general form and without a connection with a physical object action with regard to a second order oscillatory member. The skills in practical amplitude-phase-frequency characteristic location assessment are significant in this matter. In all the cases the free oscillation equation (4) is original. The case when the coefficient, c, is a variable parameter, i.e. $v(t) = c(t) = c_0 + c_1(t)$, was investigated in the section Parametric oscillations.

Example 3. Assume that the p^2 held parameter is a periodically variable parameter, i.e.

$$v(t) = a(t) = a_0 + a_1(t) = a_0 + a_1 \sin\Omega t .$$

As hereinbefore replacing the time-dependent part by the amplitude-phase characteristic $W(\psi) = 0.5ja_1e^{-j\psi}$ according to (3) and transposing that to the right side of (4) we derive

$$a_0 p^2 + bp + c = -0.5p^2 ja_1 e^{-j\psi} .$$

The following first parametric resonance excitation condition is derived as a result of the inversion of both sides of the equality and the substitution p=jω=jΩ/2:

$$\frac{(j\omega)^2}{a_0(j\omega)^2 + b(j\omega) + c} = \frac{2}{ja_1}e^{-j\psi}.$$

The right side is the inverse parametric resonance circle 1/ W(jψ) and its left side is the amplitude-phase-frequency characteristic W(jω). Refer to Figure 2.5 to assess the frequency characteristic location in the complex plane. The characteristic shown in Figure 2.5. differs from that in Figure 2.8. in the numerator only. Multiplication of the frequency characteristic by jω corresponds to its +π/2 rad counterclockwise turn and replacement of the origin ω=0 to the coordinate origin. The frequency characteristic is multiplied by jω two times as compared with the curve in Figure 2.5. so its rotation angle is +π rad. Thus originating from the coordinate origin the derived frequency characteristic curve passes throughout the both left and right upper quadrants and ends ($\omega \to \infty$) at the point $1/a_0$ of the real positive axis (see Figure 2.8.).

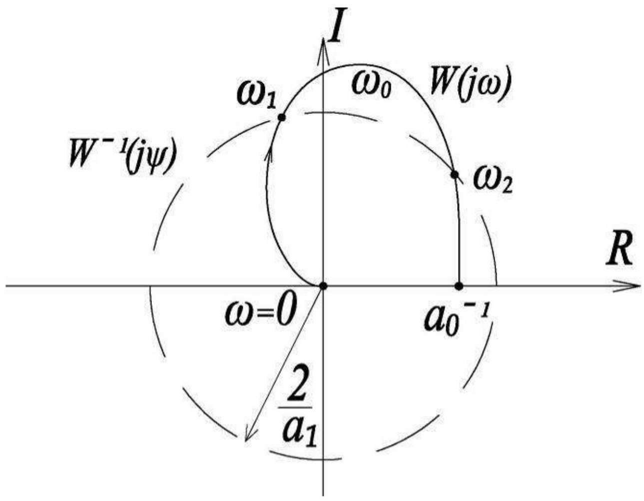


Figure 2.8. Example of parametric resonance excitation condition

This means that there are two points of intersection between the frequency characteristic and the inverse resonance circle: 1) at the sufficiently great $a_1 < a_0$ and 2) at the maximum frequency characteristic maxW(jω)>1/a_0, of the oscillatory object. The two intersection points, ω_1, ω_2, define the frequency range $\Omega_1 = 2\omega_1, \Omega_2 = 2\omega_2$ in which the first parametric resonance is excited.

It could be seemed that parametric resonance could be realized in the oscillation object containing any sufficiently great periodic parameter. Such is not the case. The following example is an illustration of the last statement.

Example 4. Assume that the b coefficient of the coordinate variation rate in the oscillation object equation (4) is a periodic variable parameter. This results in the resonance excitation condition different from that of the previous case:

$$\frac{j\omega}{a(j\omega)^2 + b_0(j\omega) + c} = -\frac{2}{jb_1}e^{-j\psi}.$$

In this case the frequency characteristic corresponding to the equation (3) turns counterclockwise at the π/2 rad angle. Originating from the coordinate origin the frequency characteristic passes through both first and fourth quadrants and it ends at the coordinate origin again (see Figure 2.9.). Moreover the maximum frequency characteristic modulus is as mash as $1/b_0$, and this value is reached at the positive real semiaxis point agree with the natural frequency of $\omega_0 = \sqrt{c/a}$. As a purely mathematical matter parametric resonance occurs at $b_1 \geq 2b_0$ and the variable parameter b(t) periodically takes negative values. In real objects in contrast to coordinates, negative physical parameters, such as length, mass, inductance, capacity, resistance, etc, are found too uncommon. So assuming the physical restriction b(t)>0 we conclude that parametric resonance cannot be excited during the parameter oscillations b(t).

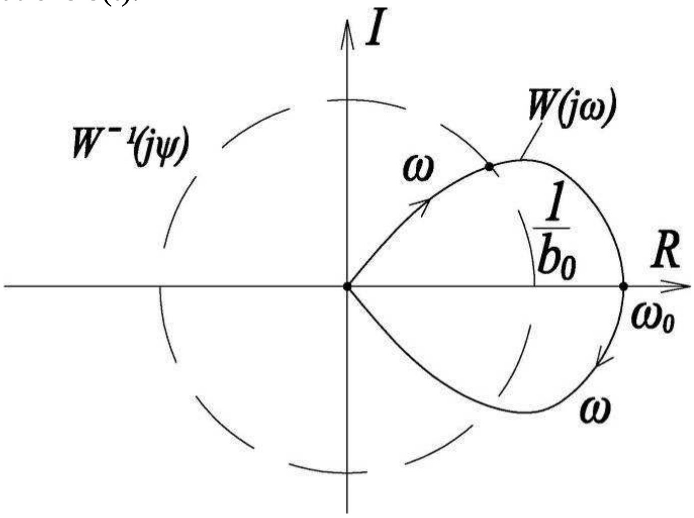


Figure 2.9. Frequency characteristic of Example 4

In mechanics and electronics the lack of parametric oscillation excitation under the resistance variations of pendulum environment and bearings is explained from the point of view of energy just as in the case of resistance fluctuations in a RLC-circuit. Namely both environmental and electrical resistances are not energy carriers and they cannot store energy. Evidently that is right although there are counter-examples as regards of more complex objects. But analyzing the system equation in this example we in no way connected the equation with a power oscillation aspect. And what is the matter? “Who” is guilty? Where and why does parametric resonance disappear? There is a simple explanation as follows: increasing the amplitude, b_1, and thereby decreasing the radius of the inverse amplitude-phase characteristic $W^{-1}(j\psi)$, we simultaneously have to enlarge the constant constituent, b_0 by virtue of b(t)>0, which leads to the diminution of the maximum frequency characteristic modulus. So the intersection of inverse amplitude-phase characteristic with frequency characteristic is not possible. In other words the conditions $b(t) = b_0 + b_1 \sin \Omega t > 0$ and $b_1 \geq 2b_0$ are not compatible with each other.

Add the concluding remark to the examples. In many situations an alternating parameter is concurrently included in several coefficients of the oscillatory object equation. That takes place with respect to so-called synchronous multiparametric systems. An analysis of the latter is similar to that in the considered examples in which the frequency characteristics are altered. Thus, If any alternating parameter β(t)= $\beta_0 + \beta_1$ sin Ωt, for instance, is simultaneously included in both coordinate and first coordinate derivative in a linear manner, the free oscillation operator equation takes the form:

$$ap^2 + b\beta(t)p + c\beta(t) = 0 .$$

Then the frequency parametric resonance excitation condition is

$$\frac{b(j\omega) + c}{a(j\omega)^2 + b\beta_0(j\omega) + c\beta_0} = -\frac{2}{j\beta_1} e^{j\psi} ,$$

where ω=Ω/2 for the first parametric resonance.

The final example relating to a swing is given below.

Ordinary swing

Assume an ordinary swing as an illustrative and well-known example of the application of parametric oscillations. The mathematical formulation of the latter differs a little from that of an ordinary pendulum. Both the pendulum with an oscillating suspension point and the swing with an oscillating effective suspension length are periodically unstable oscillatory/parametric systems. That is why the first rough description approximation regarding an ideal swing can be derived from the ideal pendulum equation (see Example 1) after removing the suspension oscillations at $a = 0$ and substituting the constant length for the variable ($\ell = \ell(t)$):

$$ml^2(t)p^2\alpha(t) + mgl(t)\alpha(t) = 0 .$$

Assuming, first, a harmonic rule of the length variation $\ell(t) = \ell_0 + \ell_1 \sin \Omega t$ we pass to the first parametric resonance oscillation excitation condition at the frequency ω=Ω/2. Then

$$\ell_0 p^2 + g = -p^2 \frac{\ell_1}{2j} e^{-j\psi} .$$

Hence the frequency resonance excitation condition at p=jω takes the form

$$\left| g - \ell_0 \omega^2 \right| = \omega^2 \ell_1 / 2 .$$

And taking into account the natural frequency $\omega_0^2 = g / \ell_0$ and denoting $\gamma = \omega / \omega_0$, $\lambda = \ell_1 / \ell_0$:

$$\lambda > 2 \frac{\left|1 - \gamma^2\right|}{\gamma^2} .$$

So in case of an ideal swing the parametric oscillation excitation at the natural frequency (γ=1) occurs without a threshold ($\ell_1 = 0$).

Let us consider now the effective pendulum length jump, which is close to practice, by the example of more rigorous pendulum model description.

The above swing formulation was derived in assuming that the oscillation amplitude of pendulum length oscillations is small, when $\ell_1 << \ell_0$, and so small inertia moment variations follow the same effective pendulum length variations. Otherwise it is necessary to take into account that the pendulum moment inertia variations $J(t) = ml^2(t)$ result in proportional variations of the angular velocity $\dot{\alpha}$ =ω. The relation is visually demonstrated by masters of figure skating beautifying final parts of their performances with their ice-rotations. Initially taking off the figure skater shows a slow rotation under a great inertia moment while he moves apart his hands and legs and sags. Then he, diminishing inertia moment, abruptly brings his legs and hands (overhead) together, thereby he effectively augments his rotation velocity. His rotation is ended with restraddling his hands and legs.

So the more rigorous formulation of a swing has the form:

$$J(t)p^2\alpha(t) + \left[\frac{dJ(t)}{dt}\right] p\alpha(t) + mgl(t)\alpha(t) = 0 .$$

Here the multiplier in the square brackets is the rate of inertia moment variation, which generates corresponding change in the angular velocity pα(t).

Substituting J(t) of m$l^2(t)$ and dividing the equation by the constant parameter, m, and the coordinate α(t) we obtain the following operator equation:

$$l^2(t)p^2 + \left[\frac{dl^2(t)}{dt}\right] p + gl(t) = 0 .$$

Assume a step-wise law of the effective pendulum length variation (see Figure 2.10). (Sometimes such kind of

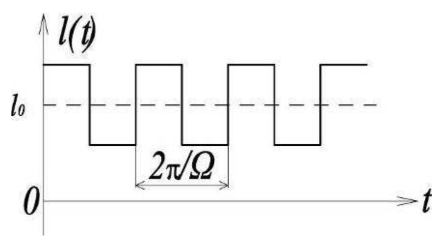


Figure 2.10. Step-wise law of effective pendulum length variation

excitation is called as a blowing excitation).

As follows from Figure 2.10.

$$l(t) = l_0 (1 + \lambda \,\mathrm{sgn} \sin \Omega t)\,,$$

$$l^2(t) = l_0^2 (1 + \lambda^2 + 2\lambda \,\mathrm{sgn} \sin \Omega t)\,.$$

Single out the first/main harmonic from the laws of parameter variations. The first harmonic amplitude is higher than the step-wise oscillation amplitude, λ, by a factor of 4/π. So

$$l(t) = l_0 [1 + (4\lambda / \pi) \sin \Omega t],$$

$$l^2(t) = l_0^2 [1 + (4\lambda / \pi)^2 + (8\lambda / \pi) \sin \Omega t].$$

Then

$$\frac{d[l^2(t)]}{dt} = \ell_0^2 \frac{8\lambda\Omega}{\pi} \cos \Omega t\,.$$

Substituting these variable parameters in the operator equation convert it to the frequency plate by replacing p by jω and Ω by 2ω. Substituting the time functions for their amplitude-phase equivalents $\sin \Omega t \to -e^{-j\varphi}/2j$ and $\cos \Omega t \to e^{-j\varphi}/2$ the frequency equality is derived as follows:

$$(j\omega)^2 (1 + \lambda^2 - \frac{4\lambda}{j\pi} e^{-j\psi}) + j\omega \frac{8\lambda\omega}{\pi} e^{-j\psi} + \omega_0^2 (1 - \frac{2\lambda}{j\pi} e^{-j\psi}) = 0\,.$$

We displace the exponential terms to the right side of the equality and equate the moduluss of left and right parts. Just now we have the first parametric resonance excitation condition in the form:

$$\left|1 - \nu^2 (1 + \lambda^2)\right| = \frac{2\lambda}{\pi} (1 + 2\gamma^2)\,,$$

where $\lambda = \ell_1 / \ell_0$, $\gamma = \omega / \omega_0$, $\omega_0^2 = g / \ell_0$.

It is worth reminding in conclusion that the derived parametric resonance excitation condition is the first parametric resonance excitation condition of an ideal swing, i.e. when the free swing oscillations are continuous. Of course, environmental/air resistance and suspension friction exist in respect of a real swing. Those resistance forces are taken into account through adding the δ / m term to the square brackets in the operator equation. Reproducing the above operator equation transformations we get the parametric resonance excitation condition for a real swing in the form:

$$\left|1 - \gamma^2 (1 + \lambda^2) + j\gamma\xi\right| = \frac{2\lambda}{\pi} (1 + 2\gamma^2)\,, \;\; \xi = \delta / m\ell_0^2 \omega_0\,.$$

In case of the oscillations close to natural ones when $\gamma \cong 1$ we have

$$\lambda^4 - \frac{36}{\pi^2} \lambda^2 + \xi^2 = 0\,.$$

The following simple rough excitation condition can be written when λ<1 and $\lambda^2 > \lambda^4$:

$$\lambda \ge \frac{\pi}{6} \xi \approx 0{,}5\xi\,.$$

The threshold condition is applied to explain the parametric resonance in the swing presented in Part 1.

Elusive avengers

So far we considered forced oscillations in linear stationary systems and parametric oscillations in linear periodically nonstationary systems. It was noted that the parametric resonance excitation in the second kind of systems results in equilibrium instability of a stable oscillatory object and also can stabilize an unstable object.

It should be noted that linear dynamic objects and systems are, as a rule, the results of idealizing. In technical practice dynamic objects starting with pendulums, swings, inductors, capacitors, let alone complex aerohydrodymanic processes, etc. are predominantly nonlinear.

In nonlinear stationary dynamic systems parameters depend on coordinates. Calculations of such systems even not complicated are extremely laborious, and they are generally realizable by only numerical methods. Simultaneously parametric oscillations and parametric resonance are very important in the main matter of oscillatory stability of nonlinear systems. The lack of notice to that matter is strictly penalized by “elusive avengers”, such as various parametric resonances. The term “elusive” means that parametric resonances are coincident with stable object oscillations and so they cannot be observed. This problem is concerned hereinafter.

Forced oscillations of nonlinear systems. Assume the following formulation of a simplest nonlinear dynamic system in which a free member is a nonlinear function, i.e.

$$ap^2 x(t) + bpx(t) + cx(t) + F[x(t)] = x_{\text{in}}(t).$$

As usual we shall solve the equation in the form $x(t) = A\sin(\omega t - \varphi)$,where $x_{in}(t) = A_{\text{in}} \sin \omega t$ is a harmonic perturbation. In this case the nonlinear function $F[A\sin(\omega t - \varphi)]$ is periodical and it can be presented as the Fourier series. Replacing the nonlinear function of periodic argument by the first harmonic of A_1 amplitude we write down as follows:

$$F[x(t)] = F[A\sin(\omega t - \varphi)] \approx A_1 \sin(\omega t - \varphi) = \frac{A_1}{A} A(\sin \omega t - \varphi) = W_{\text{h}}(A)x(t),$$

where $W_{\text{h}}(A) = A_1 / A$ is a harmonic linearization coefficient or nonlinear element transfer coefficient with respect to the first harmonic. This coefficient can be either calculated or determined graphically. Moreover it can be determined on the basis of calculation tables containing the coefficients for many types of nonlinear functions. Generally the harmonic linearization coefficient is a complex value. Its real and imaginary parts can depend on not only the amplitude x(t) but frequency, a constant constituent, and coordinate derivatives. In a simplest way the harmonic linearization coefficient is a real function of the amplitude, A, as regards the symmetrical single-valued function F(x).

Turn from the equation to its frequency form using the harmonic linearization coefficient, the exponential notation of harmonic coordinates, and p=jω as follows:

$$W^{-1}(j\omega) - [-W_{\text{r}}(A)] = \frac{A_{\text{in}}}{\text{A}} e^{j\varphi}, \qquad (6)$$

where $W^{-1}(j\omega) = a(j\omega)^2 + b(j\omega) + c$. In the plane of the reverse amplitude-phase-frequency characteristic $W^{-1}(j\omega) = R_0(\omega) + jI_0(\omega)$ the frequency equation of forced nonlinear system oscillations has the simple graphical interpretation (see Figure 2.11.). The harmonic linearization coefficient values are along the negative real semiaxis for all the positive values, A. The difference modulus of the vectors $-W_{\text{h}}(A)$ and W(jω) is A_{in} / A then. It follows from the modulus balance in both sides that

$$A_{in} = A\sqrt{[R_0(\omega) + W_{\text{h}}(A)]^2 + I_0^2(\omega)}.$$

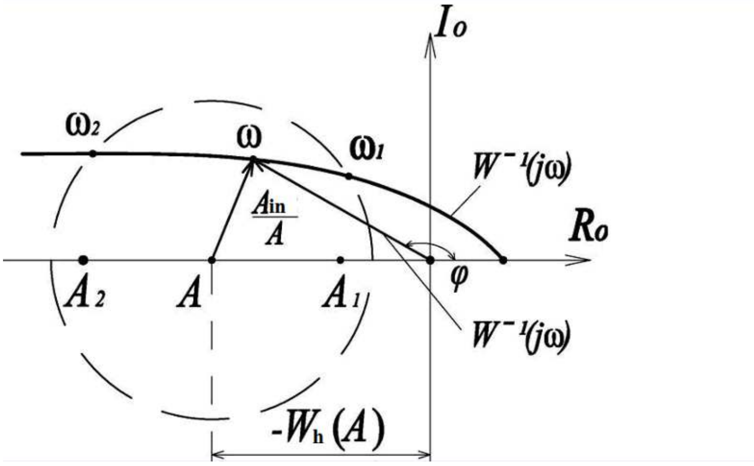


Figure 2.11. Forced oscillation frequency conditions

The condition dA_{in} / dA=0 gives the circle equation in the form:

$$[R_0(\omega) + v_0(A)]^2 + I_0^2(\omega) = v_1^2(A),$$

And the circle radius is

$$|v_1(A)| = \frac{A}{2}\left|\frac{dW_h(A)}{dA}\right|, \qquad (7)$$

And the circle center shift along the real axis is as follows:

$$v_0(A) = W_h(A) + \frac{A}{2}\frac{dW_h(A)}{dA}. \qquad (8)$$

Because dA_{in} / dA=0 is realized at the vertical tangent points of the amplitude curve $A(A_{out})$ the forced oscillation jumps occur at those points (see Figure 2.12.). For the reason given above those jumps were initially called step-wise resonance. In reality the derived circle is that of the first parametric resonance, which the reader will get to know a little later. The circle passes through the point $-W_h(A)$ and has its center on the left if $dW_h(A)/dA > 0$ and the center is on the right if $dW_h(A)/dA < 0$. By the way, it follows from this situation that the self-oscillation excitation condition (6) coincides with the boundary of parametric resonance excitation at $A_{in} = 0$ (see Section 1, Self-excited oscillation or parametric oscillation?).

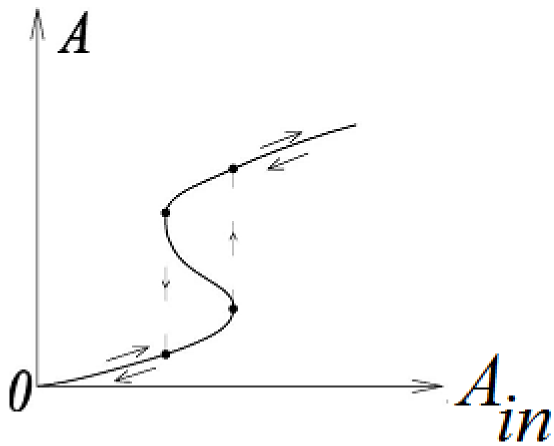


Figure 2.12. Step-wise resonance illustration

Thus, if the frequency,ω, of forced oscillations at the reversed amplitude-phase-frequency characteristic falls into the parametric resonance circle, the forced oscillations lose their stability.

According to the general postulate laid by A.M. Lyapunov to the basis of stability theory the motion x(t) of the dynamic system

$$Q(p)x(t)+G(p)F[x(t)]=S(p)y(t)$$

(where Q,G,S are polynomials of the operator, p, F is a nonlinear coordinate function, y(t) is an external action/perturbation) is stable if the zero equilibrium state is stable in respect of the linear system described by the following small perturbation equation:

$$Q(p)\Delta x(t) + G(p)\left.\frac{dF(x)}{dx}\right|_{x=x(t)} \Delta x(t) = 0,$$

where $\frac{dF(x)}{dx}$ is the transfer coefficient of increment Δx(t). If we substitute the coordinate, x, for the time function x(t), this coefficient becomes the time-varying parameter v(t). When the motion is periodic, i.e. x(t)=Asinωt, the parameter/transfer coefficient is periodic too. Its oscillation period is equal to half the forced motion period, i.e. π/ω, for the odd-symmetrical nonlinear functions $F(-x) = -F(x)$.

So the incremental equation for periodic forced oscillations is the formulation of linear periodically nonstationary system losing its stability as soon as the parametric resonance is excited. The parameter variation frequency Ω=2ω, and as long as the first parametric resonance condition is met the frequency of parametric oscillations indefinitely increasing in a nonlinear system is congruent with that of forced oscillations.

Having blended, both forced and parametric oscillations become indistinguishable and thus parametric oscillations cannot be singled out and observed. Only the jumps of forced oscillation phase and amplitude or step-wise resonance are observed. Therein lays parametric resonance non-detectability. Well, the "penalty" for taking insufficient notice to parametric resonance is not long in coming. The conclusion that step-wise resonance is forced oscillation instability initiated by parametric oscillation excitation was first provided by I.M. Smirnova sixty years ago [6].

The above-derived center (8) $v_0(A)$ and the radius (7) $v_1(A)$ of the parametric resonance circle depend on the forced oscillation amplitude, A, of a nonlinear system, and the parametric resonance excitation conditions depend on the forced oscillation amplitude, A, and the parametric excitation phase, ψ, as follows:

$$W(A,\psi) = v_0(A) - v_1(A)e^{-j\psi}.$$

A more rigorous proof can be found in [2,3]. Consider the following example.

"Buoy" parametric resonance. The case in point is a strange behavior of a river buoy in a strong current of the Neva in the region of the Ivanovsky rapids (see Part 1). We apply the ordinary pendulum equation with damping, ξ, for an waterproof cylinder tied to a cable and submerged (as the result of a spring water flood) in the form:

$$Jp^2\alpha + \xi p\alpha + [M_0 + \Delta M(\alpha)] = 0,$$

where M_0 is an initial buoyancy moment at α=0, ΔM(α) is a buoyancy moment increment concerned with an immersion at the deviation, α. By unfolding the moments the equation can be written in the form:

$$Jp^2\alpha + \xi p\alpha + [(F_0 - P)\ell + \Delta F(\alpha)\ell]\sin\alpha = 0,$$

where F_0 is an initial buoyancy force at $\alpha = 0$, $\Delta F(\alpha)$ buoyancy force increment owing to oscillations of a cylindrical buoy submerged, ℓ is a cable length, P – buoy weight. We write the cylinder parameters: s (a base area), h_0 (an initial immersion value), d (specific weight of the liquid), and $\Delta h(\alpha) = \ell(1-\cos\alpha)$ is an immersion value under deviations. Then

$$Jp^2\alpha + \xi p\alpha + \ell[(sh_0 d - mg)\sin\alpha + sd\ell(1-\cos\alpha)\sin\alpha] = 0.$$

Denoting $b = \xi J^{-1}$, $c_0 = \ell J^{-1}(sh_0 d - mg)$, $c_1 = J^{-1}sd\ell^2$ we have the following equation

$$p^2\alpha + bp\alpha + c_0\sin\alpha + c_1(\sin\alpha - 0{,}5\sin 2\alpha) = 0.$$

By linearizing the nonlinear equation up to α^3 for $\left|\alpha > \left|\alpha^3\right|\right|$ we derive the approximate equation as follows:

$$p^2\alpha + bp\alpha + c_0\alpha + c_1\frac{\alpha^3}{2} = 0.$$

The approximate equation is known as the Duffing equation. The tabular format of the harmonic linearization coefficient for a nonlinear function is as follows:

$$W_{\text{h}}(A) = 3c_1A^2/8.$$

Then the radius and the shift of the parametric resonance circles are as follows:

$$\left|v_1(A)\right| = \frac{A}{2}\left|\frac{dW_{\text{h}}(A)}{dA}\right| = 3c_1A^2/8, \quad v_0(A) = W_{\text{h}}(A) + \frac{A}{2}\frac{dW_{\text{h}}(A)}{dA} = 3c_1A^2/4.$$

In Figure 2.13. the following reversed amplitude-phase-frequency characteristic of the linear part is plotted in the plane (R_0, I_0):

$$W^{-1}(j\omega) = (j\omega)^2 + b(j\omega) + c_0 = -\omega^2 + jb\omega + c_0$$

together with the parametric resonance circle.

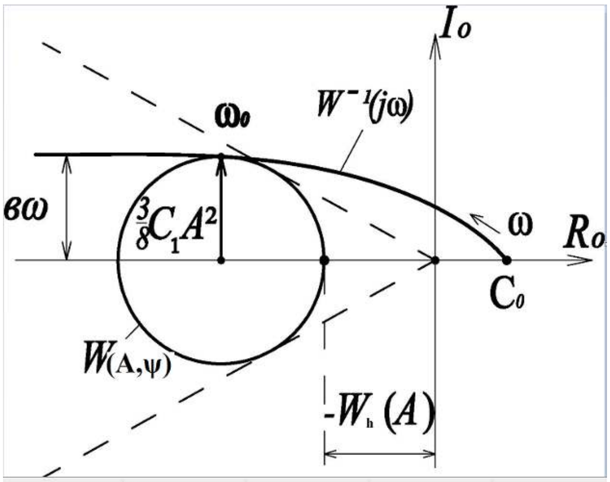


Figure 2.13. Parametric resonance excitation condition

It directly follows from Figure 2.13 that the frequency, ω_0, and the amplitude, A_0, of which parametric resonance excitation begins result from the equalities

$$2\,\mathrm{Re}W^{-1}(j\omega) = \mathrm{Im}W^{-1}(j\omega) = v_1(A).$$

From here it follows

$$\omega_0 = b + \sqrt{b^2 + c_0}, \quad A_0^2 = \frac{8}{3}\frac{\omega_0 b}{c_1}.$$

Hence the excitation conditions in terms of the buoy parameters or the conditions of “buoy” parametric resonance excitation and forced oscillation instability can be written in the form:

$$A^2 \geq \frac{8}{3}\frac{b\omega_0}{c_1}, \quad \omega > \omega_0 \cong b + \sqrt{b^2 + c_0}$$

From the point of view of formal mathematics the forced oscillation amplitude jumps to infinity. As a physical matter the infinite jump cannot be realized. The fact is that the assumed mathematical buoy formulation is completed as soon as the buoy lays on the water surface. Submerging during parametric oscillations the buoy tilts on the water surface before its complete submersion under the action of a current. Once the buoy totally spends its motion energy it extrudes out of the water again. In the condition of motionless water (without a current) buoy parametric resonance is also feasible in any vertical plane passing through the suspension axis. And in such situation forced oscillation jumps as large as infinity cannot be realized too since a “parametric force” vanishes as soon as the buoy submerges completely and the circle radius becomes zero because Δh =0.

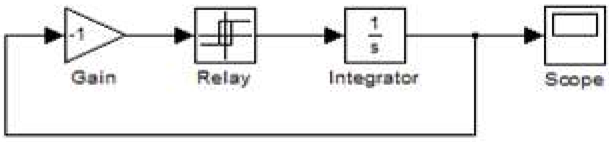
The careful reader has a right to ask the question, “And where are the forces which cause forced oscillations?” There can be several reasons for occurring forced oscillations. They are, e.g., wind gusts, wind waves, waveformations from moving ships. Moreover at the current periodic water vortexes known can separate in the process of flow the submerged buoy.

In conclusion assuming the rough buoy parameters P=50 kg, $F_0 = 55$ kg, $\ell = 10m$, J=500 $kgms^2$, s=0.3m ^{2}, b=0.1s^{-1} , $c_0 = 0.05s^{-2}$, $c_1 = 0.06s^{-2}$, we have the results close to reasonable one: $\omega_0 \approx 0.35 rads^{-1}$ and $A_0 \approx 0.5 rad$.

Oscillation process numerical simulations

As it was mentioned before in many instances analytical study of nonlinear dynamic systems is associated with dire difficulties. That is why dynamic process numerical simulations are widely used in an engineering practice. For this purpose the Matlab and Simulink software packages are applied. They allow for to composite a flow diagram of a model and investigate the problem numerically on the basis of its mathematical formulation.

Self-excited oscillation or parametric oscillation? The rough model of most simple self-exited oscillation system is shown in Figure 2.14 (here and in the next figures operator p is denouted by s). This is the Tantalus vessel. The model does not account for liquid velocity variations at the drain stage and saw-tooth self-oscillations occur.



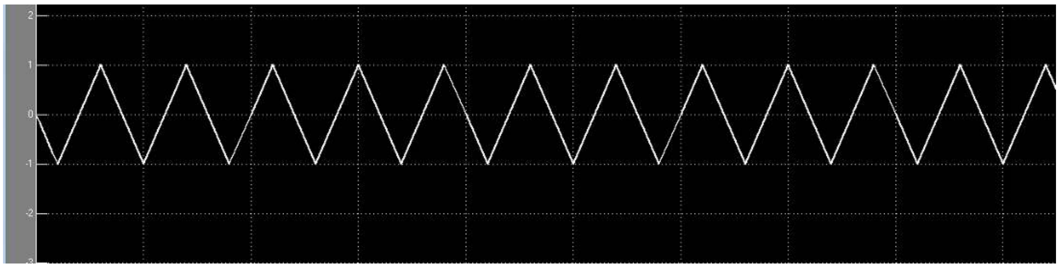
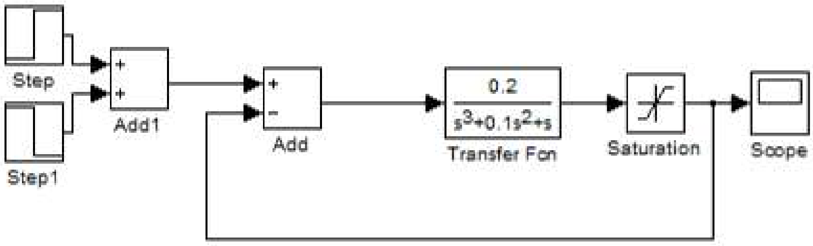


Figure 2.14. Rough model of self-excited system

In order to answer the question if the self- oscillation excitation conditions coincide with the first parametric resonance boundary we consider a third-order nonlinear self oscillatory system. The system model is given in Figure 2.15. The object/oscillatory link has the transfer function $W(p) = 0{,}2(p^3 + 0.1p^2 + p)^{-1}$; the nonlinear link of a saturation type has a linear section with a ± 1 slop. A horizontal section with zero slop is beyond the section.



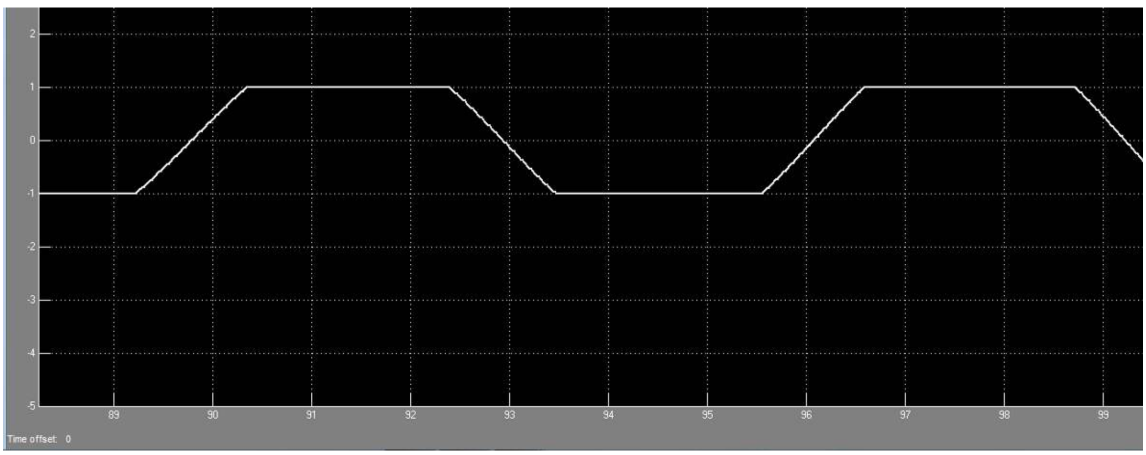


Figure 2.15. Third-order dynamic system model

The initial condition is specified in the form of a single narrow pulse. The pulse is formed by the two blocks - Step and Step1. A self-excited oscillation period is 6.3 s. Thus, the period of parameter variation from 0 to 1 is 3.15 s and the duration of parameter unit value is τ=1s. The parametric system model comprising the same object is given in Figure 2.16.

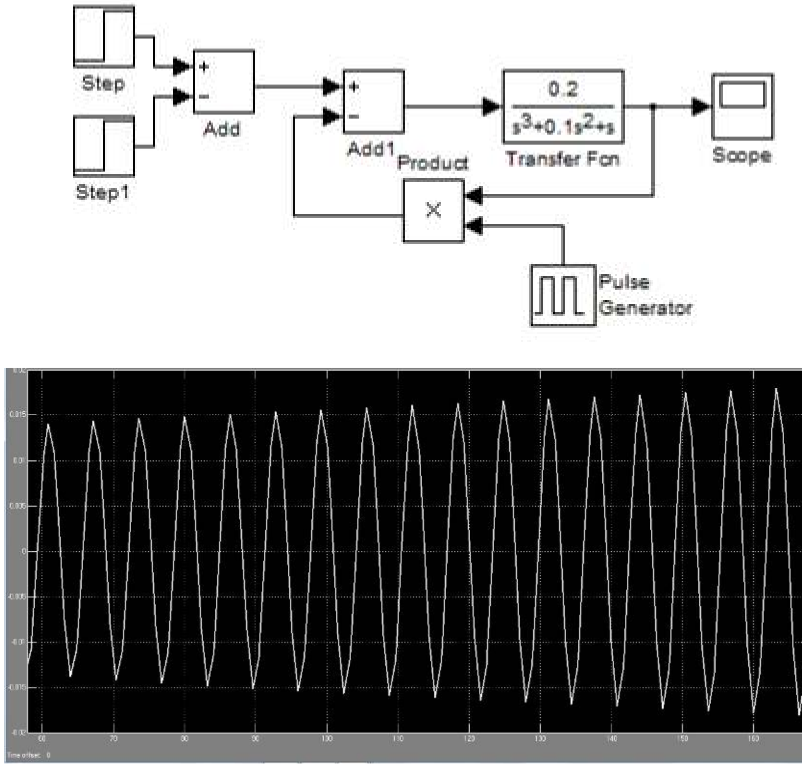


Figure 2.16. Parametric resonance excitation boundary

The multiplier Product is a 3s periodic parameter, which is unevenly changed from zero to 1. The multiplier is controlled by the 1 s unit pulse Generator. The time diagram illustrates parametric resonance beginning/boundary. So the conditions of self-excited oscillation initiation is the parametric resonance excitation boundary.

Taking into account that flexure-torsion oscillations often arise in constructions related to mechanics and flow mechanics we consider a spring pendulum model. A spring pendulum underlying those processes is a two freedom nonlinear dynamic system. The freedoms are vertical forced motions along the x-axis and α-angle parametric deviations. We consider step-by-step the formation of the model.

Parametric excitation of variable length pendulum. The block diagram for studying parametric pendulum excitation is given in Figure 2.17. Here the Transfer Fcn1 is the model of the ordinary pendulum of constant length ℓ_0 = 1 m. Pendulum weight is 1 kg. We write the characteristic equation of constant length pendulum (2) in the form:

$$p^2 + \frac{b}{a} p + \frac{c}{a} = 0,$$

or denoting $a = ml^2, c = mgl$

$$p^2 + \frac{b}{ml^2} p + \frac{g}{l} = 0.$$

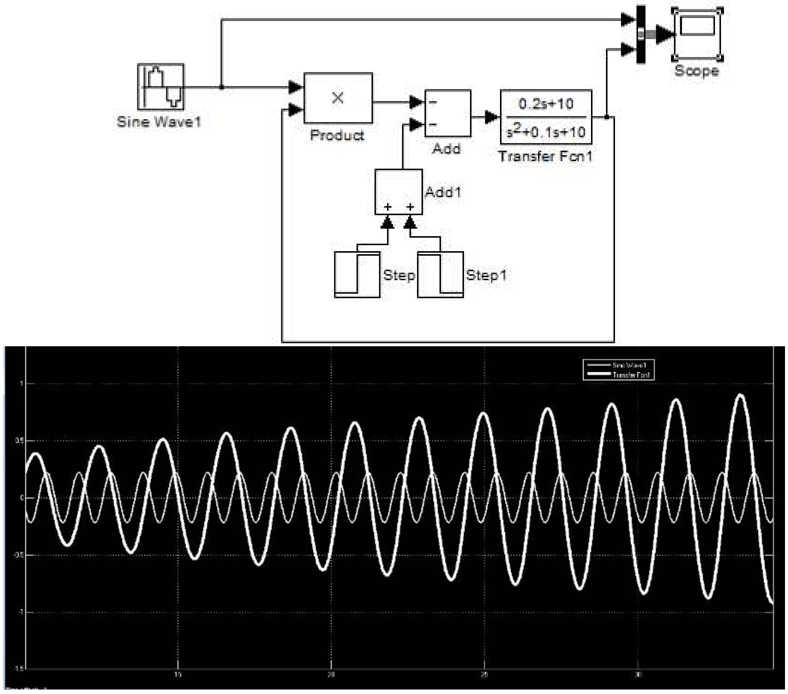


Figure 2.17. Variable length pendulum model

In terms of an oscillatory object with small damping (b=0.1) we shall derive the first approximation equation of the pendulum of variable length $l = l_0 + \Delta l(t)$ under the numerical values assumed above:

$$p^2 + 0.1(l_0^{-2} - 2\ell_0^{-3}\Delta l)p + 10(l_0^{-1} - \ell_0^{-2}\Delta l) \cong 0 . \qquad (44)$$

The variable length pendulum equation at $\ell_0 = 1m$ takes the form:

$$(p^2 + 0.1p + 10)\alpha(p) = (0.2p + 10)\Delta\ell(p) .$$

Just now we can write the transfer function corresponding to that in Figure 2.17:

$$W(p) = \frac{\alpha(p)}{\Delta\ell(p)} = \frac{0.2p + 10}{p^2 + 0.1p + 10} .$$

The harmonic signal corresponding to the oscillations of the pendulum length $\Delta l(t)$ is received by the modulator Product of oscillations $\alpha(t)$, and the two blocks Step (positive and negative) form a unit narrow pulse of initial conditions. The jscillations are recorded by the block Skope and they lack as long as the pendulum excitation threshold $|\Delta\ell(t)| < 0.21$. The parametric resonance excitation pattern at $|\Delta\ell(t)| = 0.23$ is presented in Figure 2.17.

The pendulum length/parameter oscillations have the 6 rad/s frequency (in a thin line) and 0.23 m amplitude (in a heavy line). The excited ramp parametric oscillations $\alpha(t)$ (in a heavy line) appear at the 3rad/s frequency and they are the first parametric resonance oscillations.

Spring pendulum oscillations. The spring pendulum model is shown in Figure 2.18.

The parametric oscillation circuit on the right side of the diagram (see Figure 2.18.) is borrowed from Figure 2.17. The spring circuit of forced oscillations is added on the left side of the same diagram. The jscillations are excited by the Sine Wave generator. The 1 kg spring-suspended bob at the 40 N/m spring rate is described by the transfer function Transfer Fcn

$$W(p) = \frac{F(p)}{\Delta\ell(p)} = \frac{1}{p^2 + p + 40} ,$$

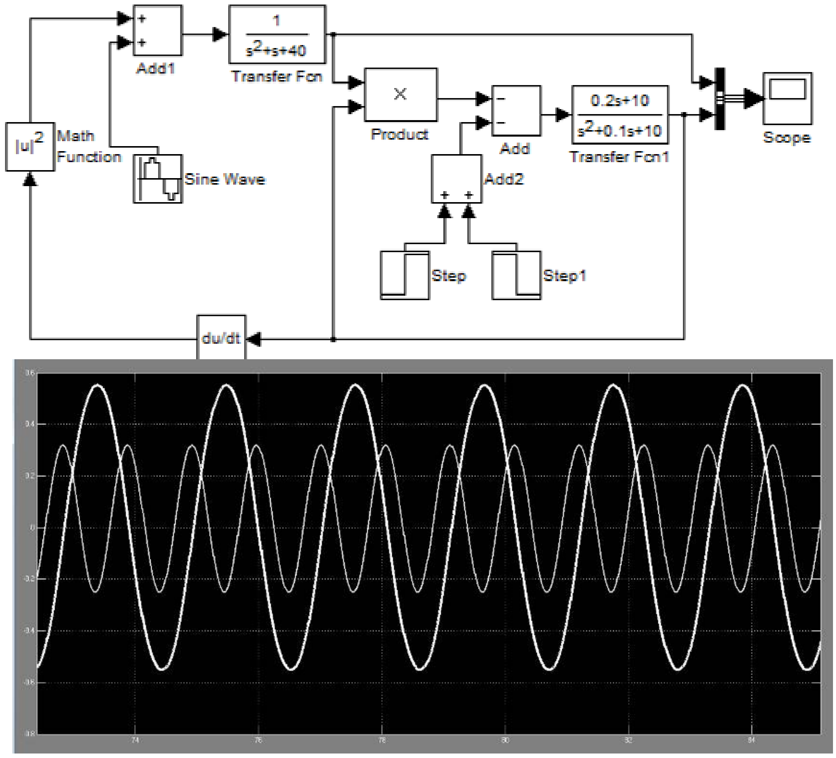


Figure 2.18. Spring pendulum model

where F(p) is an operator notation of the harmonic disturbing force f(t). The Sine Wave signal is added to the centrifugal force originating from the α-angle oscillations $f(t) = mv^2 / l_0 = m\omega^2 l_0 = ml_0(\frac{d\alpha}{dt})^2$. For the reason the Math Function squared modulus and du/dt are included in the forced oscillation circuit. The continuous, i.e. sustained and nonincreasing, spring pendulum oscillations are displayed in Figure 2.18 under the block-diagram (in a heavy line) while the 0.3 length oscillation amplitude (in a thin line) exceeds the parametric resonance excitation threshold of the circuit. The spring hinders the parametric resonance excitation and brings the spring pendulum to ***sustained parametric oscillations***. The reason is that the input Add1 adder signals are antiphased and they are subtracted but not added.

On applicability of pendulum models. The important conclusion results from Figure 2.18: during the spring pendulum oscillation excitation gaining the constant constituent the forced oscillations (see in a thin line) grow unsymmetrical. By-turn the constant constituent increase leads to the rise of Product transfer coefficient and parametric circuit amplification. And as soon as the Sine Wave amplitude becomes sufficiently great the sustained parametric oscillations lose their stability. In practical situations a physical spring pendulum can lose its oscillatory stability while it transfers to a rotation mode, which is similar to an ordinary physical pendulum. But the considered models are not applicable in that case because they are mathematical pendulum models and their domain of applicability is restricted to small angular oscillations, for example |α|<1 rad as in Figure 2.18.

CONCLUSION

That is all. The popular story about parametric resonance is approaching to its completion. As a start of the end we repeat the following question asked in Part 1, “Would a human being use parametric oscillations at present?” Try just now to specify that answer as follows, “Yes and no”. Surely parametric resonance has taught a human being a lot and he applies obtained knowledge in technical creating. And could anyone assert that a human being has “tamed” parametric resonance and placed it at his service? That really seems dubious. The point is that the short history of relations between a human being and parametric resonance was rather connected with cognizing and suppressing parametric oscillations and resonance. We again refer to [1]. The book does not contain the notions of parametric oscillations and resonance, and oscillation self-excitation of dynamic systems is discussed in it. Of course, a great authority like the book author has a right to have his own view on oscillation classifications and definitions. But still, that testifies to a certain extent the resent opinion unsettled in the scientific community. The examples given in this paper in respect of applications of parametric resonance, such as generating oscillations by mechanically displacing condenser disks, ferroresonance, etc, are not practically used. There is no choice except, possibly, microwave structures and an ordinary swing. Thus, parametric resonance still remains an antagonist regarding a human being and causes more pain than joy. That is why the following answer to the above question will be legitimate, “Likely no.”

Let us ask the other question, “Why does the phase corresponding to the crossing point of the parametric resonance circle and the frequency characteristic self-settle but not another one, e.g. opposite, i.e. at which damping of oscillations, set in itself while exciting parametric oscillations occur?” Why, in the second or opposite case parametric oscillations could diminish but not increase the forced oscillation/jump amplitude and there would be more joy than pain.

Such answer can be given the above question. First, the physical principle of virtual potential energy maximum known in mechanics is in force in dynamic systems. According to the principle the phase corresponding to the maximum degree of oscillation excitation is automatically set and rest potential energy converts into oscillatory kinetic energy. Thus, from a number of voluntary coordinate oscillation phases that are introduced by the Ω-frequency alternating parameter, the phase is set at which a total phase shift between an oscillatory object and a parameter is 180 degrees. Furthermore the $\Omega/2$-frequency of coordinate oscillation is called critical and parametric system has its maximum equilibrium stability. Parametric resonance is excited as soon as the parameter oscillation amplitude exceeds its threshold value.

There is no doubt that a human being will place parametric resonance at his service. One of the efforts is, for example, the parametric correction of oscillatory dynamic system phase [7]. Unlike the coordinate feedback known from classical theory of automatic control a new kind of feedback is used. That is parametric one. By choosing a signal lag value in a parametric feedback circuit either oscillating or damping regulators can be installed.

In conclusion it should be applied to the readers that had the patience to get to the end. All information that was succeeded in the statement about parametric oscillations and resonance is only a little part of knowledge cumulated by humanity. A mere bibliography with rundowns could take the most of the paper. So, young people taking an interest in parametric resonance will learn a lot of interesting things in macro and microworlds. Elliptic orbits are, for example, a source of parametric oscillations and, quite likely, the mysterious nature of ball lightning has a parametric origin, etc. And lastly it is not a difficult matter to transform a home computer into a research laboratory on dynamic system attributes and solving new problems using numerical simulations. It remains to wish the patient readers for sufficient strengths and future advancements.

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LIST OF FIGURES