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Stress-strain state of clamped rectangular Reissner plates

Напряженно-деформированное состояние защемленной прямоугольной пластины Рейсснера

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Abstract. The paper focuses on obtaining numerical results for a rectangular Reissner plate with clamped contour under the influence of a uniform load using the iteration superposition method of four types of trigonometric series (correcting functions). The initial function of bendings is selected as a quartic polynomial which turns into zero on the contour and is a specific solution to the main bending equation. Discrepancies in rotation angles from the initial polynomial are eliminated in turn on parallel edges by pairs of correcting functions of bendings and stresses which cause angular discrepancies themselves. During an infinite process of the superposition of these pairs, all discrepancies tend to zero, which gives a precise solution at the limit. The paper presents results of bending computations, bending moments, and shearing forces for square plates different thickness. The obtained results are compared with the results of other authors, as well as with Kirchhoff theory. It is shown that with the relative thicknesses less than 1/20, the results gained with both theories are almost the same.

Аннотация. В статье получены численные результаты для защемленной по контуру прямоугольной пластины Рейсснера под действием равномерной нагрузки итерационным методом суперпозиции четырех видов тригонометрических рядов (исправляющих функций). Начальная функция прогибов выбирается в виде многочлена четвертой степени, который обращается в нуль на контуре и является частным решением основного уравнения изгиба. Невязки по углам поворота от начального многочлена поочередно устраняются на параллельных краях парами исправляющих функций прогибов и напряжений, которые сами порождают угловые невязки. В ходе бесконечного процесса суперпозиции этих пар все невязки стремятся к нулю, что в пределе дает точное решение. Приведены результаты расчетов прогибов, изгибающих моментов и перерезывающих сил для квадратных пластин различной толщины. Дается сравнение с результатами других авторов, а также с теорией Кирхгофа. Показано, что при относительных толщинах, меньших 1/20 результаты по обеим теориям практически совпадают.

Introduction

Modern structures widely use metallic and non-metallic materials (composite, synthetic, etc.) which have increased pliability to an interlaminar shear. Such materials are often used for making plates (panels, slabs) which are main elements in ship, aero-, and other structures as well as in nanoengineering.

Solution to 3D problems of the elasticity theory, which include problems of the plates' elastic behavior, is connected with solving a complex system of differential equations and boundary conditions. It caused the necessity of shifting from 3D problems to more simple 2D ones. Historically, a simplified theory of thin plates, based on the hypotheses of Kirchhoff-Love, was the first to put forward; it is called the classical theory. Many engineering problems were successfully solved using this theory. However, it provides poor accuracy near the plate's contour, around the points of sharp change in boundary conditions and the points of applied concentrated forces, as well as when making computations for plates of average thickness. Therefore, there emerged a problem of shifting to more precise two-dimensional theories with using the altered hypotheses by Kirchhoff-Love. These theories were called refined theories (intermediate between the classical two-dimensional theory and the three-dimensional one).

Today, a variety of refined theories are developed and used, including theories that take into account the influence of transverse shear strain on the bending. Timoshenko [1] was the first to note the necessity of considering this influence when solving rod vibration problems. The number of refined theories today is quite large because there is no universal theory providing acceptable results for all types of problems.

The linear plate bending theory, which qualitatively refined the classical theory, was firstly put forward by Reissner [2]. Author rejected the hypothesis of the rectilinear element normality to the median surface and suggested replacing it with a hypothesis of rectilinearity of this element and introducing a law of stress variation based on thickness of the plate. Reissner, using a balance equation of the three-dimensional elasticity theory, compatibility conditions, and Castiglian's principle of minimum strain-energy, obtained new differential equations of the plate bending and the corresponding boundary conditions allowing for the transverse shear effect. The fundamental system consists of two equations. The first equation of the fourth order characterizes the plate bending. The second equation of the second order describes the stress state which is of local character and disappears quickly when moving away from the plate's edge. It increased the system's order to the sixth which allowed satisfying three boundary conditions (instead of two in the classical theory). The given and similar shear theories are often called Reissner - Mindlin [3] - Timoshenko [4] theories due to their similarity. Particularly, the difference between the theories by Reissner and Mindlin is basically values of the transverse shear coefficient: Reissner has it equal to 5/6 ($\approx$ 0.833) and Mindlin to $\pi^2/12$ ($\approx$ 0.822), which is very close.

Variant of shear theories presented in the work Ambartsumyan [5].

Applicability limits of the theories by Kirchhoff-Love, Poisson, and Reissner, as well as revision of refined theories are discussed in the works Goldenveizer et al. [6, 7], Vasiliev [8, 9], Zhilin [10, 11] an others.

Goldenveizer et al. [6, 7] in the refined theory divide the stress state into internal and edge. Researchers use the asymptotic method in combination with the variational principle. The authors state that the Reissner system of basic equations is incorrect because it does not result from the asymptotic method. Vasiliev [8, 9] notes that there are problems which cannot be solved with the Kirchhoff theory. The author reckons that the asymptotic method of the Goldenveizer refined theory is ambiguous and approximate. In the work Vasiliev [9], the author makes an attempt to show that with the help of certain transformations the sixth order refined theories can be presented as the modern form of the classical plate theory. Zhilin [10] points out that the Reissner theory is in line with the three-dimensional elasticity theory and the Kirchhoff theory should be considered as an asymptotic consequence of the Reissner theory. In the work Zhilin [11] author warns about possible negative consequences of the formal use of the classical plate theory in the Finite Difference Method (FDM), the Finite Element Method (FEM) and other computing systems if spatial structures have rectangular plates with free support on the framework. Actually, in the shear plate theory support reactions coincide with contour transverse forces Q_x and Q_y, which balance the pressure on the plate. It excludes any angular forces in the case of free contour support which takes place in the Kirchhoff theory.

Revision, refinement and generalization of the theory of Reissner-Mindlin-Timoshenko and dedicated work in recent years [12–19].

There is little information about numerical results of the bending problem of a rectangular plate with a clamped edge using shear theories due to the problem's complexity. Let us note the works [20–29].

A rigidly clamped uniformly-loaded plate was examined in the work of Rudiger [20]. The author used hyperbolic-trigonometric series. The series' ratios were found by the principle of virtual displacements. Numerical computations for two kinds of rectangular plates show that allowing for the transverse shear deformation significantly affects the plate bending (no computations were done for a square plate). The works [21, 22] is based on the Ambartsumyan [5] shear theory. To solve the problem, the author used trigonometric series with hyperbolic functions in a different coordinate. Indefinite coefficients are found from the problem's boundary conditions. The problem reduces to solving an infinite system of linear algebraic equations.

In the works [23–29] various modifications of FEM were used.

Xu [23, 24] used a triangular finite element. Values of bendings and bending moments in the center of a square plate with the relative thickness of 0.1 were obtained.

In the work by Zienkiewicz et al. [25], FEM with linear quadrilateral elements is used. Numerical results are obtained for a square plate with a clamped edge with under a uniform load for relative thicknesses 0.001, 0.01, and 0.1. The number of the elements increased from 4 to 1024.

In the work Weiming and Guangsong [26] "Rational FEM" is used for Reissner plates with various boundary conditions under a uniform load and a central force. The accuracy of computations with the number of elements up to 64 is studied.

Ayad et al. [27] used the hybrid-mixed variational FEM with triangular and quadrilateral elements which is based on the Hellinger-Reissner variational principle. There are numerical results, particularly the graphs of bendings in the center of a square plate for different relative plate thicknesses when dividing the plate into 144 elements.

The work Dhananjaya [28] presents a closed form solution for equilibrium and flexibility matrices of the Mindlin–Reissner plates using the Integrated Force Method (IFM) based on 4 node rectangular elements. The author obtained the numerical results for square clamped plates with the relative thicknesses of 0.01 and 0.2 as the graphs of bendings and moments in the center for different numbers of finite elements, but, unfortunately, the scale of images is small.

In the work Aghdam et al. [29], an approximate solution is obtained for the bending of a rectangular Reissner plate with clamped edges. Resolving equations are a system of three differential equations of the second order. The solution procedure is based on using the extended Kantorovich method (EKM) to transform resolving systems of equations into ordinary differential equations.

In [30] uses the method of Bergan-Wang for moderately thick plates (modified finite integral transform method – FIT method). The results of a clamped square plate are compared with the results of the classical plate theory, Reissner-Mindlin theory and the three dimensional theory of elasticity for different relative thickness of the plate.

The goal of this work is to obtain reliable numerical results on the stress-strain state of a rectangular plate with a clamped edge allowing for the transverse shear deformation within the Reissner theory, to compare with the classical theory and with works of other authors, to determine applicability limits of the classical theory.

Methods

The fundamental system of differential equations of the Reissner elastic plate (see [2, 4]) has the form:

$$\begin{aligned} &D\nabla^2\nabla^2 W = q - \frac{H^2}{10}\frac{2-\nu}{1-\nu}\nabla^2 q\ , \\ &\nabla^2\Psi - \frac{10}{H^2}\Psi = 0. \end{aligned} \tag{1}$$

where $D = EH^3/\ [12(1\text{-}\nu^2)]$ – cylindrical stiffness; E – Young's modulus; H – plate’s thickness; ν – Poisson’s ratio; ∇^2 – Laplace two-dimesional operator; $W(X,\ Y)$ – function of bending of the middle surface of the plate; $X,\ Y$ – coordinates; $q(X,\ Y)$ – transverse load; $\Psi(X,\ Y)$ – stress function (edge potential).

For the uniform transverse load, directed at the negative side of oz axis, the system (1) in its dimensionless form will look as follows:

$$\begin{aligned} &\nabla^2\nabla^2 w(x,y) = -1, \\ &\psi(x,y) - \alpha\,\nabla^2\psi(x,y) = 0. \end{aligned} \tag{2}$$

where $w(x,\ y) = W/(qb^4/D)$ – dimensionless bending function; b – the width of the plate; $x = X/b,\ y = Y/b$- dimensionless coordinates; $\psi(x,\ y) = \Psi(X,\ Y)\ /qb^2$ – dimensionless stress function; $\alpha = h^2/10$ - shear factor; $h = H/b$ – dimensionless plate’s thickness.

Boundary conditions of the rectangle plate with a clamped edge $x = \pm\,\gamma/2,\ y = \pm\ 1/2$ have the form:

$$w = 0\ , \qquad \varphi_x = 0\ , \qquad \varphi_y = 0 \tag{3}$$

where $\gamma = a/b$ – ratio of the plate’s sides; a – length of the plate; $\varphi_x,\ \varphi_y$ – the angles of rotation of sections:

$$\varphi_x = \frac{\partial}{\partial x}\left(w + \alpha_1 \nabla^2 w\right) - \alpha_1 \frac{\partial \psi}{\partial y}; \qquad \varphi_y = \frac{\partial}{\partial y}\left(w + \alpha_1 \nabla^2 w\right) + \alpha_1 \frac{\partial \psi}{\partial x}; \qquad \left(\alpha_1 = \frac{2\alpha}{1-\nu}\right).$$

The task is set to find bending functions w and stress functions ψ, satisfying fundamental Eq. (2) and the given conditions (3) on each edge.

To solve the problem, we use the system of functions:

$$w_0(x,y) = -\frac{1}{8}\left(x^2 - \frac{\gamma^2}{4}\right)\left(y^2 - \frac{1}{4}\right) \tag{4}$$

$$w_{1n}(x,y) = \sum_{k=1,3,...}^{\infty} (-1)^{k^*} \frac{A_{kn}}{\cosh \tilde{\lambda}_k}\left(x \sinh \lambda_k x - \frac{\gamma}{2} \tanh \tilde{\lambda}_k \cosh \lambda_k x\right) \cos \lambda_k y \tag{5}$$

$$w_{2n}(x,y) = \sum_{s=1,3,...}^{\infty} (-1)^{s^*} \frac{B_{sn}}{\cosh \tilde{\mu}_s}\left(y \sinh \mu_s y - \frac{1}{2} \tanh \tilde{\mu}_s \cosh \mu_s y\right) \cos \mu_s x \tag{6}$$

$$\psi_{1n}(x,y) = \sum_{k=1,3,...}^{\infty} (-1)^{k^*} C_{kn} \sinh \beta_k x \sin \lambda_k y \tag{7}$$

$$\psi_{2n}(x,y) = \sum_{s=1,3,...}^{\infty} (-1)^{s^*} D_{sn} \sinh \xi_s y \sin \mu_s x \tag{8}$$

where $w_0(x, y)$ is the initial bending function (null approximation); w_{1n} , w_{2n} , ψ_{1n} , ψ_{2n} are the correcting bending functions w and stress functions ψ; n is the number of the iteration; A_{kn} , B_{sn} , C_{kn} , D_{sn} are indefinite coefficients;

$$\lambda_k = k\pi, \quad \mu_s = \frac{s\pi}{\gamma}, \quad k^* = \frac{k+1}{2}, \quad s^* = \frac{s+1}{2}.$$

$$\tilde{\lambda}_k = \frac{\lambda_k \gamma}{2}, \quad \tilde{\mu}_s = \frac{\mu_s}{2}, \quad \beta_k = \sqrt{\lambda_k^2 + \frac{1}{\alpha}}, \quad \xi_s = \sqrt{\mu_s^2 + \frac{1}{\alpha}}.$$

The correcting bending functions are biharmonic and turn into zero on the plate's contour; the stress functions satisfy the second basic Eq. (2).

The initial function $w_0(x, y)$ is an isolated solution to the first Eq. (2). It equals to zero on the plate's contour but causes the contour to turn, i.e. creates the main discrepancies φ_x , φ_y in boundary conditions (4) which should be expanded into the Fourier series (on the edges $x = -\gamma/2$ and $y = -1/2$ they differ in signs):

$$\begin{aligned}
\varphi_{x0}\Big|_{x=\frac{\gamma}{2}} &= -\frac{\gamma}{8}\left(y^2 - \frac{1}{4} + 2\alpha_1\right) = \sum_{k=1,3....}^{\infty} (-1)^{k^*} a_{k0} \cos \lambda_k y, \\
\varphi_{y0}\Big|_{x=\frac{\gamma}{2}} &= -\frac{\alpha_1 y}{2} = \sum_{k=1,3,...}^{\infty} (-1)^{k^*} b_{k0} \sin \lambda_k y, \\
\varphi_{x0}\Big|_{y=\frac{1}{2}} &= -\frac{\alpha_1 x}{2} = \sum_{s=1,3...}^{\infty} (-1)^{s^*} u_{s0} \sin \mu_s x, \\
\varphi_{y0}\Big|_{y=\frac{1}{2}} &= -\frac{1}{8}\left(x^2 - \frac{\gamma^2}{4} + 2\alpha_1\right) = \sum_{s=1,3,...}^{\infty} (-1)^{s^*} g_{s0} \cos \mu_s x,
\end{aligned} \tag{9}$$

where a_{k0}, $bk0$, u_{s0}, g_{s0} – the coefficients of the decomposition.

The correcting functions during the infinite iteration process of their superposition must reduce these discrepancies to zero.

The idea of an infinite superposition of functions to elimination of the main deviations (residuals) from a private solution belongs to V.Z. Vasiliev [31].

The second discrepancy (9) (the first discrepancy will be allowed for in the next iteration to improve the series convergence) is eliminated by the first pair of correcting functions w_{11} and ψ_{11} with satisfying the conditions on the edges $x = \pm \gamma/2$ at the expense of coefficients A_{k1} and C_{k1}.

However, the functions themselves cause angular discrepancies on the edges $y = \pm 1/2$:

$$\varphi_{y11}\big|_{y=\frac{1}{2}} = \sum_{k=1,3,...}^{\infty}\left[\frac{\lambda_k A_{k1}}{\cosh\tilde{\lambda}_k}\left(x\sinh\lambda_k x - \frac{\gamma}{2}\tanh\tilde{\lambda}_k\cosh\lambda_k x + 2\alpha_1\lambda_k\cosh\lambda_k x\right) - \alpha_1 C_{k1}\beta_k\cosh\beta_k x\right]. \quad (10)$$

They should be expanded into the Fourier series in $\cos\mu_s x$, we should invert the summation, plug expressions for the coefficients A_{k1}, C_{k1}, in them and put them together with the corresponding discrepancies $\varphi_{y0|y=1/2}$ from the initial polynomial (the fourth function (9)), i.e. transform into

$$\varphi^*_{y11}\big|_{y=\frac{1}{2}} = \varphi_{y11}\big|_{y=\frac{1}{2}} + \varphi_{y0}\big|_{y=\frac{1}{2}} = \sum_{s=1,3,...}^{\infty}(-1)^{s^*} g^*_{s1}\cos\mu_s x. \quad (11)$$

where $g_{s1}{*} = g_{s0} + g_{s1}$ are the series' ratios.

Discrepancy (11) and the third discrepancy (9) are compensated by the second pair of correcting functions w_{21}, ψ_{21} at the expense of coefficients B_{s1} and D_{s1}.

Besides, functions w_{21} and ψ_{21} on the edges $x = \pm\gamma/2$ also create angular discrepancies:

$$\varphi_{x21}\big|_{x=\frac{\gamma}{2}} = \sum_{s=1,3,...}^{\infty}\left[\frac{\mu_s B_{s1}}{\cosh\tilde{\mu}_s}\left(y\sinh\mu_s y - \frac{1}{2}\tanh\tilde{\mu}_s\cosh\mu_s y + 2\alpha_1\mu_s\cosh\mu_s y\right) + \alpha_1 D_{s1}\xi_s\cosh\xi_s y\right]. \quad (12)$$

which should be expanded into the Fourier series in $\cos\lambda_k y$ we should invert the summation, plug expressions for the coefficients B_{s1}, D_{s1} in them and put them together with the corresponding discrepancies $\varphi_{x0|x=\gamma/2}$ from the initial polynomial (the first function (9)), i.e. transform into

$$\left(\varphi_{x0} + \varphi_{x21}\right)_{x=\frac{\gamma}{2}} = -\sum_{k=1,3,...}^{\infty}(-1)^{k^*} a^*_{k1}\cos\lambda_k y;. \quad (13)$$

where $a^*_{k1} = a_{k0} + a_{k1}$.

Discrepancies (13) are compensated by the correcting pair of w12 and ψ 12 of the second iteration when satisfying the boundary conditions on the edges $x = \pm\gamma/2$. This gives the system of two equations to determine the coefficients A_{k2}, C_{k2}.

The discrepancies of this pair $\varphi_{y12|y=1/2}$ will have the form similar to (10, 11):

$$\varphi_{y12}\big|_{y=\frac{1}{2}} = -\sum_{s=1,3,...}^{\infty}(-1)^{s^*} g_{s2}\cos\mu_s x. \quad (14)$$

Then series w_{22} and ψ_{22} are used to eliminate the discrepancies of this pair.

And then the process is repeated.

The convergence of the method

During the iteration process, discrepancies in boundary conditions should tend to zero, i.e. the iteration process should be convergent. Due to linearity of the problem, it is sufficient to prove, for example, that

$$\lim_{n\to\infty} A_{kn} = 0 \quad (k = 1,3,...;\ n = 1,2,...). \quad (15)$$

It is established that the coefficients A_k of two adjacent iterations are linked linearly. The dependence of A_{kn+1} on A_{kn} is a homogeneous infinite system of linear algebraic equations. This system should be regular [32]. Then, successive approximations will lead to a trivial solution from whatever initial values of the coefficients A_k, limited in total, we would start.

Since the discrepancy coefficients linearly depend on coefficients A_k, during the iteration process they will also tend to zero.

Analysis of the series convergence for bending moments and shearing forces

Moments M, and shearing forces Q according to [4] will have the following form:

$$M_x = -\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \alpha_2\frac{\partial^2}{\partial x^2}\nabla^2 w\right) + \alpha_2\frac{\partial^2\psi}{\partial x\partial y} + \alpha_3,$$

$$M_y = -\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} + \alpha_2\frac{\partial^2}{\partial y^2}\nabla^2 w\right) - \alpha_2\frac{\partial^2\psi}{\partial x\partial y} + \alpha_3,$$

$$M_{xy} = (1-\nu)\frac{\partial^2 w}{\partial x\partial y} + \alpha_2\frac{\partial^2}{\partial x\partial y}\nabla^2 w - \alpha\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right).$$

$$Q_x = -\frac{\partial}{\partial x}\nabla^2 w + \frac{\partial\psi}{\partial y}\ ,\ Q_y = -\frac{\partial}{\partial y}\nabla^2 w - \frac{\partial\psi}{\partial x}\ .$$

Here, the moments are referred to value qb2, shearing forces – to value qb; $\alpha_2 = 2\alpha$; $\alpha_3 = \nu\alpha/(1-\nu)$.

Let us show final expressions for bending moments M_x, torsion moments M_{xy} and shearing forces Q_x which were used for computing:

$$\begin{aligned}
M_x &= \frac{\alpha_1}{2} + \frac{1}{4}\left[y^2 - \frac{1}{4} + \nu\left(x^2 - \frac{\gamma^2}{4}\right)\right] \\
&- \sum_{k=1,3,\ldots}^{\infty}(-1)^{k^*}\lambda_k\left\{[2\cosh\lambda_k x + 4\alpha\lambda_k^2\left(\cosh\lambda_k x - \frac{\cosh\tilde\lambda_k}{\cosh\tilde\beta_k}\cosh\beta_k x\right)\right. \\
&\left.+(1-\nu)\left(\lambda_k x\sinh\lambda_k x - \tilde\lambda_k\tanh\tilde\lambda_k\cosh\lambda_k x\right)]\frac{A_{k\Sigma}}{\cosh\tilde\lambda_k} + \frac{4\alpha}{\lambda_k^2}\frac{\cosh\beta_k x}{\cosh\tilde\beta_k}\right\}\cos\lambda_k y \\
&+ \sum_{s=1,3,\ldots}^{\infty}(-1)^{s^*}\mu_s\left\{[-2\nu\cosh\mu_s y + 4\alpha\mu_s^2\left(\cosh\mu_s y - \frac{\cosh\tilde\mu_s}{\cosh\tilde\xi_s}\cosh\xi_s y\right)\right. \\
&\left.+(1-\nu)\left(\mu_s y\sinh\mu_s y - \tilde\mu_s\tanh\tilde\mu_s\cosh\mu_s y\right)]\frac{B_{s\Sigma}}{\cosh\tilde\mu_s} + \frac{4\alpha}{\gamma\mu_s^2}\frac{\cosh\xi_s y}{\cosh\tilde\xi_s}\right\}\cos\mu_s x,
\end{aligned} \tag{16}$$

$$\begin{aligned}
M_{xy} &= -\frac{1-\nu}{2}xy - \sum_{k=1,3,\ldots}^{\infty}(-1)^{k^*}\left\{\left(\left[(1-\nu)\left(1-\tilde\lambda_k\tanh\tilde\lambda_k\right) + 4\alpha\lambda_k^2\right]\sinh\lambda_k x + (1-\nu)\lambda_k x\cosh\lambda_k x\right.\right. \\
&\left.\left.-2\alpha\frac{\lambda_k\cosh\tilde\lambda_k}{\beta_k\cosh\tilde\beta_k}\left(\lambda_k^2+\beta_k^2\right)\sinh\beta_k x\right)\frac{\lambda_k A_{k\Sigma}}{\cosh\tilde\lambda_k} + 2\alpha\frac{\lambda_k^2+\beta_k^2}{\lambda_k^2\beta_k\cosh\tilde\beta_k}\sinh\beta_k x\right\}\sin\lambda_k y \\
&- \sum_{s=1,3,\ldots}^{\infty}(-1)^{s^*}\left\{\left(\left[(1-\nu)\left(1-\tilde\mu_s\tanh\tilde\mu_s\right) + 4\alpha\mu_s^2\right]\sinh\mu_s y + (1-\nu)\mu_s y\cosh\mu_s y\right.\right. \\
&\left.\left.-2\alpha\frac{\mu_s\cosh\tilde\mu_s}{\xi_s\cosh\tilde\xi_s}\left(\mu_s^2+\xi_s^2\right)\sinh\xi_s y\right)\frac{\mu_s B_{s\Sigma}}{\cosh\tilde\mu_s} + \frac{2\alpha}{\gamma}\frac{\mu_s^2+\xi_s^2}{\mu_s^2\xi_s\cosh\tilde\xi_s}\sinh\xi_s y\right\}\sin\mu_s x,
\end{aligned} \tag{17}$$

$$\begin{aligned}
Q_x &= \frac{x}{2} - 2\sum_{k=1,3,\ldots}^{\infty}(-1)^{k^*}\left[\lambda_k^2\left(\sinh\lambda_k x - \frac{\lambda_k}{\beta_k}\frac{\cosh\tilde\lambda_k}{\cosh\tilde\beta_k}\sinh\beta_k x\right)\frac{A_{k\Sigma}}{\cosh\tilde\lambda_k} + \frac{1}{\lambda_k\beta_k}\frac{\sinh\beta_k x}{\cosh\tilde\beta_k}\right]\cos\lambda_k y \\
&+ 2\sum_{s=1,3,\ldots}^{\infty}(-1)^{s^*}\left[\mu_s^2\left(\cosh\mu_s y - \frac{\cosh\tilde\mu_s}{\cosh\tilde\xi_s}\cosh\xi_s y\right)\frac{B_{s\Sigma}}{\cosh\tilde\mu_s} + \frac{1}{\gamma}\frac{1}{\mu_s^2}\frac{\cosh\xi_s y}{\cosh\tilde\xi_s}\right]\sin\mu_s x,
\end{aligned} \tag{18}$$

where $A_{k\Sigma} = A_{k1} + A_{k2} + ... + A_{kn}$; ...; $B_{s\Sigma} = B_{s1} + B_{s2} + ... + B_{sn}$ are overall coefficients in all iterations; $\tilde{\beta}_k = \beta_k \gamma / 2$, $\tilde{\xi}_s = \xi_s / 2$.

Let us study the convergence of functional series that occur in formulae (16), (18).

The fastest to converge are the series of bending moments (16) in the center of the plate, where general terms have the order $O\left(1/\cosh\tilde{\lambda}_k\right)$ or $O\left(1/\cosh\tilde{\mu}_s\right)$, and the slowest – in the middle of clamped edges, where expressions for bending moments will take the form:

$$M_x\left(\frac{\gamma}{2};0\right) = \frac{2-\nu}{2}\alpha_1 - \frac{1}{16} - 2\sum_{k=1,3,...}^{\infty}(-1)^{k^*}\lambda_k A_{k\Sigma},$$
$$M_x\left(0;\pm\frac{1}{2}\right) = \nu\left(\frac{\alpha_1}{2} - \frac{\gamma^2}{16}\right) - 2\nu\sum_{s=1,3,...}^{\infty}(-1)^{s^*}\mu_s B_{s\Sigma}. \tag{19}$$

The coefficients A_{kn} (5) and B_{sn} (6) have similar estimations $A_{kn} = O(1/k^2)$, $B_{sn} = O(1/s^2)$, and the corresponding series that occurs in (19), starting from some number, converges not worse than alternating series $\sum_{m=1,3,...}^{\infty}(-1)^{m^*}/m$. Although such a series converges slowly, it is good for computations because pursuing the Leibniz theory, it is possible to estimate the inaccuracy of computing its sum (from the moment when the series terms start to decay).

Let us note that in angular points of the plate

$$M_x\left(\pm\frac{\gamma}{2};\pm\frac{1}{2}\right) = \frac{\alpha_1}{2} = \frac{h^2}{10(1-\nu)}, \tag{20}$$

while for the Kirchhoff plate these moments equal to zero.

The most slowly the series of shearing forces converges on the side $x = \pm\gamma/2$:

$$Q_x\left(\frac{\gamma}{2};y\right) = \frac{\gamma}{4} - 2\sum_{k=1,3,...}^{\infty}(-1)^{k^*}\left[\lambda_k^2\left(\tanh\tilde{\lambda}_k - \frac{\lambda_k}{\beta_k}\tanh\tilde{\beta}_k\right)A_{k\Sigma} + \frac{\tanh\tilde{\beta}_k}{\lambda_k\beta_k}\right]\cos\lambda_k y$$
$$-2\sum_{s=1,3,...}^{\infty}\left[\mu_s^2\left(\frac{\cosh\mu_s y}{\cosh\tilde{\mu}_s} - \frac{\cosh\xi_s y}{\cosh\tilde{\xi}_s}\right)B_{s\Sigma} + \frac{1}{\gamma\mu_s^2}\frac{\cosh\xi_s y}{\cosh\tilde{\xi}_s}\right], \tag{21}$$

It is proved that the series for the shear forces converge no worse than numerical series $\sum_{m=1,3,...}^{\infty} 1/m^2$.

Similar conclusions are also valid for bending moments M_y, torsion moments M_{xy} and shearing forces Q_y.

Thus, a series of moments and shear forces are quite suitable for computer calculations.

Results and Discussion

Numerical results were obtained for square plates with relative thicknesses h = 0.05, 0.1, 0.2, 0.3 and Poisson's ratio ν = 0.3. Up to 150 terms were held in the series depending on the speed of convergence of a particular series. The process converged in a geometrical progression with the ratio ≤ 1/3 for all considered examples. The discrepancy coefficients were printed out in every iteration. The calculation stopped after ten iterations, when all discrepancies were nearly equal to zero; in the process, the overall coefficients $A_{k\Sigma}$ and $B_{s\Sigma}$ were calculated (due to linearity of the problem), using which the bendings, bending moments M_x, and shearing forces Q_x in different points of plates were obtained. Near the contour, computational points clustered in order to refine the influence of ends.

In Table 1 the first five coefficients $A_{k\Sigma}$ (= $B_{s\Sigma}$) are given, as well as their values with k = 299 (150 terms of the series) for different relative thicknesses of a square plate.

The table shows that the highest are the first coefficients; the second are lower in an absolute value approximately by two orders, then the coefficients decay, keeping the negative sign.

Tables 2–5 show values of relative bendings; Tables 6–9 show values of bending moments M_x; Tables 10–13 show values of shearing forces Q_x for square Reissner plates with the relative thicknesses of 0.05, 0.1, 0.2, 0.3.

Table 1. Values of coefficients $A_{k\Sigma}$ for the bending functions of a square plate (Reissner - CCCC, q = const)

h	k					
	1	3	5	7	9	299
0.05	1.774×10^{-2}	-1.063×10^{-4}	-3.966×10^{-5}	-1.173×10^{-5}	-3.971×10^{-6}	-1.013×10^{-9}
0.1	1.726×10^{-2}	-5.582×10^{-5}	-1.923×10^{-5}	-4.715×10^{-6}	-1.564×10^{-6}	-4.941×10^{-9}
0.2	1.543×10^{-2}	-5.191×10^{-5}	-3.400×10^{-5}	-2.000×10^{-5}	-1.356×10^{-5}	-2.229×10^{-8}
0.3	1.248×10^{-2}	-2.531×10^{-4}	-1.231×10^{-4}	-6.974×10^{-5}	-4.492×10^{-5}	-5.282×10^{-8}

Table 2. Values of bendings referred to value qb^4 / D $\times10^{-5}$ of a square plate h = 0.05 (Reissner - CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-132.70	-123.80	-98.33	-60.94	-21.70	-15.11	-9.40	-4.79	-1.56	0
0.1	-123.80	-115.50	-91.84	-57.01	-20.35	-14.19	-8.83	-4.51	-1.47	0
0.2	-98.33	-91.84	-73.23	-45.68	-16.43	-11.48	-7.17	-3.67	-1.21	0
0.3	-60.94	-57.01	-45.67	-28.70	-10.42	-7.30	-4.57	-2.35	-0.78	0
0.4	-21.70	-20.35	-16.43	-10.42	-3.77	-2.62	-1.62	-0.82	-0.26	0
0.42	-15.11	-14.19	-11.48	-7.30	-2.62	-1.82	-1.11	-0.55	-0.17	0
0.44	-9.40	-8.83	-7.17	-4.57	-1.62	-1.11	-0.67	-0.32	-0.09	0
0.46	-4.79	-4.51	-3.67	-2.35	-0.82	-0.55	-0.32	-0.14	-0.03	0
0.48	-1.56	-1.47	-1.21	-0.78	-0.26	-0.17	-0.09	-0.03	-0.001	0
0.5	0	0	0	0	0	0	0	0	0	0

Table 3. Values of bendings referred to value qb^4 / D $\times10^{-5}$ of a square plate h = 0.1 (Reissner - CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-150.50	-140.90	-113.60	-72.98	-28.88	-21.07	-14.04	-8.01	-3.23	0
0.1	-140.90	-132.00	-106.50	-68.56	-27.21	-19.88	-13.26	-7.57	-3.06	0
0.2	-113.60	-106.50	-86.19	-55.75	-22.32	-16.35	-10.94	-6.28	-2.55	0
0.3	-72.98	-68.56	-55.75	-36.36	-14.73	-10.83	-7.28	-4.20	-1.72	0
0.4	-28.88	-27.21	-22.32	-14.74	-6.03	-4.44	-2.98	-1.72	-0.70	0
0.42	-21.08	-19.88	-16.35	-10.83	-4.44	-3.26	-2.19	-1.26	-0.51	0
0.44	-14.04	-13.26	-10.94	-7.28	-2.98	-2.19	-1.46	-0.83	-0.34	0
0.46	-8.01	-7.57	-6.28	-4.20	-1.72	-1.26	-0.83	-0.47	-0.19	0
0.48	-3.23	-3.06	-2.55	-1.72	-0.70	-0.51	-0.34	-0.19	-0.07	0
0.5	0	0	0	0	0	0	0	0	0	0

Table 4. Values of bendings referred to value $qb^4 / D \times 10^{-5}$ of a square plate $h = 0.2$ (Reissner - CCCC, $q = const$)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-217.20	-205.40	-171.00	-118.20	-55.60	-43.20	-31.22	-19.87	-9.38	0
0.1	-205.40	-194.30	-161.90	-112.10	-52.91	-41.14	-29.77	-18.97	-8.96	0
0.2	-171.00	-161.90	-135.40	-94.29	-44.94	-35.04	-25.43	-16.26	-7.72	0
0.3	-118.20	-112.10	-94.29	-66.38	-32.24	-25.27	-18.45	-11.88	-5.68	0
0.4	-55.60	-52.91	-44.94	-32.24	-16.22	-12.85	-9.50	-6.21	-3.02	0
0.42	-43.20	-41.14	-35.04	-25.27	-12.85	-10.22	-7.59	-4.99	-2.45	0
0.44	-31.22	-29.77	-25.43	-18.45	-9.50	-7.59	-5.68	-3.76	-1.87	0
0.46	-19.87	-18.97	-16.26	-11.88	-6.21	-4.99	-3.76	-2.53	-1.28	0
0.48	-9.38	-8.96	-0.77	-5.68	-3.02	-2.45	-1.87	-1.28	-0.67	0
0.5	0	0	0	0	0	0	0	0	0	0

Table 5. Values of bendings referred to value $qb^4 / D \times 10^{-5}$ of a square plate $h = 0.3$ (Reissner - CCCC, $q = const$)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-324.60	-309.10	-263.40	-190.70	-98.19	-78.38	-58.47	-38.64	-19.07	0
0.1	-309.10	-294.40	-251.20	-182.20	-94.11	-75.19	-56.14	-37.13	-18.34	0
0.2	-263.40	-251.20	-215.00	-157.00	-81.92	-65.62	-49.13	-32.59	-16.16	0
0.3	-190.70	-182.20	-157.00	-116.00	-61.80	-49.78	-37.50	-25.04	-12.51	0
0.4	-98.19	-94.11	-81.92	-61.80	-34.29	-27.96	-21.36	-14.50	-7.38	0
0.42	-78.38	-75.19	-65.62	-49.78	-27.96	-22.90	-17.59	-12.02	-6.16	0
0.44	-58.47	-56.14	-49.13	-37.50	-21.36	-17.59	-13.60	-9.38	-4.87	0
0.46	-38.63	-37.13	-32.59	-25.04	-14.50	-12.02	-9.38	-6.55	-3.46	0
0.48	-19.07	-18.34	-16.16	-12.51	-7.38	-6.16	-4.87	-3.46	-1.90	0
0.5	0	0	0	0	0	0	0	0	0	0

Table 6. Values of bending moments M_x referred to value $qb^2 \times 10^{-6}$ of a square plate h = 0.05 (Reissner -CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-23110	-21450	-15860	-4413	16220	21860	28090	34980	42520	50670
0.1	-21460	-19950	-14830	-4244	15040	20340	26220	32720	39870	47610
0.2	-16570	-15480	-11690	-3555	11790	16090	20890	26230	32160	38640
0.3	-8553	-8087	-6281	-1850	7450	10150	13200	16630	20470	24730
0.4	2189	1948	1463	1554	3772	4614	5593	6693	7905	9298
0.42	4603	4222	3282	2498	3311	3802	4400	5080	5821	6691
0.44	7088	6570	5183	3542	2984	3142	3389	3696	4033	4465
0.46	9643	8991	7167	4688	2800	2645	2574	2563	2589	2717
0.48	12290	11510	9251	5945	2745	2290	1927	1646	1445	1466
0.5	15120	14210	11520	7342	2713	1944	1270	720	355	357

Table 7. Values of bending moments M_x referred to value $qb^2 \times 10^{-6}$ of a square plate h = 0.1 (Reissner -CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-23630	-21920	-16210	-4677	15810	21340	27420	34060	41260	48940
0.1	-22040	-20470	-15200	-4482	14730	19930	25670	31960	38790	46090
0.2	-17300	-16130	-12130	-3732	11760	16030	20760	25970	31670	37810
0.3	-9528	-8962	-6858	-1970	7824	10610	13740	17230	21100	25370
0.4	952	822	704	1454	4523	5566	6788	8212	9870	11830
0.42	3343	3072	2500	2400	4104	4800	5650	6681	7942	9514
0.44	5840	5428	4398	3444	3787	4140	4622	5274	6145	7307
0.46	8469	7913	6422	4600	3567	3573	3687	3956	4498	5432
0.48	11280	10580	8611	5876	3417	3067	2806	2635	2729	3348
0.5	14360	13540	11020	7338	3196	2620	1806	1456	472	1429

Table 8. Values of bending moments M_x referred to value $qb^2 \times 10^{-6}$ of a square plate h = 0.2 (Reissner -CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-25290	-23450	-17440	-5692	14220	19410	25050	31120	37630	44580
0.1	-23830	-22120	-16500	-5461	13380	18320	23680	29480	35700	42330
0.2	-19470	-18120	-13630	-4635	11080	15260	19840	24810	30190	35910
0.3	-12250	-11440	-8666	-2823	8012	11010	14330	18000	22030	26380
0.4	-2112	-1969	-1306	598	5157	6620	8348	10380	12750	15520
0.42	321	318	516	1554	4679	5780	7138	8809	10840	13400
0.44	2925	2770	2487	2627	4242	4946	5887	7155	8833	10880
0.46	5732	5418	4630	3834	3867	4127	4573	5333	6581	8066
0.48	8787	8305	6982	5196	3587	3366	3240	3296	3800	5341
0.5	12220	11510	9532	6654	3494	2992	2037	878	322	5714

Table 9. Values of bending moments M_x referred to value $qb^2 \times 10^{-6}$ of a square plate h = 0.3 (Reissner -CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	-27650	-25710	-19460	-7598	11770	16720	22050	27780	33890	40490
0.1	-26300	-24480	-18580	-7348	11120	15860	20980	26490	32380	38680
0.2	-22230	-20750	-15890	-6489	9337	13460	17950	22800	28020	33500
0.3	-15340	-14370	-11160	-4692	6847	9979	13430	17230	21360	25720
0.4	-5251	-4951	-3860	-1340	4120	5845	7875	10250	12980	16170
0.42	-2763	-2613	-2000	-380	3566	4915	6572	8595	11040	14270
0.44	-82	-88	31	715	3028	3942	5148	6751	8832	11260
0.46	2818	2649	2252	1970	2552	2958	3594	4616	6205	7570
0.48	5965	5628	4696	3416	2215	2050	2007	2187	2871	4716
0.5	9552	8928	7253	4879	2245	1964	613	-1191	-1472	12860

Table 10. Values of shearing forces Q_x referred to value qb of a square plate h = 0.05 (Reissner -CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	0	0.052	0.112	0.191	0.297	0.322	0.349	0.376	0.404	0.429
0.1	0	0.047	0.103	0.177	0.279	0.303	0.329	0.356	0.384	0.408
0.2	0	0.033	0.076	0.136	0.224	0.246	0.269	0.294	0.320	0.343
0.3	0	0.011	0.030	0.068	0.132	0.149	0.166	0.186	0.206	0.227
0.4	0	-0.021	-0.034	-0.027	0.010	0.020	0.031	0.041	0.052	0.067
0.42	0	-0.028	-0.048	-0.049	-0.016	-0.007	0.003	0.013	0.022	0.036
0.44	0	-0.035	-0.062	-0.070	-0.042	-0.032	-0.022	-0.012	-0.003	0.010
0.46	0	-0.039	-0.072	-0.087	-0.063	-0.053	-0.042	-0.031	-0.021	-0.008
0.48	0	-0.035	-0.066	-0.083	-0.066	-0.057	-0.047	-0.036	-0.027	-0.013
0.5	0	0	0	0	0	0	0	0	0	0

Table 11. Values of shearing forces Q_x referred to value qb of a square plate h = 0.1 (Reissner - CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	0	0.051	0.112	0.189	0.292	0.316	0.340	0.366	0.390	0.412
0.1	0	0.047	0.103	0.176	0.274	0.298	0.322	0.346	0.370	0.392
0.2	0	0.034	0.077	0.137	0.222	0.243	0.265	0.287	0.309	0.331
0.3	0	0.013	0.034	0.072	0.136	0.152	0.169	0.187	0.207	0.226
0.4	0	-0.016	-0.023	-0.012	0.024	0.035	0.046	0.059	0.074	0.092
0.42	0	-0.021	-0.033	-0.028	0.002	0.012	0.022	0.035	0.049	0.066
0.44	0	-0.024	-0.041	-0.041	-0.016	-0.008	0.001	0.012	0.025	0.043
0.46	0	-0.024	-0.043	-0.047	-0.029	-0.022	-0.014	-0.005	0.006	0.023
0.48	0	-0.018	-0.033	-0.038	-0.027	-0.023	-0.018	-0.013	-0.005	0.009
0.5	0	0	0	0	0	0	0	0	0	0

Table 12. Values of shearing forces Q_x referred to value qb of a square plate h = 0.2 (Reissner - CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	0	0.051	0.110	0.184	0.277	0.298	0.319	0.340	0.362	0.382
0.1	0	0.047	0.103	0.172	0.262	0.282	0.303	0.324	0.345	0.366
0.2	0	0.036	0.080	0.139	0.218	0.237	0.256	0.276	0.296	0.316
0.3	0	0.018	0.044	0.084	0.146	0.162	0.179	0.197	0.215	0.235
0.4	0	-0.002	0.003	0.019	0.055	0.065	0.078	0.092	0.109	0.128
0.42	0	-0.004	-0.004	0.008	0.037	0.046	0.057	0.070	0.086	0.105
0.44	0	-0.006	-0.008	-0.001	0.021	0.028	0.037	0.048	0.063	0.081
0.46	0	-0.007	-0.010	-0.007	0.008	0.013	0.019	0.028	0.040	0.057
0.48	0	-0.005	-0.008	-0.007	0.000	0.003	0.006	0.010	0.018	0.032
0.5	0	0	0	0	0	0	0	0	0	0

Table 13. Values of shearing forces Q_x referred to value qb of a square plate h = 0.3 (Reissner - CCCC, q = const)

y	x									
	0	0.1	0.2	0.3	0.4	0.42	0.44	0.46	0.48	0.5
0	0	0.051	0.109	0.179	0.266	0.285	0.304	0.324	0.344	0.365
0.1	0	0.048	0.102	0.169	0.253	0.272	0.291	0.311	0.331	0.351
0.2	0	0.038	0.082	0.140	0.216	0.234	0.252	0.271	0.290	0.310
0.3	0	0.023	0.052	0.093	0.154	0.170	0.186	0.204	0.222	0.242
0.4	0	0.006	0.017	0.036	0.073	0.083	0.096	0.110	0.127	0.146
0.42	0	0.004	0.011	0.026	0.055	0.064	0.075	0.089	0.104	0.123
0.44	0	0.002	0.006	0.016	0.038	0.046	0.055	0.066	0.080	0.099
0.46	0	0.000	0.002	0.008	0.023	0.028	0.034	0.043	0.055	0.072
0.48	0	0.000	0.000	0.003	0.010	0.012	0.015	0.020	0.027	0.042
0.5	0	0	0	0	0	0	0	0	0	0

Hereinafter CCCC -plate is clamped on all four edges.

Figure 1 illustrates bending lines of square Reissner plates under a uniform load at the section $y = 0$. Curve 1(the dotted line) represents the Kirchhoff plate, the following numbers are given to the Reissner plates with relative thickness h = 0.05, 0.1, 0.2, 0.3. Figures 2, 3 illustrate curves of bending moments M_x for these plates at the clamped section $x = \pm \gamma/2$, and Figures 4, 5 – on the adjacent side $y = \pm 1/2$. The curves numeration is similar to Figure 1.

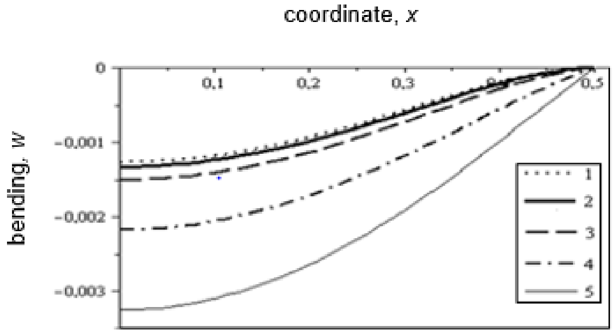

Figure 1. Lines of relative bendings of square plates (Reissner -CCCC, q = const) at the section y = 0

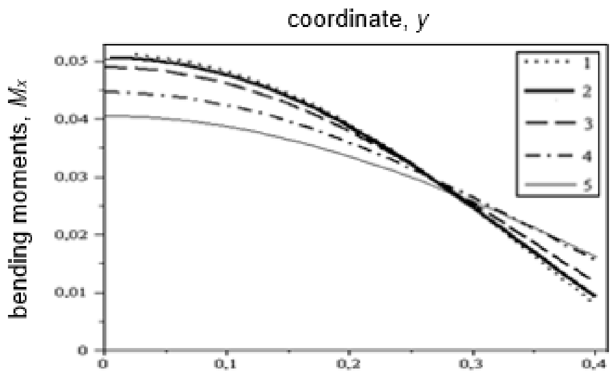

Figure 2. Curves of bending moments M_x of square plates (Reissner -CCCC, q = const) at the section $x = \pm \gamma/2$

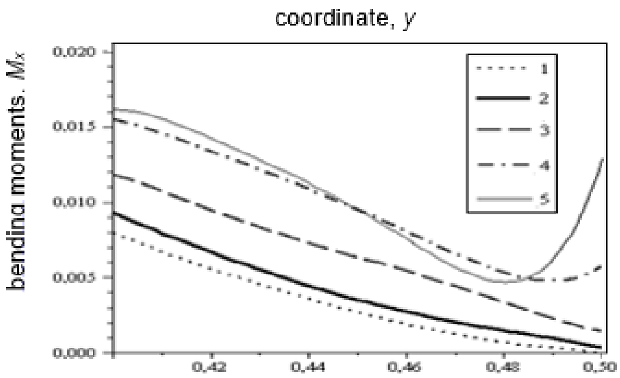

Figure 3. Magnified fragment of the curve of bending moments M_x of square plates (Reissner -CCCC, q = const) at the section $x = \pm \gamma/2$ near the plate's angle

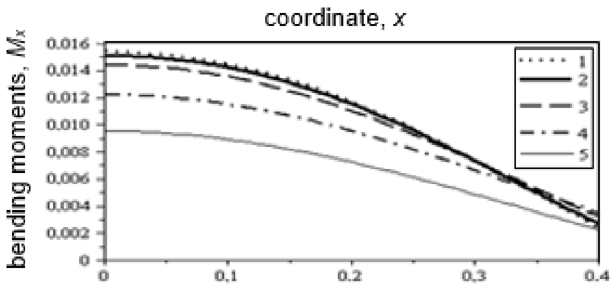

Figure 4. Curves of bending moments M_x of square plates (Reissner -CCCC, q = const) at the section $y = \pm 1/2$

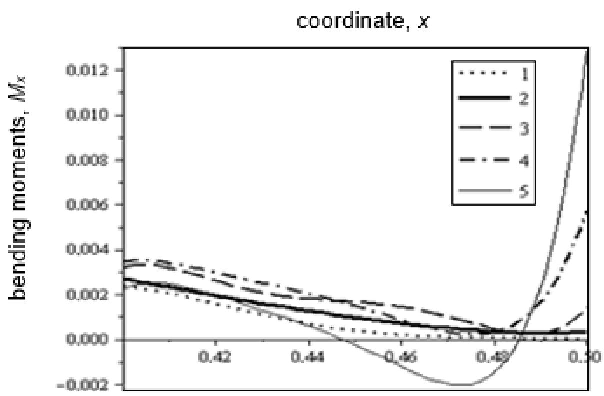

Figure 5. Magnified fragment of the curve of bending moments M_x of square plates (Reissner -CCCC, q = const) at the section $y = \pm 1/2$ near the plate's angle

The computations and graphs show that with small relative thicknesses h ≤ 1/20, the results for the Kirchhoff and Reissner plates are almost equal. With the increase in the relative thickness, relative bendings also increase. Absolute bendings, of course, decrease, because they are obtained by multiplying relative bendings with the expression $qb^4/D = 12(1\text{-}\nu^2)\ qb/(Eh^3)$. If the bending in the center for the square Kirchhoff plate equals to 0.00126 [4], for the Reissner plates with thickness h = 0.05, 0.1, 0.2, 0.3 it amounts to 0.001327, 0.001505, 0.002172, 0.003246 respectively.

Thus, the Kirchhoff plate can be considered as a limit behavior of the Reissner plate, when $h \to 0$.

Bending moments in the middle of clamped edges decrease when h increases, but they rise when closer to angles of the plate. In angular points the bending moments different from zero and increase in a proportion to the square of relative thickness (see (17)). This is the fundamental difference from the Kirchhoff plate.

In the center of the plate, bending moments slightly increase in an absolute value when h increases; shearing forces change moderately.

The Shirakawa work [21] presents calculated correlations w/w_{cl} of the plate's bendings within the shear theory to the bendings within the classical theory for central points of the median z_0/h = 0 and top z_0/h = 0.5 surfaces. For a square plate with relative thicknesses h/a = 0.1, 0.2, 0.3 these values amounted to ≈ 1.25, 1.85, 2.95 and 1.2, 1.7, 2.6 respectively. In this paper, average values of thickness amounted to 1.2, 1.7, 2.6, i.e. were equal to the corresponding values [21] on the plate's surface.

In [22] shows diagrams of shear forces on the contour of the uniformly loaded clamped square plates with the relative thickness of 0.001, 0.04, 0.1 and 0.3. These results practically coincide with those obtained in the present work.

In the works of X_u [23, 24], for a square plate with the relative thickness of 0.1 the bending in the center amounted to 0.001499 and the bending moment amounted to 0.0231. In the work [25] of Zienkiewicz *et al*. 1993 with the grid of 1024 elements these values amounted 0.00150442 and 0.023195 respectively, opposing to 0.0015050 and 0.023630 in our work. It indicates a good agreement of the results.

In the work [26] by Weiming and Guangsong, the bending in the center of a square plate with the relative thickness h/a = 0.3 amounted to 0.0028997 and the bending moment amounted to 0.023538, while in this work – to 0.0032460 and 0.027650 respectively. The values in the aforementioned work were obtained using FEM with the grid of 8×8 elements; however, they poorly correlate with our results.

In the work of Ayad et al. [27], the maximum bending for a square clamped plate with the relative thickness of 0.1 amounted to ≈ 0.001575 (according to the graph).

The work of Dhananjaya [28] provides numerical results for square plates with the relative thicknesses of 0.01 and 0.2, represented as graphs of bendings and moments in the center depending on the number of finite elements. The scale of the images does not allow making a proper comparison, although the proximity of the results is obvious.

In the article [30] for a square plate with the relative thickness of 0.1 the bending in the center amounted to 0.0013636 (method FIT), 0.0015040 (FEM, theory of Reissner - Mindlin), 0.0014918 (FEM, 3D solution). The last two values are in good agreement with the value 0.0015050 obtained in the present work.

Conclusions

1. In the present work the iterative process of superposition of hyperbolic-trigonometric series to solve the problem of bending rectangular Reissner plate clamped along the contour as a result of the action of a uniform load is constructed and its convergence to the exact solution of the problem is proved.

2. Increasing the number of members in the ranks and the number of iterations, we can obtain the numerical solution with high accuracy having used a simple algorithm.

3. The convergence of the series and their suitability for computations of bending moments and shear forces an investigated.

4. Numerous examples of calculating deflections, bending moments and shear forces for square plates with different relative thickness are given.

5. It is shown that in case of small relative thicknesses theories of Reissner and Kirchhoff produced the same results.

6. We also analyzed the differences of the above theories when changing the relative thickness.

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