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Nonlinear deformation and stability of geometrically exact elastic arches

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Ключевые слова: stability of structures, buckling, geometrically exact theory, dead load, round arch, stiffness, stationary point, Lagrange functional

Abstract. In the present paper a plane round double-hinged arch under the potential dead load is investigated. To describe the stress-strain state and the equilibrium stability the geometrically exact theory is used. According to this theory every point of the bar has two translational degrees of freedom and one rotational, which is independent from the previous two. To solve the problem no displacements are simplified and all the stiffnesses are used: axial, shear and bending. Exact nonlinear differential equations are found for the static problem. A variational definition for the problem is defined as finding a stationary point of Lagrange functional. The match of the differential and variational formulations is shown. Exact stability equations accounting non-linear geometric deformations in pre-buckling state were worked out. The problem of the equilibrium stability of the round arch under the potential dead load was solved using the obtained equations regarding all the bar's stiffnesses. The characteristic transcendental equation and its asymptotic solution as simple formulas, suitable for practical application, were worked out. The comparison of described solution which regards all the bar's stiffnesses and classical solution, based on bending stiffness, was made.

1. Introduction

Arches are one of the most widespread structural systems. On the one hand, this is due to their architectural expression; on the other hand, they are efficient at mechanics due to their curvature which can neglect the effect of the bending moment. As a result, the arch can be rather flexible as the size of the cross section can be relatively small. That is why the problem of arch equilibrium stability is one of the main problems for engineers to consider.

Historically, the most popular problem in the theory of stability of arches is the problem of stability of round arch under the radial pressure. This implication is typical for different underground structures – tunnels, pipelines and hull ribs of the submarines. The solution for this problem for the semi-ring was derived according to the fact that the loads, despite the bending of the axis of the bar, maintain the line of action. Besides, lines of action don't move in case of buckling [1–4]:

$$q_T = 3\frac{EI}{R^3}. \tag{1}$$

The solution for the problem of the stability of the plane arch under the radial pressure, when the load maintains the line of action, but the application points move with the axis of the arch were worked out by N.V. Kornoukhov [5] and A.N. Dinnik [6]. In case of the semi-ring the critical force is:

$$q_K = 3.27\frac{EI}{R^3}. \tag{2}$$

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In practice arches usually suffer different loads. A lot of problems were solved about the non-linear stability and the post-buckling deformation of round arches under the single force in the center [7–9]; under vertical or horizontal pressure, distributed on whole length [10–14] or located only on the part of the arch [15, 16]. Some other problems and their solutions can be found in the papers of V.N. Paimushin [17, 18], I.A. Karnovsky [19, 20], V.V. Galishnikova [21].

The critical force value is influenced not only by external loads but by the other parameters: flexibility of the supports, material properties and shape of the axis of the arch. The effect of the horizontal and vertical support stiffness on the stability of the round arch and frames was analyzed in [22, 23]. The effect of the physical properties, inconstant through the cross-section was analyzed with the help of functionally graded materials in [9, 14, 24, 25]. The problems of linear stability of plane parabolic arches can be found in [12, 13, 26]. Experimental researches of pre-buckling deformation, ultimate equilibrium and the failure behavior in case of buckling are described in [27–30].

Despite the big amount of analytical study, all the mentioned researches consider only the bending stiffness of the bar, and for this reason should be considered as approximate. There are no problem formulations and their solutions about the stability of arches, considering both axial and shear stiffness.

Variational method as the principle of virtual displacements is the most popular method to research the problems of stability [16–18, 31]. However, the variational definition for the problem as finding a stationary point of some functional was used only for straight bars [32], not for the arches. Note, that exact stability equations can be obtained from the second variation of the functional [31].

Thus, the purpose of this paper is to solve the problem of stability of plane double-hinged arch under the potential dead load regarding all the stiffnesses of the bar: axial, shear and bending by variational approach.

The aims of this paper are:

1. to work out a variational definition for the problem of deformation of the geometrical non-linear plane elastic round arch regarding axial, shear and bending stiffnesses as finding the stationary point of Lagrange functional;

2. to work out the stability equations as the result of the second variation of the Lagrange functional;

3. solving the problem of the stability of the arch with the help of the obtained equations under the potential dead load regarding all the stiffnesses of the bar: axial, shear and bending.

4. comparing the obtained solution with the Kornoukhov-Dinnik solution (2), where only bending stiffness was considered.

2. Methods

This paper is based on geometrically exact bar theory [32–37], whereby each point of the plane bar has two translational degrees of freedom – displacements u, w and one rotational – angle φ, which is independent form the previous ones.

Consider a plane round double-hinged arch with radius R under the potential dead load: uniformly applied forces and moments. Each point of the arch can be described with the local trihedron (t, n, k): tangent basis vector t is directed towards the increasing θ, normal basic vector n is away from the center of curvature C. The direction of the basic binormal vector k can be found using the vectoral product $t \times n = k$. All the unknowns, describing the stress-strain state of the bar can be found via angular coordinate θ, $0 \le \theta \le \Theta$, where Θ is the central angle (Figure 1).

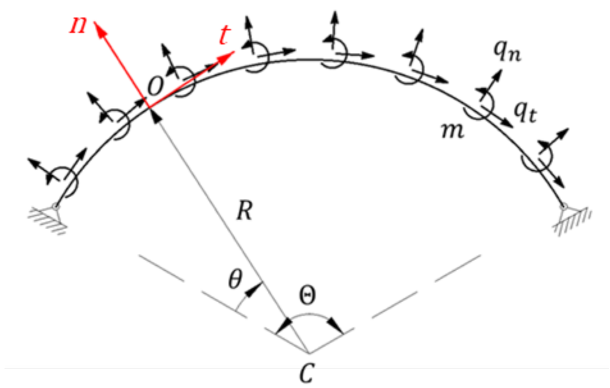


Figure 1. Arch state in basic condition (condition before deformation).

Definition of the problem of geometrically non-linear deformation of the round arch consists of three groups of differential equations: static (equilibrium equations), geometrical and physical. The equations for the static problem in the vector form were derived in [32–33]. The scalar form for the vector equations in the curvilinear coordinates will be derived below.

Equilibrium equations for the problem of the plane non-linear deformation of the arch are:

$$\begin{cases} \left(N\cos\varphi - Q\sin\varphi\right)' + \left(N\sin\varphi + Q\cos\varphi\right) + Rq_t = 0; \\ \left(N\sin\varphi + Q\cos\varphi\right)' - \left(N\cos\varphi - Q\sin\varphi\right) + Rq_n = 0; \\ M' + \left(u' + w + \mathrm{R}\right)\left(N\sin\varphi + Q\cos\varphi\right) - \left(w' - u\right)\left(N\cos\varphi - Q\sin\varphi\right) + Rm = 0, \end{cases} \tag{3}$$

where N is axial force;

Q is shear force;

M is bending moment;

q_t, q_n are projection of the distributed loads on the tangential and normal (radial) direction;

m is distributed moment load;

u, w are tangential and normal displacements;

φ is rotating angle. Henceforward differentiation is made according to angle $\theta\left(\ \right)' \equiv d\left(\ \right)/d\theta$.

Geometrical equations for the plane problem are:

$$\begin{cases} \varepsilon = \dfrac{1}{R}\left(u' + w + R\right)\cos\varphi + \dfrac{1}{R}\left(w' - u\right)\sin\varphi - 1, \\ \gamma = -\dfrac{1}{R}\left(u' + w + R\right)\sin\varphi + \dfrac{1}{R}\left(w' - u\right)\cos\varphi, \\ \psi = \dfrac{1}{R}\varphi', \end{cases} \tag{4}$$

where ε, γ, ψ are axial, shear and bending deformations.

Physical equations for the linear elastic material are

$$N = k_1\varepsilon; \quad Q = k_2\gamma; \quad M = k_3\psi, \tag{5}$$

where $k_1 = EA$ is axial stiffness; $k_2 = GAk$ is shear stiffness; $k_3 = EI$ is bending stiffness; E is Young's modulus; A is cross-section area of the bar; G is shear modulus; k is cross-section form coefficient; I is moment of inertia.

The equations (3)–(5) are the exact equations of geometrical non-linear round arch, taking into account all stiffnesses of round arch. To get the closed system on each end of the arch three boundary conditions are needed. For the double-hinged arch they are as follows:

$$\begin{aligned} \theta = 0: \ & u(0) = 0,\ w(0) = 0, M(0) = 0; \\ \theta = \Theta: \ & u(\Theta) = 0,\ w(\Theta) = 0, M(\Theta) = 0. \end{aligned} \tag{6}$$

3. Results and Discussion

3.1. Variational formulation of non-linear static problem

Lagrange functional can be written as follows:

$$\mathcal{L}\left(u, w, \varphi\right) = R\int_0^{\Theta}\left[\frac{1}{2}\left(k_1\varepsilon^2 + k_2\gamma^2 + k_3\psi^2\right) - q_t u - q_n w - m\varphi\right]d\theta. \tag{7}$$

It can be proved, that variational definition for the problem, defined as finding a stationary point of functional $\mathcal{L}$

$$\mathcal{L} \to \mathrm{STAT} \tag{8}$$

in case the fulfillment of the essential boundary conditions

$$u(0) = w(0) = u(\Theta) = w(\Theta) = 0 \tag{9}$$

is equivalent to the initial problem (3)–(6). The first variation of the functional (7) is:

$$\begin{gathered}\delta\mathcal{L}(u,w,\varphi) = -\int_0^{\Theta}\Big\{u_v\Big(\big(N\cos\varphi - Q\sin\varphi\big)' + \big(Q\cos\varphi + N\sin\varphi\big) + Rq_t\Big) + \\ + w_v\Big(\big(N\sin\varphi + Q\cos\varphi\big)' + \big(Q\sin\varphi - N\cos\varphi\big) + Rq_n\Big) + \\ + \varphi_v\Big(M' + \big(u' + w + R\big)\big(Q\cos\varphi + N\sin\varphi\big) + \big(w' - u\big)\big(Q\sin\varphi - N\cos\varphi\big) + Rm\Big)\Big\}d\theta + \\ + \Big[u_v\big(N\cos\varphi - Q\sin\varphi\big) + w_v\big(N\sin\varphi + Q\cos\varphi\big) + \varphi_v M\Big]_0^{\Theta},\end{gathered} \tag{10}$$

where the variations are labeled as follows:

$$u_v = \delta u,\;\; w_v = \delta w,\;\; \varphi_v = \;\delta\varphi. \tag{11}$$

The solution of the variational problem are the functions u, w, φ, satisfying the essential boundary conditions (6), that $\delta\mathcal{L} = 0$ for any variations u_v, w_v, φ_v. Initial nonlinear equilibrium equations (3) are the Euler equations of the variational problem (8)–(9), according to (10).

As it can be seen from (9), variations of the displacements on the boundaries equal to zero:

$$u_v(0) = w_v(0) = u_v(\Theta) = w_v(\Theta) = 0. \tag{12}$$

Considering (12), the terms outside the integral (10) are:

$$M(\Theta)\varphi_v(\Theta) - M(0)\varphi_v(0). \tag{13}$$

From the stationary condition of the functional for any $\varphi_v(\Theta)$ and $\varphi_v(0)$ are, it can be seen, that their factors should equal to zero. So, the natural boundary conditions are:

$$M(\Theta) = 0,\;\; M(0) = 0. \tag{14}$$

Thus, the equivalence of differential (3)–(6) and variation (8)–(9) formulations is proved.

3.2. Stability problem formulation

The second variation of the functional (7) is:

$$\begin{gathered}\delta^2\mathcal{L}(u,w,\varphi) = \frac{1}{R}\int_0^{\Theta}\Big\{k_1\Big[\Big(\big(u_v' + w_v\big)\cos\varphi - \varphi_v\big(u' + w + R\big)\sin\varphi + \big(w_v' - u_v\big)\sin\varphi + \\ + \varphi_v\big(w' - u\big)\cos\varphi\Big)^2 + \Big(\big(u' + w + R\big)\cos\varphi + \big(w' - u\big)\sin\varphi - R\Big)\Big(-2\varphi_v\big(u_v' + w_v\big)\sin\varphi - \\ - \varphi_v^2\big(u' + w + \mathrm{R}\big)\cos\varphi + 2\varphi_v\big(w_v' - u_v\big)\cos\varphi - \varphi_v^2\big(w' - u\big)\sin\varphi\Big) + \\ + k_2\Big[\Big(-\big(u_v' + w_v\big)\sin\varphi - \varphi_v\big(u' + w + \mathrm{R}\big)\cos\varphi + \big(w_v' - u_v\big)\cos\varphi - \varphi_v\big(w' - u\big)\sin\varphi\Big)^2 + \\ + \Big(-\big(u' + w + \mathrm{R}\big)\sin\varphi + \big(w' - u\big)\cos\varphi\Big)\Big(-2\varphi_v\big(u_v' + w_v\big)\cos\varphi + \varphi_v^2\big(u' + w + R\big)\sin\varphi - \\ - 2\varphi_v\big(w_v' - u_v\big)\sin\varphi - \varphi_v^2\big(w' - u\big)\cos\varphi\Big)\Big] + k_3\varphi_v'^2\Big\}d\theta.\end{gathered} \tag{15}$$

Let us label $\frac{1}{2}\delta^2\mathcal{L} \equiv F_{\mathrm{ST}}^*\big(u_v,\, w_v,\, \varphi_v\big)$, where $F_{\mathrm{ST}}^*\big(u_v,\, w_v,\, \varphi_v\big)$ is static stability functional.

To derive Euler equations for the variational problem of finding a stationary point of functional $F_{\mathrm{ST}}^{*} \to \mathrm{STAT}$ in carrying out essential boundary conditions (12) the first variation of the stability functional should be computed:

$$\begin{gathered}\delta F_{\mathrm{ST}}^{*}\left(u_v, w_v, \varphi_v\right)=\int_0^{\Theta}\Big(\delta u_v\Big(-\left(N_v \cos\varphi-\varphi_v N \sin\varphi-Q_v \sin\varphi-\varphi_v Q \cos\varphi\right)'- \\ -N_v \sin\varphi-\varphi_v N \cos\varphi-Q_v \cos\varphi+\varphi_v Q \sin\varphi\Big)+\delta w_v\Big(-\left(N_v \sin\varphi+\varphi_v N \cos\varphi+\right. \\ \left.+Q_v \cos\varphi-\varphi_v \mathrm{Q} \sin\varphi\right)'+N_v \cos\varphi-\varphi_v N \sin\varphi-Q_v \sin\varphi-\varphi_v Q \cos\varphi\Big)+ \\ +\delta\varphi_v\Big(-M_v'-\left(u'+w+R\right) N_v \sin\varphi+\left(w'-u\right) N_v \cos\varphi-\left(u_v'+w_v\right) N \sin\varphi- \\ -\varphi_v\left(u'+w+R\right) N \cos\varphi+\left(w_v'-u_v\right) N \cos\varphi-\varphi_v\left(w'-u\right) N \sin\varphi- \\ -\left(u'+w+R\right) Q_v \cos\varphi-\left(w'-u\right) Q_v \sin\varphi-\left(u_v'+w_v\right) Q \cos\varphi+\varphi_v\left(u'+w+R\right) Q \sin\varphi- \\ -\left(w_v'-u_v\right) Q \sin\varphi-\varphi_v\left(w'-u\right) Q \cos\varphi\Big) d\theta+\Big[\delta u_v\left(N_v \cos\varphi-\varphi_v \mathrm{N} \sin\varphi-Q_v \sin\varphi-\right. \\ \left.-\varphi_v Q \cos\varphi\right)+\delta w_v\left(N_v \sin\varphi+\varphi_v N \cos\varphi+Q_v \cos\varphi-\varphi_v Q \sin\varphi\right)+\delta\varphi_v M_v\Big]_0^{\Theta},\end{gathered} \tag{16}$$

where the following labels are introduced:

$$\begin{gathered}N_v=k_1\varepsilon_v, \quad Q_v=k_2\gamma_v, \quad M_v=k_3\psi_v; \\ \varepsilon_v=\frac{1}{R}\left(\left(w_v'-u_v\right)\cos\varphi-\varphi_v\left(u'+w+R\right)\sin\varphi+\left(w_v'-u_v\right)\sin\varphi+\varphi_v\left(w'-u\right)\cos\varphi\right), \\ \gamma_v=\frac{1}{R}\left(-\left(u_v'+w_v\right)\sin\varphi-\varphi_v\left(u'+w+R\right)\cos\varphi+\left(w_v'-u_v\right)\cos\varphi-\varphi_v\left(w'-u\right)\sin\varphi\right), \\ \psi_v=\frac{1}{R}\varphi_v'.\end{gathered} \tag{17}$$

Euler equations resulting from the condition $\delta F_{\mathrm{ST}}^{*}=0$ are the further equations:

$$\begin{gathered}\varphi_v\left(N\cos\varphi-Q\sin\varphi\right)+\left(N_v\sin\varphi+Q_v\cos\varphi\right)-\Big(\varphi_v\left(N\sin\varphi+Q\cos\varphi\right)- \\ -\left(N_v\cos\varphi-Q_v\sin\varphi\right)\Big)'=0; \\ \varphi_v\left(N\sin\varphi+Q\cos\varphi\right)-\left(N_v\cos\varphi-Q_v\sin\varphi\right)+\Big(\varphi_v\left(N\cos\varphi-Q\sin\varphi\right)+ \\ +\left(N_v\sin\varphi+Q_v\cos\varphi\right)\Big)'=0; \\ M_v'+\left(u_v'+w_v\right)\left(N\sin\varphi+Q\cos\varphi\right)-\left(w_v'-u_v\right)\left(N\cos\varphi-Q\sin\varphi\right)+ \\ +\left(u'+w+R\right)\Big(\varphi_v\left(N\cos\varphi-Q\sin\varphi\right)+\left(N_v\sin\varphi+Q_v\cos\varphi\right)\Big)+ \\ +\left(w'-u\right)\Big(\varphi_v\left(N\sin\varphi+Q\cos\varphi\right)-\left(N_v\cos\varphi-Q_v\sin\varphi\right)\Big)=0.\end{gathered} \tag{18}$$

Expression (18) is the system of equations involving functions u_v, w_v, φ_v. Functions u, w, φ, N, Q, M are known and are the solution of nonlinear static problem, the stability of which is being researched.

Equations (18) are the exact stability equations of the elastic round arch under the potential dead load, taking into account all stiffnesses of round arch. To get these equations no hypothesis was made about the value of displacements and type of stress-strain state of the bar.

The natural boundary conditions can be derived from the terms outside the integral (16) regarding the essential boundary conditions (12):

$$M_v(\Theta)=0, \; M_v(0)=0. \tag{19}$$

Thus, the formulation of the stability problem consists of stability equations (18) and six boundary conditions (12), (19). The exact solution for the problem of stability can be derived if the exact solution of the nonlinear static problem (3)–(6) is put in system (18). There are no exact analytical solutions for the nonlinear static problems of the curvilinear bars. That is why the solution of the stability problem is derived in a linearized formulation [31]. This means, that solution of the original static problem in linear formulation is put in system (18).

3.3. Solving the problem of arch equilibrium stability

Consider the problem of half ring equilibrium (an arch with central angle $\Theta = \pi$) of radius R, under the dead radial pressure (Figure 2).

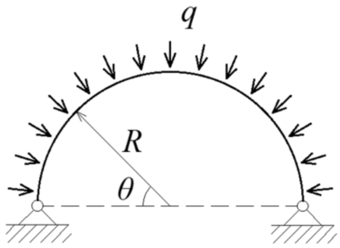


Figure 2. Structural model of the half-ring under the radial pressure.

Statically acceptable solution in linear formulation [19] can be written as follows:

$$N = -qR, \;\; Q = 0, \;\; M = 0. \tag{20}$$

As the physical equations (5) and functionals (7) and (15) are valid only for the elastic material, then the distributed pressure shouldn't outnumber the following values:

$$q \le \frac{\sigma_y A}{R}, \tag{21}$$

where σ_y is elastic limit.

Substitution of the static solution (20) into stability equations (18) leads to the following system:

$$\begin{cases} \left(N\varphi_v + Q_v\right) + \left(N_v\right)' = 0; \\ -N_v + \left(N\varphi_v + Q_v\right)' = 0; \\ M_v' - N\left(w_v' - u_v\right) + R\left(N\varphi_v + Q_v\right) = 0, \end{cases} \tag{22}$$

where, according to the (17), are made the following labels:

$$N_v = \frac{k_1}{R}\left(u_v' + w_v\right), \;\; Q_v = \frac{k_2}{R}\left(-R\varphi_v + \left(w_v' - u_v\right)\right), \;\; M_v = \frac{k_3}{R}\varphi_v'.$$

The solution of the system (22) are the following functions:

$$\begin{gathered} u_v = \frac{1}{2}GC_1\cos\theta + \frac{1}{2}GC_1\theta\sin\theta + \frac{1}{2}GC_2\theta\cos\theta + HC_3\cos\sqrt{A}\theta + HC_4\sin\sqrt{A}\theta + \\ + C_5\cos\theta + C_6\sin\theta; \\ w_v = -\frac{1}{2}GC_1\theta\cos\theta + \frac{R}{k_1}C_1\sin\theta + \frac{R}{k_1}C_2\cos\theta + \frac{1}{2}GC_2\theta\sin\theta - \frac{1}{2}GC_2\cos\theta + \\ + \sqrt{A}HC_3\sin\sqrt{A}\theta - \sqrt{A}HC_4\cos\sqrt{A}\theta + C_5\sin\theta - C_6\cos\theta; \\ \varphi_v = \frac{R}{k_3}HC_1\cos\theta - \frac{R}{k_3}HC_2\sin\theta + C_3\cos\sqrt{A}\theta + C_4\sin\sqrt{A}\theta, \end{gathered} \tag{23}$$

where C_i is integration constants and the following labels are used:

$$A = \left(1 + \frac{qR}{k_2}\right)\frac{qR^3}{k_3}, \;\; H = \frac{\left(1 + \frac{qR}{k_2}\right)R}{\left(1 + \frac{qR}{k_2}\right)\frac{qR^3}{k_3} - 1}, \;\; G = \left(\frac{1}{k_1} + \frac{1}{k_2} - H\left(1 + \frac{qR}{k_2}\right)\frac{R}{k_3}\right). \tag{24}$$

Boundary conditions (12), (19) lead to the system of linear equations involving integration constants. After the equivalent transformations it can be derived:

$$\begin{cases} GC_1 = 0; \\ \sqrt{A}H \sin\left(\sqrt{A}\pi\right)C_3 = 0; \\ \begin{cases} -H\dfrac{R}{k_3}C_2 + \sqrt{A}\sin\left(\sqrt{A}\pi\right)C_4 = 0; \\ -\dfrac{1}{2}G\pi\sqrt{A}\sin\left(\sqrt{A}\pi\right)C_2 + 2\sqrt{A}H\sin\left(\sqrt{A}\pi\right)\left(\cos\left(\sqrt{A}\pi\right)+1\right)C_4 = 0. \end{cases} \end{cases} \tag{25}$$

According to the numerical test, a critical load, calculated from the first two equations of the set (25), outnumbers the critical load from the last system of two equations. Hence the transcendental equation relative to the minimal value of the critical load can be derived:

$$\sqrt{A}H^2\frac{R}{k_3}\left(\cos\left(\sqrt{A}\pi\right)+1\right) - \frac{\pi}{4}AG\sin\left(\sqrt{A}\pi\right) = 0, \tag{26}$$

Using the labels from (24) and trigonometric transformations, (26) can be rewritten:

$$\mathrm{tg}\left(\frac{\pi}{2}\sqrt{\left(1+\frac{qR}{k_2}\right)\frac{qR^3}{k_3}}\right) = \\ = \frac{4}{\pi}\frac{\left(1+\dfrac{qR}{k_2}\right)^2\dfrac{R^2}{k_3}}{\sqrt{\left(1+\dfrac{qR}{k_2}\right)\dfrac{qR^3}{k_3}}\left(\dfrac{1}{k_1}+\dfrac{1}{k_2}+\dfrac{\left(1+\dfrac{qR}{k_2}\right)^2\dfrac{R^2}{k_3}}{1-\left(1+\dfrac{qR}{k_2}\right)\dfrac{qR^3}{k_3}}\right)\left(1-\left(1+\dfrac{qR}{k_2}\right)\dfrac{qR^3}{k_3}\right)^2}. \tag{27}$$

Transcendental equation (27) makes it possible to determine the value for the critical load q_{cr} for the circle arch under the dead pressure considering all the stiffnesses of the bar. Equation (27) can be solved numerically for any arch with any cross-section, though you can't get a general correlation between loading and bar stiffness. Moreover, it is very uncomfortable to use such an equation in practice. To get the simple form for the critical load an asymptotic solution for the equation (27) will be done.

Consider the labels for the non-dimensional values:

$$b = \sqrt{\frac{qR^3}{k_3}}, \quad \xi_1 = \frac{k_3}{k_2R^2}, \quad \xi_2 = \frac{k_3}{k_1R^2}. \tag{28}$$

For the cross-sections, widely used in practice, shear and axial stiffnesses are much more than bending stiffness, that is why ξ_1, ξ_2 can be considered as small parameters: ξ_1, $\xi_2 \ll 1$.

According to (28), (27) can be written as follows:

$$\mathrm{tg}\left(\frac{\pi}{2}b\sqrt{\left(1+b^2\xi_1\right)}\right) = \frac{4}{\pi}\frac{\left(1+b^2\xi_1\right)^2}{b\sqrt{\left(1+\mathrm{b}^2\xi_1\right)}\left(\xi_2+\xi_1+\dfrac{\left(1+b^2\xi_1\right)^2}{1-\left(1+b^2\xi_1\right)b^2}\right)\left(1-\left(1+b^2\xi_1\right)b^2\right)^2}. \tag{29}$$

The parameter ξ_2 can be considered to be dependent on ξ_1 through the k coefficient, where k is a certain constant value.

$$\xi_2 = \xi_1 k. \tag{30}$$

According to (30), (29) can be written as follows:

$$\mathrm{tg}\left(\frac{\pi}{2}b\sqrt{\left(1+b^2\xi_1\right)}\right)=\frac{4}{\pi}\frac{\left(1+b^2\xi_1\right)^2}{b\sqrt{\left(1+b^2\xi_1\right)}\left(\xi_1\left(k+1\right)+\frac{\left(1+b^2\xi_1\right)^2}{1-\left(1+b^2\xi_1\right)b^2}\right)\left(1-\left(1+b^2\xi_1\right)b^2\right)^2}. \tag{31}$$

The unknown b can be estimated as an asymptotic series with a small parameter ξ_1:

$$b=b_0+b_1\xi_1+b_2\xi_1^2+\ldots, \tag{32}$$

where $b_0=\sqrt{q_K R^3/k_3}$ relatives the value of the critical force q_K neglecting axial and shear yielding, i.e. when $k_1\to\infty$, $k_2\to\infty$, which is equivalent to $\xi_2\to 0,\ \xi_1\to 0$.

In fact, when $\xi_1\to 0,\ \xi_2\to 0$, (29) transforms into

$$\mathrm{tg}\left(\frac{\pi}{2}b_0\right)=\frac{4}{\pi}\frac{1}{b_0\left(1-b_0^2\right)}. \tag{33}$$

The minimal positive root of the transcendental equation (33) is $b_0\approx 1.80866$. Using the first expression in (28) a Kornoukhov-Dinnik solution (2) can be derived:

$$q_{cr}=1.80866^2\frac{k_3}{R^3}\approx 3.27\frac{k_3}{R^3}. \tag{34}$$

An approximate formula for the critical force can be derived by substitution of the asymptotic series with a small parameter (32) into the equation (31), expanding both parts of the equation into a series, setting coefficients of the same powers equal and considering only terms of the first order of smallness:

$$q_{cr}=q_K\left(1-0.223\frac{q_K R}{k_1}-1.223\frac{q_K R}{k_2}\right). \tag{35}$$

The solution for the model neglecting axial stiffness (Timoshenko beam theory) can be obtained from the (35) by letting $k_1\to\infty$:

$$q_{cr}=q_K\left(1-1.223\frac{q_K R}{k_2}\right). \tag{36}$$

The solution for the model regarding only bending stiffness can be obtained from the (35) by letting $k_1\to\infty$, $k_2\to\infty$.. In that case the Kornoukhov-Dinnik solution (2) is derived.

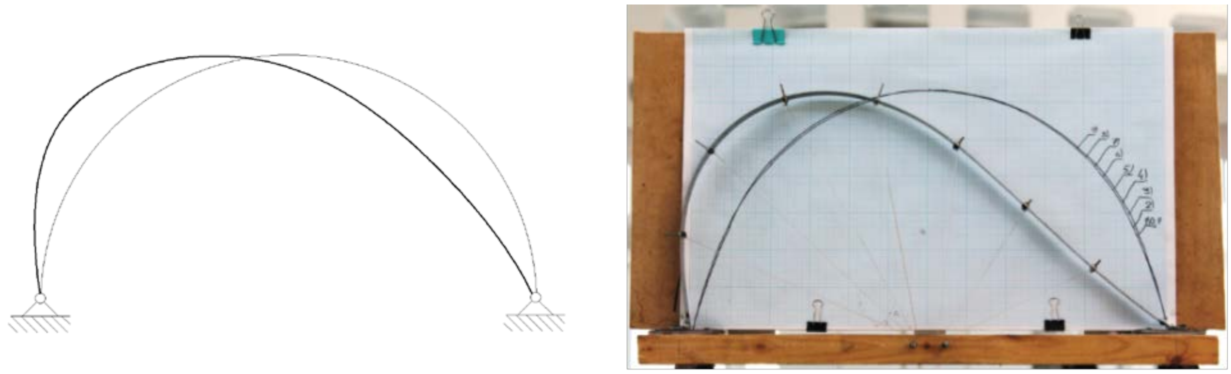
Using the value for the critical load from (27) or (35) the mode of buckling can be found:

$$\begin{cases} u_в=\frac{1}{2}G_{cr}C_2\theta\cos\theta+H_{cr}C_4\sin\sqrt{A_{cr}}\theta; \\ w_в=\frac{1}{2}G_{cr}C_2\theta\sin\theta-\frac{1}{2}G_{cr}C_2\cos\theta+\frac{R}{k_1}C_2\cos\theta-\sqrt{A_{cr}}H_{cr}C_4\cos\sqrt{A_{cr}}\theta; \\ \varphi_в=-H_{cr}\frac{R}{k_3}C_2\sin\theta+C_4\sin\sqrt{A_{cr}}\theta, \end{cases} \tag{37}$$

where A_{cr}, H_{cr}, G_{cr} are labels from (24), where the critical force value was substituted.

It is easy to prove, that each of the terms in (37) has no left-to-right symmetry, so the buckling mode is antisymmetry. This goes with the results of the experiments [38], shown in Figure 3.

The comparison of the numerical exact solution (27), asymptotic solutions (35), (36) and a Kornoukhov-Dinnik solution (2) can be made. Consider an arch with radius R = 12 m, cross-section is a thin-walled tube with a thickness of 10 mm and a variety of diameters: from 355.6 mm till 1420 mm. Geometrical and stiffness values can be found in Table 1.



a b

Figure 3. Buckling mode of the arch under the dead radial load: a – analytical model; b – experiment.

Table 1. Geometrical and stiffness values of the cross-section.

Cross-section	Outer diameter, [mm]	Cross-section area A, [cm²]	Moment of inertia I, [m⁴]	Axial stiffness $k_1 = EA$, [N]	Shear stiffness $k_2 = GAk$, [N]	Bending stiffness $k_3 = EI$, [N·m²]
355.6×10	355.60	108.57	1.62E-04	2.1714E+09	4.1758E+08	1.5698E+08
377×10	377.00	115.29	1.94E-04	2.3058E+09	4.4342E+08	1.9832E+08
406.4×10	406.40	124.53	2.45E-04	2.4906E+09	4.7896E+08	2.6780E+08
426×10	426.00	130.69	2.83E-04	2.6138E+09	5.0265E+08	3.2332E+08
478×10	478.00	147.02	4.03E-04	2.9404E+09	5.6546E+08	5.1252E+08
530×10	530.00	163.36	5.52E-04	3.2672E+09	6.2831E+08	7.7465E+08
630×10	630.00	194.77	9.36E-04	3.8954E+09	7.4912E+08	1.5465E+09
720×10	720.00	223.05	1.41E-03	4.4610E+09	8.5788E+08	2.6383E+09
820×10	820.00	254.46	2.09E-03	5.0892E+09	9.7869E+08	4.4387E+09
920×10	920.00	285.88	2.96E-03	5.7176E+09	1.0995E+09	7.0332E+09
1020×10	1020.00	317.29	4.05E-03	6.3458E+09	1.2203E+09	1.0627E+10
1120×10	1120.00	348.71	5.37E-03	6.9742E+09	1.3412E+09	1.5448E+10
1220×10	1220.00	380.12	6.96E-03	7.6024E+09	1.4620E+09	2.1749E+10
1420×10	1420.00	442.95	1.10E-02	8.8590E+09	1.7037E+09	3.9917E+10

Consider the non-dimensional values of critical forces (27), (35), (36) by dividing them by q_K and construct a plot (Figure 4). An equation (27) is solved using the iterative Newton's method with the help of Wolfram Mathematica software.

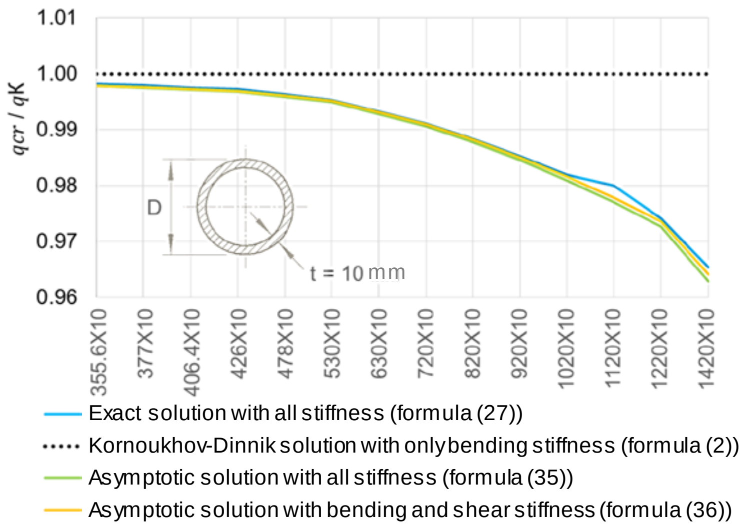


Figure 4. The comparison for the critical load values.

According to the comparison of the critical loads, the one, that considers all stiffnesses is tends to be less, than the one, that regards only bending stiffness. It is worthwile noticing, that inaccuracy between the Kornoukhov-Dinnik solution (2) and transcendental equation (27) increases as the size of the cross-section grows. Critical loads, estimated using the asymptotic formulas (35) and (36) are always smaller, than the exact values, so they improve the margin of safety. Moreover, inaccuracy between the asymptotic formulas and the solution (27) is less than 0.3 %.

The results can be used in the analytical defining of the stress-strain state of structures, where tensile-compression and shear stiffnesses make a significant contribution. Such structures include masonry and concrete arches [39–42], curvilinear elements of dams [43, 44] and long-span steel roofs [45, 46].

4. Conclusions

1. An analytical model of a geometrical non-linear deformation and stability of the plane elastic round arch taking into account all stiffnesses was worked out. This model contains:

1.1. exact non-linear equilibrium equations;

1.2. variational formulation for the problem is defined as finding stationary point of Lagrange functional;

1.3. static stability functional;

1.4. exact stability equations.

2. Based on the derived equations the problem of the equilibrium stability of the round arch under the dead radial load was solved. The characteristic transcendental equation and its asymptotic solution as a number of simple formulas, suitable for practical application, were worked out.

3. The comparison of described solution which regards all the bar's stiffnesses and classical Kornoukhov-Dinnik solution based on bending stiffness, was made. It was shown, that considering axial and shear stiffnesses leads to decreasing the values of the critical forces.

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Геометрически нелинейное деформирование и устойчивость упругих арок

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Keywords: устойчивость конструкций, потеря устойчивости, геометрически точная теория, «мертвая» нагрузка, круговая арка, жесткость, точка стационарности, функционал Лагранжа

Аннотация. В статье рассматривается плоская круговая двухшарнирная арка, нагруженная потенциальной «мертвой» нагрузкой. Для описания напряженно-деформированного состояния и устойчивости равновесия используется геометрически точная теория, в соответствии с которой каждая точка стержня имеет две трансляционные степени свободы и одну вращательную, не зависящую от трансляционных. Для получения решения не используются никакие упрощения о величинах перемещений и углов поворота, а также учитываются все жесткости стержня – продольная, сдвиговая и изгибная. Получены точные нелинейные дифференциальные уравнения статической задачи. Сформулирована вариационная постановка в виде задачи поиска точки стационарности функционала типа Лагранжа. Доказана эквивалентность дифференциальной и вариационной постановок. Получены точные уравнения устойчивости, учитывающие геометрически нелинейное деформирование в докритическом состоянии. На основе полученных уравнений решена задача устойчивости равновесия круговой арки при действии «мертвого» радиального давления с учетом всех жесткостей стержня. Получено характеристическое трансцендентное уравнение, а также асимптотическое решение этого уравнения в виде простых формул, пригодных для практического применения. Выполнено сравнение полученного решения, учитывающего все жесткости стержня, с классическим решением, учитывающим только изгибную жесткость.

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