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The functional of additional energy for stability analysis of spatial rod systems

Функционал дополнительной энергии для анализа устойчивости пространственных стержневых систем

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Key words: stability; finite element method in stresses; piecewise constant stress; an additional energy functional; the lower limit of solutions; critical force; reserve of stability

Ключевые слова: устойчивость; метод конечных элементов в напряжениях; кусочно-постоянные напряжения; функционал дополнительной энергии; нижняя граница решения; критические силы; запас устойчивости

Abstract. The problem solutions of stability of spatial rod systems by finite elements method in stresses were considered. The proposed method is based on a combination of functional additional energy and the principle of virtual displacements, used for the construction of the equilibrium equations. After discrediting of the subject field, solution of the problem is reduced to the search of the minimum of additional strain energy functional with constraints in the form of the system of linear algebraic equilibrium equations of the nodes. The equilibrium equations are included in the functional with the help of Lagrange multipliers, which are displacements of the nodes. Equations are derived for the static analysis based on approximations of internal forces (stresses) for the spatial rod systems. To solve the stability problems, in the functional of additional energy there are added additional energy the longitudinal deformations, arising due to the bending of rods. Form of the rod buckling is approximated by a linear function on finite element field. Two variants of the internal forces approximations on the finite element field: linear and piecewise constant were considered. Calculations of critical forces (loads) have been performed by the proposed method for the straight rods with different variants of the ends support and the spatial frameworks. The calculation results were compared with the analytical solutions and the solutions obtained by the method of finite elements in displacements. Analysis of the results shows that the use of piecewise constant approximations of internal forces leads to convergence to the exact values of the critical forces (loads) is strictly from below and provides solution with the reserve of stability.

Аннотация. Рассматривается решение задач устойчивости пространственных стержневых систем методом конечных элементов в напряжениях. Предлагаемая методика основывается на сочетании функционала дополнительной энергии и принципа возможных перемещений, используемого для построения уравнений равновесия узлов конечно-элементной сетки. После дискретизации предметной области, решение задачи сводится к поиску минимума функционала дополнительной энергии деформации при наличии ограничений в виде системы линейных алгебраических уравнений равновесия узлов. Уравнения равновесия включаются в функционал при помощи множителей Лагранжа, которыми являются перемещения узлов. Получены разрешающие уравнения для статического расчета пространственных стержневых систем на основе аппроксимации усилий (напряжений). Для решения задач устойчивости в функционале учитывается дополнительная энергия от продольных деформаций, возникающих за счет изгиба стержней. Форма потери устойчивости по области конечного элемента аппроксимируется линейной функцией. Рассматриваются два варианта аппроксимации внутренних усилий по области конечного элемента: линейная и кусочно-постоянная. По предложенной методике были выполнены расчеты критических сил (нагрузок) для прямых стержней при различных вариантах закрепления концов и пространственных рам. Выполнено сравнение результатов с аналитическим решениями и решениями, полученными по методу конечных элементов в перемещениях. Анализ полученных результатов показывает, что использование кусочно-постоянных аппроксимаций внутренних усилий приводит к сходимости к точному значению критических сил (нагрузок) строго снизу и позволяет получить решение в запас устойчивости.

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Introduction

The finite element method in displacements is successfully used to solve the widest range of structural mechanic problems, including stability problems [1–27]. In recent years, for rod systems the greatest attention was given to the build functional for solving the problems of stability of thin-walled rods [4, 6, 9, 13, 17, 20, 26] and given the effect of shear and longitudinal deformations [3, 4, 9, 13]. Also, in some articles the methods of solving rods stability problems [15] with considering the physical non-linearity [12, 16] and the torsion of the cross sections were introduce. A series of articles are devoted to the calculations the stability of rods based on direct solution of differential equations for compressed-bent rods [18–25], including rods with variable cross-section [20, 21, 23–25].

In [3, 4, 28–36] the solutions of problems by mixed finite elements methods and in stresses are considered. In [4], in order to solve the problems of constructions stability the mixed Reissner's functional is used. For this purpose, the functional is complemented by summand that considers the additional energy of longitudinal deformation occurring by the bending. Also, it is considered functional for solving stability problems of the open profiles rods and flexural-torsional forms of buckling spatial rods systems.

In [38] two variants of extreme energy principles to solve static problems of structural mechanics are considered. It includes the Castilian principle which involves the static equations (equilibrium) as constraints. The equations of statics were based on differential dependencies which bind forces and external loads. The equilibrium equations of the longitudinal and transverse forces are prepared for separate nodes. There were introduced different types of rod elements, the equilibrium equations are constructed using a special algorithm. Stability problems are not considered.

In [32–36] solutions of static and dynamic problems of structural mechanics by finite elements method in stresses were constructed. The equilibrium equations, formed on virtual displacement's principle, are included in the functional by the Lagrange's multipliers or using penalty functions method. It is shown that, if for approximation of the stress (forces) in the field of finite element constant or piecewise constant functions were used, then displacements of nodes seek to exact values from above by crushing the finite elements mesh. Thus, it is possible to receive the opposite, compared to the traditional finite element method in displacements, border exact solutions.

It is known [1–3] that the solution, obtained by the method of finite elements in displacements, under certain conditions, converges to the lower border of the exact values of displacements. Solutions, obtained by the minimum principle of additional energy, also under certain conditions, allow getting opposite bound of the exact values of displacements.

The purpose of this work is the construction of solution algorithm of stability problems of spatial rod systems on the basis the functional of additional energy and the principle of virtual displacements, which allow determine the lower limit of the critical loads.

Methods

In [32–36], founding by the functional of additional energy and principle of virtual displacements, solving the building structures problems by finite element analysis in stresses were built. Using constant or piecewise constant functions for the approximations of stresses (forces) in the field of finite element we will get the upper border of displacements. In general, the solution of the problem reduces to finding the minimum of the additional energy functional (1) in the presence of limitations in the form of equilibrium equations of nodes (2).

$$\Pi^c = U^* + V^* = \frac{1}{2}\int \{\sigma\}^T [E]^{-1} \{\sigma\} d\,\Omega - \int \{T\}^T \{\overline{\Delta}\} dS \to min, \quad (1)$$

$$\begin{aligned} \{C_{i,x}\}^{\mathrm{T}}\{\overline{\sigma}_i\} + \overline{P}_{i,x} = 0, \quad & i \in \Xi_x, \\ \{C_{i,y}\}^{\mathrm{T}}\{\overline{\sigma}_i\} + \overline{P}_{i,y} = 0, \quad & i \in \Xi_y, \\ \{C_{i,z}\}^{\mathrm{T}}\{\overline{\sigma}_i\} + \overline{P}_{i,z} = 0, \quad & i \in \Xi_z. \end{aligned} \quad (2)$$

U^* – additional energy of the strains, V^*– potential boundary forces corresponding to the specified displacements [1]; $\{\overline{\sigma}_i\}$ – vector of unknown node stresses (forces) of finite elements adjacent to the node i; Ξ_x, Ξ_y, Ξ_z – sets of nodes that have free displacements along the axes X, У и Z respectively; $\{\overline{\Delta}\}$ – vector given displacements of nodes; $\{T\}$ – vector boundary forces; S – boundary surface, on which the displacement nodes are given; $\{C_{i,x}\}, \{C_{i,x}\}, \{C_{i,z}\}$ – vectors, which elements are the

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coefficients (multipliers) of the unknown node stresses (forces) of finite elements adjacent to the node i; $\overline{P}_{i,x}, \overline{P}_{i,y}, \overline{P}_{i,z}$ – external loads potential corresponding to the virtual unit displacements of the node i along axes x, y, z respectively. The equations of equilibrium (2) are formed using the principle of virtual displacements for all admissible displacements of nodes along the coordinate axes.

In order to go on to unconstrained minimization problem, we use the method of Lagrange's multipliers. Then advanced functional of additional energy takes the following form:

$$\Pi_c^u = U^* + V^* + \sum_{j=x,y,z} \sum_{i \in \Xi_j} u_{i,j} \left(\left\{ C_{i,j} \right\}^{\mathrm{T}} \{ \overline{\sigma}_i \} + \overline{P}_{i,j} \right) \rightarrow min. \quad (3)$$

$u_{i,j}$ – the actual displacement of the node *i* towards *j*, which is the Lagrange's multiplier for the corresponding equilibrium equation. When using the functional (3) there is not necessary to use a stress field that satisfies the differential equations of equilibrium, as required by the principle of minimum additional energy. The equilibrium equations will be carried out in discrete sense – in the form of the equilibrium equations of the finite element mesh nodes.

Let us consider the application of the proposed approach to solve static problems of the spatial rod systems. Using the notations for rods systems, the functional (1) without the given displacements of nodes will be as follows:

$$\Pi^c = \sum_{i=1}^{n} \left(\frac{1}{2} \int_0^l \frac{M_y(x)^2}{EI_y} dx + \frac{1}{2} \int_0^l \frac{M_z(x)^2}{EI_z} dx + \frac{1}{2} \int_0^l \frac{M_k(x)^2}{GI_k} dx + \frac{1}{2} \int_0^l \frac{N(x)^2}{EA} dx \right) \rightarrow min. \quad (4)$$

EI_y, EI_z – bending stiffness, GI_k – torsional stiffness; $M_y(x)$, $M_z(x)$ – the bending moments directed around axes Y_1 and Z_1 respectively (fig. 1b), $M_k(x)$ – torque (directed around the axis X_1); EA – longitudinal stiffness; $N(x)$ – longitudinal force; l – length of the finite element; n – number of finite elements.

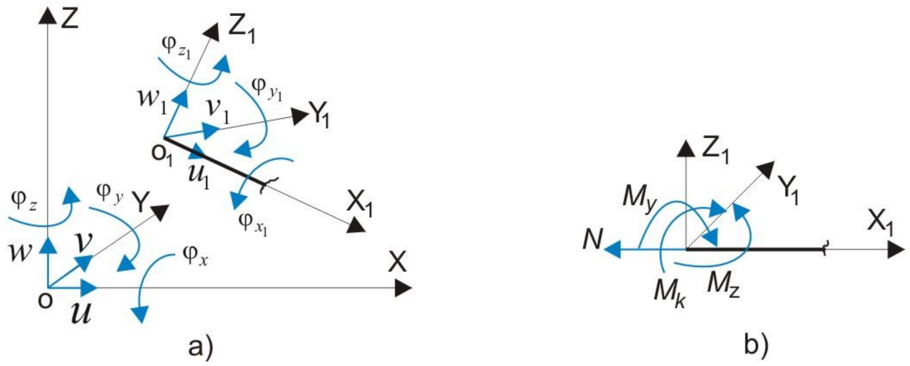

Figure 1. Positive directions of nodal displacements: a) global XYZ coordinate system and local coordinate system $X_1Y_1Z_1$ at the beginning of finite element; b) positive directions of the nodal internal forces at the beginning of finite element

The approximations of internal forces (longitudinal forces and moments) will take linear (5) or piecewise constant (6).

$$S(x) = S_1 \left(1 - \frac{x}{l} \right) + S_2 \frac{x}{l}, \quad (5)$$

$$S(x) = \left\{ \begin{matrix} S_1, & x \in [0, l/2] \\ S_2, & x \in [l/2, l] \end{matrix} \right\}. \quad (6)$$

In (5) and (6) under the symbol S any of the internal forces – N, M_y, M_z, M_k, is meant. The positive directions for the beginning finite element are shown in Figure 1b.

Substituting (5) or (6) into (4) we obtain the expression of the finite element additional energy in matrix form:

$$\Pi^c = \frac{1}{2} \{S_e\}^T [D_e] \{S_e\}, \quad \{S_e\}^T = (M_{y,1} \quad M_{y,2} \quad M_{z,1} \quad M_{z,2} \quad M_{k,1} \quad M_{k,2} \quad N_1 \quad N_2). \quad (7)$$

$\{S_e\}$ – vector of unknown nodal forces for finite element. Flexibility matrix $[D_e]$ of finite element for the case of linear – $[D_e]_L$ (8), and piecewise constant approximations – $[D_e]_C$ (9), will be as follows:

$$[D_e]_L = \begin{bmatrix} \frac{l}{3EI_y} & \frac{l}{6EI_y} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{l}{6EI_y} & \frac{l}{3EI_y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l}{3EI_z} & \frac{l}{6EI_z} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l}{6EI_z} & \frac{l}{3EI_z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{l}{3GI_k} & \frac{l}{6GI_k} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{l}{6GI_k} & \frac{l}{3GI_k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{l}{3EA} & \frac{l}{6EA} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{l}{6EA} & \frac{l}{3EA} \end{bmatrix}, \qquad (8)$$

$$[D_e]_C = \begin{bmatrix} \frac{l}{2EI_y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{l}{2EI_y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l}{2EI_z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{2EI_z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{l}{2GI_k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{l}{2GI_k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{l}{2EA} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{l}{2EA} \end{bmatrix}. \qquad (9)$$

Note, that the unknown nodal forces are accepted independently for each finite element. Therefore, the size of the global vector of unknown nodal forces $\{S\}$ will be equal to 8n. Global flexibility matrix $[D]$ to the whole system, and its inverse$[D]^{-1}$, will have a block-diagonal (or diagonal) form:

$$[D] = \begin{bmatrix} [D_1] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & [D_n] \end{bmatrix}, [D]^{-1} = \begin{bmatrix} [D_1]^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & [D_n]^{-1} \end{bmatrix} \qquad (10)$$

Using (10), the functional (4) can be written as follows:

$$\Pi^c = \frac{1}{2}\{S\}^T[D]\{S\} \to min. \qquad (11)$$

To form the equilibrium equations nodes of finite element we consider displacements of nodes in the local coordinate system (Figs. 2–3) and obtain corresponding expressions the strain energy of finite element.

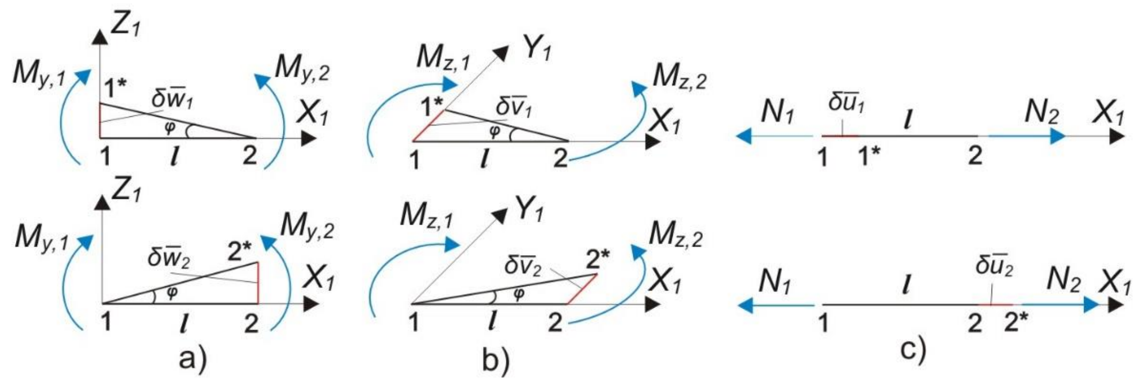

Figure 2. Possible displacements finite element nodes along the local axes: a) along the Z_1 – $\delta\overline{w}$; b) along the Y_1 – $\delta\overline{v}$; c) along the X_1 – $\delta\overline{u}$

We assume that the possible displacements are changed along length of finite element according to linear law. The displacements $\delta\overline{w}$ and $\delta\overline{v}$ will give the rotations of finite element, as rigid body rotations. The displacement of node $\delta\overline{w}_1$ causes the angle of finite element rotation:

$$\varphi = \frac{\delta\overline{w}_1}{l}. \tag{12}$$

When the element is rotated at an angle φ, the nodal moments will perform the work $\delta\overline{A}_{\overline{w},1}$ as the external forces:

$$\delta\overline{A}_{\overline{w},1} = -M_{y,1}\varphi + M_{y,2}\varphi = \delta\overline{w}_1\left(\frac{-M_{y,1}+M_{y,2}}{l}\right). \tag{13}$$

The internal moments are opposite in sign, so the energy of deformations:

$$\delta\overline{U}_{\overline{w},1} = -\delta\overline{A}_{\overline{w},1} = \delta\overline{w}_1\left(\frac{M_{y,1}-M_{y,2}}{l}\right). \tag{14}$$

Similarly, we obtain

$$\delta\overline{U}_{\overline{w},2} = \delta\overline{w}_2\left(\frac{-M_{y,1}+M_{y,2}}{l}\right),\ \delta\overline{U}_{\overline{v},1} = \delta\overline{v}_1\left(\frac{M_{z,1}-M_{z,2}}{l}\right),\ \delta\overline{U}_{\overline{v},2} = \delta\overline{v}_2\left(\frac{-M_{z,1}+M_{z,2}}{l}\right). \tag{15}$$

At possible displacement $\delta\overline{u}_1$ along the axis X_1:

$$u(x) = \delta\overline{u}_1\left(1-\frac{x}{l}\right),\ \ \varepsilon(x) = \frac{du(x)}{dx} = \frac{-\delta\overline{u}_1}{l}. \tag{16}$$

Then, the energy of deformations

$$\delta\overline{U}_{\overline{u},1} = \frac{-\delta\overline{u}_1}{l}\int_0^l N(x)dx. \tag{17}$$

If we substitute in (16) the expression for N (x) from (5) or (6) the result is the same:

$$\delta\overline{U}_{\overline{u},1} = \delta\overline{u}_1\frac{-(N_1+N_2)}{2}. \tag{18}$$

The similar expression can be obtained for a possible displacement of node 2

$$\delta\overline{U}_{\overline{u},2} = \delta\overline{u}_2\frac{(N_1+N_2)}{2}. \tag{19}$$

Next, let us consider possible displacements, as nodes rotation (Fig. 3a) and as element rotation (Fig. 3b), around axes of the local coordinate system

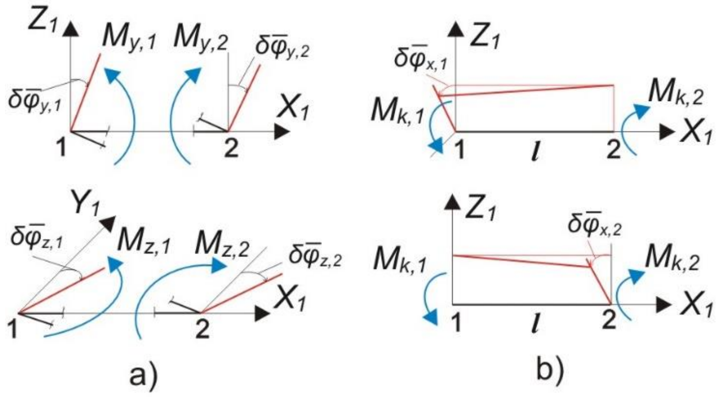

Figure 3. The possible turns: a) $\delta\overline{\varphi}_y, \delta\overline{\varphi}_z$ – around Y1 and Z1 axes; b) $\delta\overline{\varphi}_x$ – around the axis X1

For rotations around the axes Y1 and Z1, expressions of strain energy will be written simply:

$$\delta\overline{U}_{\overline{\varphi}_{y,1}} = -M_{y,1}\delta\overline{\varphi}_{y,1}, \qquad \delta\overline{U}_{\overline{\varphi}_{y,2}} = M_{y,2}\delta\overline{\varphi}_{y,2},$$
$$\delta\overline{U}_{\overline{\varphi}_{z,1}} = -M_{z,1}\delta\overline{\varphi}_{z,1}, \qquad \delta\overline{U}_{\overline{\varphi}_{z,2}} = M_{z,2}\delta\overline{\varphi}_{z,2}. \tag{20}$$

For rotations around the axis X1 (Fig. 3b) strain energy expressions are like equations (17) and (18) with substitution longitudinal forces by torques:

$$\delta\overline{U}_{\overline{\varphi}_{x,1}} = \delta\overline{\varphi}_{x,1}\frac{-(M_{k,1}+M_{k,2})}{2}, \quad \delta\overline{U}_{\overline{\varphi}_{x,2}} = \delta\overline{\varphi}_{x,2}\frac{(M_{k,1}+M_{k,2})}{2}. \tag{21}$$

The possible nodal displacements in the global and the local coordinate systems are connected by a matrix of the direction cosines $[t]$:

$$\begin{Bmatrix}\delta u_1\\ \delta v_1\\ \delta w_1\end{Bmatrix} = [t]\begin{Bmatrix}\delta\overline{u}_1\\ \delta\overline{v}_1\\ \delta\overline{w}_1\end{Bmatrix}, \begin{Bmatrix}\delta\varphi_{x,1}\\ \delta\varphi_{y,1}\\ \delta\varphi_{z,1}\end{Bmatrix} = [t]\begin{Bmatrix}\delta\overline{\varphi}_{x,1}\\ \delta\overline{\varphi}_{y,1}\\ \delta\overline{\varphi}_{z,1}\end{Bmatrix}, \begin{Bmatrix}\delta u_2\\ \delta v_2\\ \delta w_2\end{Bmatrix} = [t]\begin{Bmatrix}\delta\overline{u}_2\\ \delta\overline{v}_2\\ \delta\overline{w}_2\end{Bmatrix}, \begin{Bmatrix}\delta\varphi_{x,2}\\ \delta\varphi_{y,2}\\ \delta\varphi_{z,2}\end{Bmatrix} = [t]\begin{Bmatrix}\delta\overline{\varphi}_{x,2}\\ \delta\overline{\varphi}_{y,2}\\ \delta\overline{\varphi}_{z,2}\end{Bmatrix}. \tag{22}$$

$$[t] = \begin{bmatrix}t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{bmatrix}. \tag{23}$$

The energies of strains in the global and the local coordinate systems are connected also. Formation of the matrix of direction cosines $[t]$ is executed as in the calculation of spatial rod systems by finite elements method in displacements [1–5]. The deformation energy values for possible displacements of finite element nodes in the global coordinate system are placed into the vector $\{\delta U_e\}$ as follows:

$$\{\delta U_e\}^T = \{\delta U_{u,1}\ \delta U_{v,1}\ \delta U_{w,1}\ \delta U_{\varphi_{x,1}}\ \delta U_{\varphi_{y,1}}\ \delta U_{\varphi_{z,1}}\ \delta U_{u,2}\ \delta U_{v,2}\ \delta U_{w,2}\ \delta U_{\varphi_{x,2}}\ \delta U_{\varphi_{y,2}}\ \delta U_{\varphi_{z,2}}\}^T. \tag{24}$$

Possible displacements of finite element nodes in the global coordinate system, in the same order, are presented by square diagonal matrix $[\delta y_e]$.

$$[\delta y_e] = \begin{bmatrix}\delta u_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \delta\varphi_{z,2}\end{bmatrix}. \tag{25}$$

Using the vector of unknown nodal forces $\{S_e\}$, introduced in (7), we can write the following expression:

$$\{\delta U_e\} = [\delta y_e][L_e]\{S_e\}. \tag{26}$$

Matrix $[L_e]$, which may be called as matrix of equilibrium of finite element, considering the expressions (14–15), (20–23) is as follows:

$$\begin{bmatrix} \frac{-t_{31}}{l} & \frac{t_{31}}{l} & \frac{t_{21}}{l} & \frac{-t_{21}}{l} & 0 & 0 & \frac{-t_{11}}{2} & \frac{-t_{11}}{2} \\ \frac{-t_{32}}{l} & \frac{t_{32}}{l} & \frac{t_{22}}{l} & \frac{-t_{22}}{l} & 0 & 0 & \frac{-t_{12}}{2} & \frac{-t_{12}}{2} \\ \frac{-t_{33}}{l} & \frac{t_{33}}{l} & \frac{t_{23}}{l} & \frac{-t_{23}}{l} & 0 & 0 & \frac{-t_{13}}{2} & \frac{-t_{13}}{2} \\ -t_{21} & 0 & -t_{31} & 0 & \frac{-t_{11}}{2} & \frac{-t_{11}}{2} & 0 & 0 \\ -t_{22} & 0 & -t_{32} & 0 & \frac{-t_{12}}{2} & \frac{-t_{12}}{2} & 0 & 0 \\ -t_{23} & 0 & -t_{33} & 0 & \frac{-t_{13}}{2} & \frac{-t_{13}}{2} & 0 & 0 \\ \frac{t_{31}}{l} & \frac{-t_{31}}{l} & \frac{-t_{21}}{l} & \frac{t_{21}}{l} & 0 & 0 & \frac{t_{11}}{2} & \frac{t_{11}}{2} \\ \frac{t_{32}}{l} & \frac{-t_{31}}{l} & \frac{-t_{22}}{l} & \frac{t_{22}}{l} & 0 & 0 & \frac{t_{12}}{2} & \frac{t_{12}}{2} \\ \frac{t_{33}}{l} & \frac{-t_{33}}{l} & \frac{-t_{23}}{l} & \frac{t_{23}}{l} & 0 & 0 & \frac{t_{13}}{2} & \frac{t_{13}}{2} \\ 0 & t_{21} & 0 & t_{31} & \frac{t_{11}}{2} & \frac{t_{11}}{2} & 0 & 0 \\ 0 & t_{22} & 0 & t_{32} & \frac{t_{12}}{2} & \frac{t_{12}}{2} & 0 & 0 \\ 0 & t_{23} & 0 & t_{33} & \frac{t_{13}}{2} & \frac{t_{13}}{2} & 0 & 0 \end{bmatrix}. \quad (27)$$

From the local equilibrium matrices $[L_e]$, in accordance with the numbering of nodes and finite elements, global matrix $[L]$ of the system equilibrium equations of nodes will be formed. If the number of finite elements is equal to *n*, number of nodes – *k*, and number of kinematic links – *s*, the matrix [L] would be having $(6k-s)$ lines and *8n* columns. From the vector of unknown nodal forces of finite element $\{S_e\}$ vector of unknown forces for the whole system $\{S\}$ is formed. It consists of *8n* elements.

In order to form the equilibrium equations it is also necessary to get expression of the work of external forces from the possible displacements. At possible node displacements, the work is performed by concentrated vertical forces and moments in the nodes and by loads, that is distributed along the element. If evenly distributed along the finite element loads q_x, q_y, q_z are defined in the global coordinate system, then they should be reformed into local loads using expression (28):

$$\begin{Bmatrix} \bar{q}_x \\ \bar{q}_y \\ \bar{q}_z \end{Bmatrix} = [t]^T \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}. \quad (28)$$

Next, we form the vector $\{\bar{F}_e\}$ that consists of concentrated forces and moments in the nodes:

$$\{\bar{F}_e\}^T = \left\{ \frac{\bar{q}_x l}{2} \quad \frac{\bar{q}_y l}{2} \quad \frac{\bar{q}_z l}{2} \quad \frac{\bar{q}_x l^2}{12} \quad \frac{\bar{q}_y l^2}{12} \quad \frac{\bar{q}_z l^2}{12} \quad \frac{\bar{q}_x l}{2} \quad \frac{\bar{q}_y l}{2} \quad \frac{\bar{q}_z l}{2} \quad \frac{-\bar{q}_x l^2}{12} \quad \frac{-\bar{q}_y l^2}{12} \quad \frac{-\bar{q}_z l^2}{12} \right\}^T. \quad (29)$$

Vector $\{\bar{F}_e\}$, obtained in local coordinate system, should be transformed into the vector $\{F_e\}$ in global coordinate system:

$$\{F_e\} = [t_e]\{\bar{F}_e\},$$
$$[t_e] = \begin{bmatrix} [t] & 0 & 0 & 0 \\ 0 & [t] & 0 & 0 \\ 0 & 0 & [t] & 0 \\ 0 & 0 & 0 & [t] \end{bmatrix} \quad (30)$$

From the local vectors $\{F_e\}$, according to the numbering of nodes and elements, we form the global vector of the nodal loads $\{F\}$ for all finite elements. Next, the forces and moments, that are concentrated at the nodes, are added to elements of vector $\{F\}$. Obviously, the work of the external forces is calculated as product of the elements of the vector $\{F\}$ and the corresponding possible node displacements. Thus, the system of equilibrium equations for the whole system can be written in the following form:

$$\{F\} - [L]\{S\} = 0. \tag{31}$$

For the finite element, we introduce the notation for the vector of nodal unknowns in the local coordinate system

$$\{\bar{y}_e\}^T = \{\bar{u}_1\ \bar{v}_1\ \bar{w}_1\ \bar{\varphi}_{x,1}\ \bar{\varphi}_{y,1}\ \bar{\varphi}_{z,1}\ \bar{u}_2\ \bar{v}_2\ \bar{w}_2\ \bar{\varphi}_{x,2}\ \bar{\varphi}_{y,2}\ \bar{\varphi}_{z,2}\}^T \tag{32}$$

and the global vector of the nodal unknowns for whole system $\{y\}$, which is the vector of Lagrange's multipliers for the equilibrium equations of the system (30). By means of Lagrange's multipliers, we include the equations (30) into the functional (11) and obtain:

$$\Pi^c = \frac{1}{2}\{S\}^T[D]\{S\} + \{y\}^T(\{F\} - [L]\{S\}) \rightarrow min. \tag{33}$$

Equating to zero the derivatives Π^c from vector $\{S\}$, we obtain the equations of compatibility of strains in terms of stresses:

$$[D]\{S\} - [L]^T\{y\} = 0. \tag{34}$$

The derivatives Π^c on elements of the vector $\{y\}$ are systems of equilibrium equations of nodes (31). Combining (31) and (34), we obtain the final system of linear algebraic equations:

$$\begin{bmatrix} [D] & -[L]^T \\ -[L] & [0] \end{bmatrix} \begin{Bmatrix} \{S\} \\ \{y\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\{F\} \end{Bmatrix}. \tag{35}$$

Expressing vector $\{S\}$ from the first matrix equation and using it in the second, we get

$$[L][D]^{-1}[L]^T\{y\} = \{F\}, \tag{36}$$

$$\{S\} = [D]^{-1}[L]^T\{y\}. \tag{37}$$

Let us note that for getting (35) the approximation functions for the displacements are not used. Only, there were introduced approximations for possible displacements that can be of any shape, but must satisfy the kinematic relations. The solution was based on the introduction of approximations for the internal forces (stresses). By using linear approximations, we will get the values of forces and displacements of nodes that equal to the values, obtained by the method of finite elements in displacements. Since the matrix $[D]$ has simple structure, calculating product of the matrices in (36) does not require extensive computational resources.

Let us consider the problem of determining the critical load, which leads to the loss of stability in form of the rods bulging. In this paper, more complicated flexural-torsional buckling forms are not considered. As is well known [1–4], in solving problems of rod systems stability must be counted the stretching deformations that are associated with bending:

$$\varepsilon_0 = \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2. \tag{38}$$

After buckling, the function of the transverse displacements of axis of the finite element is approximated by the linear function in the local coordinate system

$$v(x) = \bar{v}_1\left(1 - \frac{x}{l}\right) + \bar{v}_2\frac{x}{l},\ \ w(x) = \bar{w}_1\left(1 - \frac{x}{l}\right) + \bar{w}_2\frac{x}{l}. \tag{39}$$

Then

$$\varepsilon_0 = \frac{1}{2}\frac{(\bar{v}_2 - \bar{v}_1)^2}{l^2} + \frac{1}{2}\frac{(\bar{w}_2 - \bar{w}_1)^2}{l^2}. \tag{40}$$

The additional energy of deformations

$$U^*_{\varepsilon_0} = \int_0^l N(x)\varepsilon_0 dx. \tag{41}$$

Setting in (41), any (5) or (6), approximation for $N(x)$ we obtain the following matrix expression for the energy of deformations

$$U^*_{\varepsilon_0} = \frac{1}{2}\{\bar{y}_e\}^T[\bar{G}_e]\{\bar{y}_e\}, \tag{42}$$

$$[\overline{G_e}] = \frac{(N_1+N_2)}{2l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (43)$$

For getting the geometric matrix of the finite elements in the global coordinate system, the following conversions must be done:

$$[G_e] = [t_e][\overline{G_e}][t_e]^T. \quad (44)$$

From the local matrices $[G_e]$ of the finite elements we generate geometric global matrix $[G]$ for the whole system. Using $U^*_{\varepsilon_0}$, the functional (33), for solving buckling problems will be as follows:

$$\Pi^c = \frac{1}{2}\{S\}^T[D]\{S\} + \frac{1}{2}\lambda\{y\}^T[G]\{y\} - \{y\}^T[L]\{S\} \rightarrow min. \quad (45)$$

In the expression (45) the parameter λ, which is interpreted as the buckling safety factor, are introduced. Minimum of functional (45) corresponds to the existence the equilibrium of the system in deflected shape. Equating the derivatives Π^c along the vector of forces $\{S\}$ to zero, we obtain the equations the compatibility of deformations (34). The derivatives of the (45) along the vector of displacements $\{y\}$ create the equations equilibrium of the nodes after buckling with adding the influence of longitudinal forces to bending:

$$-[L]\{S\} + \lambda[G]\{y\} = 0. \quad (46)$$

Combining (34) and (46), we obtain a system of homogeneous linear algebraic equations

$$\begin{bmatrix} [D] & -[L]^T \\ -[L] & \lambda[G] \end{bmatrix}\begin{Bmatrix} \{S\} \\ \{y\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (47)$$

Let us express vector of forces $\{S\}$, from the first matrix equation, and put it into the second equation. Introducing the notation for the matrix product $[K] = [L][D]^{-1}[L]^T$, we get:

$$-[K]\{y\} + \lambda[G]\{y\} = 0. \quad (48)$$

To determine the critical value of the parameter λ_{cr} we apply the method of inverse iterations, which includes the following steps. After solving (36) and (37), we obtain the vectors $\{y_0\}$ and $\{S_0\}$. Next, we must perform the iterations:

$$\begin{cases} i = 1,2,\cdots m; \\ \{y_i\} = [K]^{-1}[G]\{y_{i-1}\}, \\ y_{max} = \max\limits_{j=1..(6k-s)}|y_{i,j}|, \\ \lambda_{cr,i} = \dfrac{1}{|y_{max}|}. \end{cases} \quad (49)$$

In (49) y_{max} – maximum in modulus element of vector $\{y_i\}$. The iterative procedure is finished after achieving the necessary accuracy of calculating $|\lambda_{cr,i} - \lambda_{cr,i-1}| < \varepsilon$.

Results and Discussion

As examples calculations stability of the spatial frameworks, shown in Figures 4–6, were performed. The calculations were performed in Mathcad 14.0. The following characteristics of cross sections stiffness have been taken: $EI_y = 10\ kNm^2, EI_z = 10\ kNm^2, GI_k = 10\ kNm^2, EA = 1000\ kN$. Geometric dimensions in meters are indicated in the figures. The critical loads were calculated as functions of the finite elements number, which divide each rod, shown in the figures.

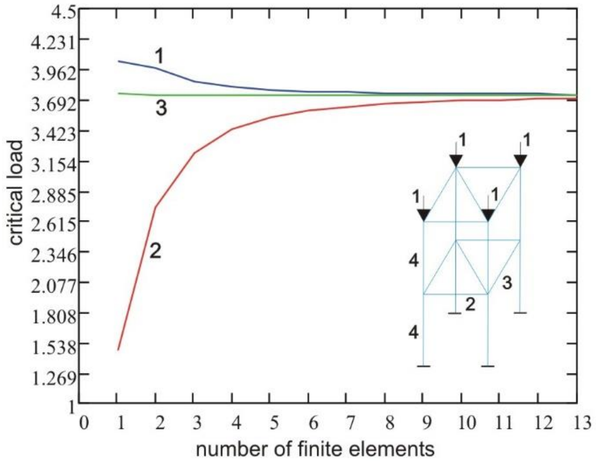

Figure 4. The critical load for rectangular framework with clamped supports

The graphs in Figures 4–6 are built on values of the critical forces, given in Tables 1–3.

Table 1. Values of the critical loads for the rectangular framework (kN). (Fig. 4)

Approximation	**Number of finite elements**												
	1	2	3	4	5	6	7	8	9	10	11	12	13
Linear	4.038	3.976	3.850	3.80	3.776	3.762	3.755	3.749	3.746	3.743	3.741	3.740	3.733
Piecewise constant	1.464	2.733	3.221	3.430	3.534	3.592	3.628	3.652	3.669	3.681	3.689	3.696	3.701
LIRA-SAPR	3.748	3.738	3.733	3.732	3.732	3.732	3.732	3.732	3.732	3.732	3.732	3.732	3.732

On figures: the number 1 (blue line) – indicate the results obtained by the linear approximations of the internal forces; number 2 (red line) – using piecewise constant approximations forces; number 3 (green line) – the results obtained by the finite elements method in the displacements on the program LIRA-SAPR 2013.

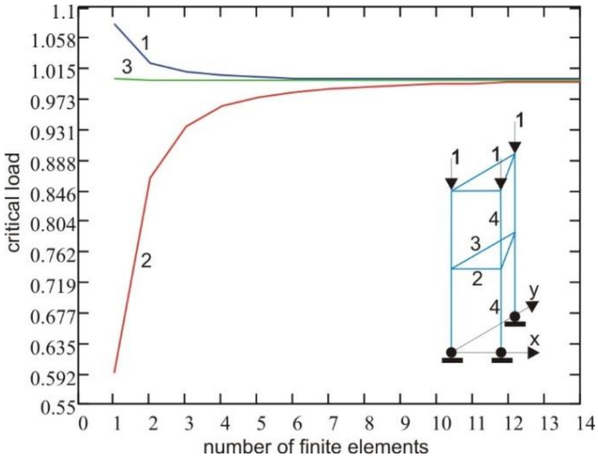

Figure 5. The critical load for the triangular framework with hinged supports

Table 2. The values of the critical loads for triangular framework with hinged supports (kN). (Fig. 5)

Approximation	Number of finite elements													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Linear	1.075	1.021	1.009	1.005	1.003	1.001	1.0008	1.0004	1.0001	0.9999	0.9997	0.9996	0.9995	0.9994
Piecewise constant	0.593	0.862	0.935	0.962	0.975	0.982	0.986	0.990	0.9915	0.9929	0.9940	0.9948	0.9954	0.9959
LIRA-SAPR	1.0009	0.9991	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990

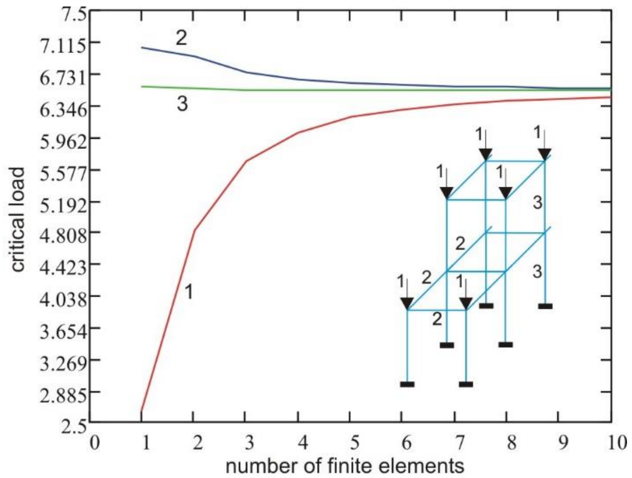

Figure 6. Critical load for the stepped framework with clamped supports

Table 3. The values of the critical load for the stepped framework with clamped supports (kN). (Fig. 6)

Approximation	Number of finite elements									
	1	2	3	4	5	6	7	8	9	10
Linear	7.0531	6.9479	6.7372	6.6526	6.6118	6.5892	6.5754	6.5665	6.5603	6.5559
Piecewise constant	2.6228	4.8323	5.6690	6.0234	6.2003	6.3000	6.3614	6.4019	6.4298	6.4499
LIRA-SAPR	6.5644	6.5465	6.5388	6.5373	6.5369	6.5368	6.5368	6.5367	6.5367	6.5366

Analysis of the results of the calculations shows that the use of piecewise constant approximations of internal forces lead to the convergence of the calculated values of critical forces (loads) to the exact values of strictly from bottom and allows you to get solutions to the stability reserve. At the same time, compared to the finite element method in the displacements, it is necessary to use the finer grids. The finite element method in displacements provides more accurate solutions with coarse grids. Necessary to consider, that the solution in displacements is more "rigid" and converges to the exact value from above as in the case of the use of linear approximations for the internal forces on the proposed method. It is known, by dividing of the finite elements grid we get values of stresses, which will tend to constant values, so for the convergence of solutions is necessary to ensure representation of the constant stresses or deformations. If solutions are get by proposed method, then this condition is performed. In the Fig. 7 shows graphs of the relative difference, in percentages, between the solutions, obtained for different approximations of internal forces, for the above examples. In the figure introduced the notation: $P_{cr,1}$ – linear approximations of internal forces; $P_{cr,2}$ – piecewise constant approximations of forces; $P_{cr,1*}$ – the minimum value, obtained by the linear approximations of forces; 1 (red line) – the results for the framework in Figure 4; 2 (blue line) – the results for the framework in Figure 5; 3 (green line) – the results for the framework in Figure 6. Reducing the difference between two solutions by the crushing of finite element mesh indicates to the convergence of solutions to the exact value. Note, that graphics for the 1st and 3rd schemes are practically the same (Table 4). Per the difference of two solutions we can assume the accuracies of calculation critical forces.

Tyukalov Yu.Ya. The functional of additional energy for stability analysis of spatial rod systems. *Magazine of Civil Engineering*. 2017. No. 2. Pp. 18–32. doi: 10.18720/MCE.70.3

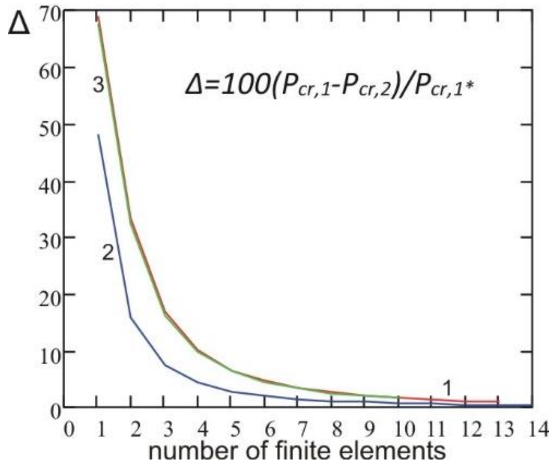

Figure 7. The relative difference between the values of the critical forces

Table 4. The relative difference of critical forces values in percentage (Fig. 7)

Scheme	Number of the crushing elements													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Fig. 4	68.9	33.3	16.8	9.9	6.5	4.6	3.4	2.6	2.1	1.7	1.4	1.2	0.85	-
Fig. 5	48.2	15.8	7.4	4.3	2.7	1.9	1.4	1.1	0.9	0.7	0.57	0.48	0.41	0.39
Fig. 6	67.6	32.3	16.3	9.6	6.3	4.4	3.3	2.5	2.0	1.6	-	-	-	-

To evaluate the accuracy and convergence of the approximate solution by proposed method, the critical forces were defined for straight rods with different types of ends fixing and for different number of finite elements. To simplify the analysis, the bending stiffness and length of the rods have been taken equal to unity. We considered the following variants of the rods: 1 – hinged rod; 2 – cantilever rod; 3 – rod with hinge and with clamped end; 4 – rod with clamped ends. The calculation results are shown in Table 5. Exact, analytically derived, values of the critical forces are taken from [37].

Table 5. Critical forces for straight rods

Variants of rods	Approximations	Number of finite elements								Exact values
		2	4	5	10	20	40	80	100	
1	Linear	12.0	10.4	10.2	9.951	9.8999	9.8746	9.87087	9.87042	9.86960
	Piecewise constant	8.0	9.38	9.55	9.789	9.8493	9.8645	9.86834	9.86879	
2	Linear	3.0	2.50	2.49	2.472	2.4687	2.4677	2.46748	2.46745	2.46740
	Piecewise constant	2.0	2.41	2.45	2.462	2.4661	2.4671	2.46732	2.46735	
3	Linear	27.4	22.4	21.6	20.53	20.275	20.212	20.1960	20.1941	20.19064
	Piecewise constant	12.0	17.8	18.6	19.79	20.089	20.165	20.1844	20.1867	
4	Linear	48.0	48.0	44.9	40.79	39.804	39.560	39.4987	39.4914	39.47842
	Piecewise constant	16.0	32.0	34.6	38.20	39.155	39.397	39.4581	39.4654	

The calculation results of the stability of straight rods confirm, as was noted above, characteristic features of the proposed method of calculation, which is based on the functional of additional energy. These characteristics allow to note the following possible fields of application of the method: getting the lower limit of the critical forces; calculation the stability of the structures such as plates, which can be strengthen by rods; the curvilinear constructions or constructions on elastic foundation; getting the solutions, which are alternative to the solutions on method of finite elements in displacements.

Conclusions

1. For problems of stability the spatial rod systems there are proposed the method, which is based on functional of the additional energy and the principle of virtual displacements. Equations for static analysis of spatial rod systems based on the approximation of the forces (stress) were obtained.

2. The examples of the calculations the critical forces for straight rods and three-dimensional frameworks for different finite element grids show that using of piecewise constant approximations of internal forces provides a lower bound of the critical forces. It is necessary to use a fine grid of finite elements. For the above examples, the required number of finite elements for achieving the same accuracy as accuracy of the solutions by finite elements method in displacements is about 5 times more. Accuracy solutions, lot less than 1 percent, can be obtained, if very fine grid of the finite elements (Table 4) is used.

3. By using linear approximations of the internal forces, we get the solutions which converge to the exact values of the critical forces from above and give an upper bound. It is possible to define accuracies of calculation of the critical forces per the difference of two solutions with linear and piecewise constant approximations.

4. Possible fields of application of the method are getting the lower limit of the critical forces; calculation the stability of the structures such as plates, which can be strengthen by rods; calculation the stability of the curvilinear constructions or constructions on an elastic foundation; getting the solutions, which are alternative to the solutions getting on method of finite elements in displacements.

References

1. Zenkevich O. *Metod konechnykh elementov v tekhnike* [The finite element method in the technique]. M.: Mir, 1975. 541 p. (rus)
2. Zenkevich O., Morgan K. *Konechnyye elementy i approksimatsiya* [Finite Elements and Approximation]. M.: Mir, 1986. 318 p. (rus)
3. Gallager R. *Metod konechnykh elementov. Osnovy* [Finite elements and approximation. foundation]. M.: Mir, 1984. 428 p. (rus)
4. Slivker V.I. *Stroitelnaya mekhanika. Variatsionnyye osnovy* [Construction mehanika. Variatsionnye bases]. M.: Izdatelstvo Assotsiatsii stroitelnykh vuzov. 2005. 736 p. (rus)
5. Rozin L.A. *Metod konechnykh elementov v primenenii k uprugim sistemam* [Finite element method, as applied to an elastic system]. M.: Stroyizdat, 1977. 129 p. (rus)
6. Senjanović I., Vladimir N., Cho D.-S. A simplified geometric stiffness in stability analysis of thin-walled structures by the finite element method. *Inter J. Nav. Archit. Oc. Engng*. 2012. Vol. 4. Pp. 313-321.
7. Ibearugbulem O.M., Eziefula U.G., Onwuka D.O. Stability analysis of uniaxially compressed flat rectangular isotropic CCSS plate. *International Journal of Applied Mechanics and Engineering.* 2015. Vol. 20. No. 3. Pp. 637–646.
8. Yevzerov I.D. Zadachi ustoychivosti dlya sterzhney i plastin [Stability problems for rods and plates]. *Magazine of Civil Engineering.* 2014. No. 1(45). Pp. 6–11. (rus)
9. Lalin V.V., Rozin L.A., Kushova D.A. Variatsionnaya postanovka ploskoy zadachi geometricheski nelineynogo deformirovaniya i ustoychivosti uprugikh sterzhney [The variational formulation of the plane problem of geometrically nonlinear deformation and stability of elastic rods]. *Magazine of Civil Engineering.* 2013. No. 1(36). Pp. 87–96. (rus)
10. Lalin V.V. Razlichnyye formy uravneniy ravnovesiya nelineynoy dinamiki uprugikh sterzhney [Various forms of the equilibrium equations of the nonlinear dynamics of elastic rods]. *Trudy SPbGPU.* 2004. No. 489. Pp. 121–128. (rus)
11. Lalin V.V., Zdanchuk Ye.V., Kushova D.A., Rozin L.A. Variatsionnyye postanovki nelineynykh zadach s nezavisimymi vrashchatelnymi stepenyami svobody [Variational formulation of nonlinear problems with independent rotational degrees of freedom]. *Magazine of*

Литература

1. Зенкевич О. Метод конечных элементов в технике. М.: Мир, 1975. 541 с.
2. Зенкевич О., Морган К. Конечные элементы и аппроксимация. М.: Мир, 1986. 318 с.
3. Галлагер Р. Метод конечных элементов. Основы. М.: Мир, 1984. 428 с.
4. Сливкер В. И. Строительная механика. Вариационные основы. М.: Издательство Ассоциации строительных вузов. 2005. 736 с.
5. Розин Л. А. Метод конечных элементов в применении к упругим системам. М.: Стройиздат, 1977. 129 с.
6. Senjanović I., Vladimir N., Cho D.-S. A simplified geometric stiffness in stability analysis of thin-walled structures by the finite element method // Inter J. Nav. Archit. Oc. Engng. 2012. Vol. 4. Pp. 313–321.
7. Ibearugbulem O.M., Eziefula U.G., Onwuka D.O. Stability analysis of uniaxially compressed flat rectangular isotropic CCSS plate // International Journal of Applied Mechanics and Engineering. 2015. Vol. 20. № 3. Pp. 637–646.
8. Евзеров И.Д. Задачи устойчивости для стержней и пластин // Инженерно-строительный журнал. 2014. № 1(45). С. 6–11.
9. Лалин В.В., Розин Л.А., Кушова Д.А. Вариационная постановка плоской задачи геометрически нелинейного деформирования и устойчивости упругих стержней // Инженерно-строительный журнал. 2013. № 1(36). С. 87–96.
10. Лалин В.В. Различные формы уравнений равновесия нелинейной динамики упругих стержней // Труды СПбГПУ. 2004. № 489. С. 121–128.
11. Лалин В.В., Зданчук Е.В., Кушова Д.А., Розин Л.А. Вариационные постановки нелинейных задач с независимыми вращательными степенями свободы // Инженерно-строительный журнал. 2015. № 4(56). С. 54–65.
12. Лалин В.В., Беляев М.О. Изгиб геометрически нелинейного консольного стержня. Решение по теориям Кирхгофа и Коссера – Тимошенко // Инженерно-строительный журнал. 2015. № 1(53). С. 39–55.
13. Lalin V., Rybakov V., Alexander S. The finite elements for design of frame of thin-walled beams // Applied Mechanics

Civil Engineering. 2015. No. 4(56). Pp. 54–65. (rus)

12. Lalin V.V., Belyayev M.O. Izgib geometricheski nelineynogo konsolnogo sterzhnya. Resheniye po teoriyam Kirkhgofa i Kossera – Timoshenko [Bending geometrically nonlinear cantilever beam. The decision on the theories of Kirchhoff and Cosserat – Tymoshenko]. *Magazine of Civil Engineering.* 2015. No. 1(53). Pp. 39–55. (rus)
13. Lalin V., Rybakov V., Alexander S. The finite elements for design of frame of thin-walled beams. *Applied Mechanics and Materials.* 2014. Vol. 578–579. Pp. 858–863.
14. Perelmuter A.V., Slivker V.I. *Ustoychivost ravnovesiya konstruktsiy i rodstvennyye problemy* [Stability of equilibrium structures and related problems]. Vol. 1. M.: SKAD SOFT, 2010. 704 p. (rus)
15. Chepurenko A.S., Andreyev V.I., Yazyyev B.M. Energeticheskiy metod pri raschete na ustoychivost szhatykh sterzhney s uchetom polzuchesti [Energy method, with respect to stability in view of the rods of compressed creep]. *Vestnik MGSU.* 2013. No. 1. Pp. 101–108. (rus)
16. Kagan-Rozentsveyg L.M. O raschete uprugikh ram na ustoychivost [On calculation of the elastic frame for stability]. *Magazine of Civil Engineering.* 2012. No. 1(27). Pp. 74–78. (rus)
17. Chernov S.A. Modelirovaniye ustoychivosti podkreplennoy tonkostennymi sterzhnyami plastiny [Simulation of the stability of thin-walled stiffened plate]. *Programmnyye produkty i sistemy.* 2014. No. 4. Pp. 183–187. (rus)
18. Manuylov G.A., Kositsyn S.B., Begichev M. M. Chislennoye issledovaniye prostranstvennoy ustoychivosti uprugikh krugovykh zashchemlennykh arok [Numerical study of spatial stability of elastic clamped circular arches]. *International Journal for Computational Civil and Structural Engineering.* 2013. Vol. 9. No 1. Pp. 78–84. (rus)
19. Chamekh M., Mani-Aouadi S., Moakher M. Stability of elastic rods with self-contact. *Computer Methods in Applied Mechanics and Engineering.* 2014. Vol. 279. Pp. 227–246.
20. Dubrovin V.M., Butina T.A. Modelirovaniye ustoychivosti szhatogo i skruchennogo sterzhnya v tochnoy postanovke zadachi [Simulation of the stability of compressed and twisted rod in the exact formulation of the problem]. *Matematicheskoye modelirovaniye i chislennyye metody.* 2015. No. 3. Pp. 3–16. (rus)
21. Blyumin S.L., Zverev V.V., Sotnikova I.V., Sysoyev A.S. Resheniye zadachi ustoychivosti szhato-izgibayemykh zhestko opertykh sterzhney peremennoy zhestkosti [Solution of the problem of stability of compressed-bent rigidly simply supported rods of variable stiffness]. *Vestnik MGSU.* 2015. No. 5. Pp. 18–26. (rus)
22. Yuan W.B., Kim B., Li L.Y. Buckling of axially loaded castellated steel columns. *J. Construct. Steel Res.* 2014. No. 92. Pp. 40–45.
23. Kim B., Li L., Edmonds A. Analytical Solutions of Lateral–Torsional Buckling of Castellated Beams. *International Journal of Structural Stability and Dynamics.* 2016. Vol. 16. No. 8. Pp. 443–455.
24. Gorbachev V.I., Moskalenko O.B. Ustoychivost sterzhney s peremennoy zhestkostyu pri szhatii raspredelennoy nagruzkoy [Stability of rods with variable stiffness in compression load distributed]. *Vestnik Moskovskogo gosudarstvennogo universiteta. Seriya 1. Matematika. Mekhanika.* 2012. No. 1. Pp. 41–47. (rus)
25. Galkin A.V., Sysoyev A.S., Sotnikova I.V. Zadacha ustoychivosti szhato-izgibayemykh sterzhney so stupenchatym izmeneniyem zhestkosti [The problem of stability of compressed-bent rods with a step change in stiffness]. *Vestnik MGSU.* 2015. No. 2. Pp. 38–44. (rus)
26. Kucukler M., Gardner L., Macorini L. Flexural–torsional buckling assessment of steel beam–columns through a stiffness reduction method. *Engineering Structures*. 2015. Vol. 101. Pp. 662–676.
27. Pokrovskiy A.A. Ustoychivost sterzhnya peremennogo

and Materials. 2014. Vol. 578–579. Pp. 858–863.

14. Перельмутер А.В., Сливкер В.И. Устойчивость равновесия конструкций и родственные проблемы. Том 1. М.: СКАД СОФТ, 2010. 704 с.
15. Чепуренко А.С., Андреев В.И., Языев Б.М. Энергетический метод при расчете на устойчивость сжатых стержней с учетом ползучести// Вестник МГСУ. 2013. № 1. С. 101–108.
16. Каган-Розенцвейг Л. М. О расчете упругих рам на устойчивость // Инженерно-строительный журнал. 2012. № 1(27). С. 74–78.
17. Чернов С.А. Моделирование устойчивости подкрепленной тонкостенными стержнями пластины// Программные продукты и системы. 2014. № 4. С. 183–187.
18. Мануйлов Г.А., Косицын С.Б., Бегичев М.М. Численное исследование пространственной устойчивости упругих круговых защемленных арок // International Journal for Computational Civil and Structural Engineering. 2013. Vol. 9. № 1. Pp. 78–84.
19. Chamekh M., Mani-Aouadi S., Moakher M. Stability of elastic rods with self-contact// Computer Methods in Applied Mechanics and Engineering. 2014. Vol. 279. Pp. 227–246.
20. Дубровин В.М., Бутина Т.А. Моделирование устойчивости сжатого и скрученного стержня в точной постановке задачи // Математическое моделирование и численные методы. 2015. № 3. С. 3–16.
21. Блюмин С.Л., Зверев В.В., Сотникова И.В., Сысоев А.С. Решение задачи устойчивости сжато-изгибаемых жестко опертых стержней переменной жесткости // Вестник МГСУ. 2015. № 5. С. 18–26.
22. Yuan W. B., Kim B., Li L. Y. Buckling of axially loaded castellated steel columns // J. Construct. Steel Res. 2014. № 92. Pp. 40–45.
23. Kim B., Li L., Edmonds A. Analytical Solutions of Lateral–Torsional Buckling of Castellated Beams // International Journal of Structural Stability and Dynamics. 2016. Vol. 16. № 8. Pp. 443–455.
24. Горбачев В.И., Москаленко О.Б. Устойчивость стержней с переменной жесткостью при сжатии распределенной нагрузкой // Вестник Московского государственного университета. Серия 1. Математика. Механика. 2012. № 1. С. 41–47.
25. Галкин А.В., Сысоев А.С., Сотникова И.В. Задача устойчивости сжато-изгибаемых стержней со ступенчатым изменением жесткости // Вестник МГСУ. 2015. № 2. С. 38–44.
26. Kucukler M., Gardner L., Macorini L. Flexural–torsional buckling assessment of steel beam–columns through a stiffness reduction method // Engineering Structures. 2015. Vol. 101. Pp. 662–676.
27. Покровский А.А. Устойчивость стержня переменного сечения по длине и его применение в структурных конструкциях // Строит. механика и расчёт сооружений. 2007. № 4. С. 39–40.
28. Yong-Lin Kuo. Stress-based Finite Element Analysis of Sliding Beams // Appl. Math. Inf. Sci. 2015. Vol. 9. № 2. Pp. 609–616.
29. Qiu W., Demkowicz L. Mixed hp-finite element method for linear elasticity with weakly imposed symmetry: stability analysis // SIAM J. Numer. Anal. 2011. Vol. 49. № 2. Pp. 619–641.
30. Guzman J. A unified analysis of several mixed methods for elasticity with weak stress symmetry // J. Sci. Comput. 2010. Vol. 44. Pp. 156–169.
31. Суходолова Ю.С., Труфанов Н.А. О конечном элементе на основе вариационного принципа Кастильяно для плоских задач теории упругости // Вестник Пермского национального исследовательского политехнического

secheniya po dline i yego primeneniye v strukturnykh konstruktsiyakh [Stability rod of variable cross section along the length of its use and structural designs]. Stroit. mekhanika i raschet sooruzheniy. 2007. No. 4. Pp. 39–40. (rus)

28. Yong-Lin Kuo. Stress-based Finite Element Analysis of Sliding Beams. *Appl. Math. Inf. Sci.* 2015. Vol. 9. No. 2. Pp. 609–616.
29. Qiu W., Demkowicz L. Mixed hp-finite element method for linear elasticity with weakly imposed symmetry: stability analysis. *SIAM J. Numer. Anal.* 2011. Vol. 49. No. 2. Pp. 619–641.
30. Guzman J. A unified analysis of several mixed methods for elasticity with weak stress symmetry. *J. Sci. Comput.* 2010. Vol. 44. Pp. 156–169.
31. Sukhodolova Yu.S., Trufanov N.A. O konechnom elemente na osnove variatsionnogo printsipa Kastilyano dlya ploskikh zadach teorii uprugosti [On the finite element based on Castigliano variational principle for plane elasticity problems]. *Vestnik Permskogo natsionalnogo issledovatelskogo politekhnicheskogo universiteta. Mekhanika.* 2012. No. 1. Pp. 168–178. (rus)
32. Tyukalov Yu.Ya. Raschet plit na dinamicheskiye vozdeystviya s uchetom plasticheskikh deformatsiy [Calculation of the plates on the dynamic effects of considering plastic deformation]. *Seysmostoykoye stroitelstvo. Bezopasnost sooruzheniy.* 2005. No. 2. Pp. 24–26. (rus)
33. Tyukalov Yu.Ya. Raschet obolochek proizvolnoy formy metodom konechnykh elementov v napryazheniyakh [Calculation of shells of arbitrary shape by finite element method in stresses]. *Stroitelnaya mekhanika i raschet sooruzheniy.* 2006. No. 1. Pp. 65–74. (rus)
34. Tyukalov Yu.Ya. Resheniye ploskoy zadachi teorii uprugosti metodom konechnykh elementov v napryazheniyakh [The solution of the plane problem of elasticity theory of finite elements method in stresses]. *Stroitelnaya mekhanika i raschet sooruzheniy.* 2006. No. 2. Pp. 34–38. (rus)
35. Tyukalov Yu.Ya. Variatsionno-setochnyy metod resheniya zadach izgiba plit v napryazheniyakh[Variational-grid method for solving the problems of plate bending in stresses]. *Izvestiya vuzov. Stroitelstvo.* 2006. No. 8. Pp. 13–20. (rus)
36. Tyukalov Yu.Ya. Ispolzovaniye kusochno-postoyannykh napryazheniy dlya resheniya obyemnykh zadach teorii uprugosti [Using the piecewise constant stresses for solutions of the volume elasticity problems]. *Izvestiya vuzov. Stroitelstvo.* 2008. No. 4. Pp. 4–9. (rus)
37. Kiselev V.A. *Stroitelnaya mekhanika. Dinamika i ustoychivost sooruzheniy* [Structural mechanics. The dynamics and stability of structures]. M.: Stroyizdat, 1980. 616 p. (rus)
38. Chiras A.A. *Stroitelnaya mekhanika: Teoriya i algoritmy* [Structural Mechanics: Theory and Algorithms]. M.: Stroyizdat, 1989. 255 p.(rus)

университета. Механика. 2012. № 1. С. 168–178.

32. Тюкалов Ю.Я. Расчет плит на динамические воздействия с учетом пластических деформаций // Сейсмостойкое строительство. Безопасность сооружений. 2005. № 2. С. 24–26.
33. Тюкалов Ю.Я. Расчет оболочек произвольной формы методом конечных элементов в напряжениях // Строительная механика и расчет сооружений. 2006. № 1. С. 65–74.
34. Тюкалов Ю.Я. Решение плоской задачи теории упругости методом конечных элементов в напряжениях // Строительная механика и расчет сооружений. 2006. № 2. С. 34–38.
35. Тюкалов Ю.Я. Вариационно-сеточный метод решения задач изгиба плит в напряжениях // Известия вузов. Строительство. 2006. № 8. С. 13–20.
36. Тюкалов Ю.Я. Использование кусочно-постоянных напряжений для решения объемных задач теории упругости // Известия вузов. Строительство. 2008. № 4. С. 4–9.
37. Киселев В.А. Строительная механика. Динамика и устойчивость сооружений. М.: Стройиздат, 1980. 616 с.
38. Чирас А.А. Строительная механика: Теория и алгоритмы. М.: Стройиздат, 1989. 255 с.

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