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A REGULARIZATION OF THE HARTLE-HAWKING WAVE FUNCTION

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The paper puts forward a modification of the no-boundary Hartle-Hawking wave function in which, in the general case, the Euclidean functional integral can be described by an inhomogeneous universe. The regularization of this integral is achieved in arbitrary canonical calibration by abandoning integration over the lapse and shift functions. This makes it possible to 'correct' the sign of the Euclidean action corresponding to the scale factor of geometry. An additional time parameter associated with the canonical calibration condition then emerges. An additional condition for the stationary state of the wave function's phase after returning to the Lorentzian signature, serving as the quantum equivalent of the classical principle of the least action, was used to find this time parameter. We have substantiated the interpretation of the modified wave function as the amplitude of the universe's birth from 'nothing' with the additional parameter as the time of this process. A homogeneous model of the universe with a conformally invariant scalar field has been considered. In this case, two variants of the no-boundary wave function which are solutions of the Wheeler-DeWitt equation have been found.

Key words: general relativity; universe; quantum cosmology; Euclidean action; Lorentzian singature; Hartle-Hawking wave function

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РЕГУЛЯРИЗАЦИЯ ВОЛНОВОЙ ФУНКЦИИ ХАРТЛА – ХОКИНГА

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Предложена модификация неограниченной (no-boundary) волновой функции Хартла – Хокинга, в которой, в общем случае, евклидов функциональный интеграл можно определить неоднородной Вселенной. Регуляризация этого интеграла достигается в произвольной канонической калибровке отказом от интегрирования по функциям следования и сдвига. Это позволяет «исправить» знак евклидова действия, отвечающего масштабному фактору геометрии. При этом возникает дополнительный параметр времени, связанный с каноническим калибровочным условием. Для нахождения этого параметра времени использован квантовый аналог классического принципа наименьшего действия. Им служит дополнительное условие стационарности фазы волновой функции после возвращения к лоренцевой сигнатуре. Обоснована интерпретация модифицированной волновой функции как амплитуды рождения Вселенной из «ничего» с указанным дополнительным параметром в качестве времени этого процесса. Рассмотрена однородная модель Вселенной с конформно-инвариантным скалярным полем. В этом случае найдены два варианта волновой функции no-boundary, которые являются решениями уравнения Уиллера – Девитта.

Ключевые слова: квантовая вселенная; евклидово действие; лоренцева сигнатура; волновая функция Хартла – Хокинга

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Introduction

The Hartle-Hawking no-boundary wave function of the universe [1, 2] is a unique construction in quantum cosmology which has been put forward to describe the early stages of the universe evolution. It is possible that this function describes the whole universe evolution defining the probability measure on classical spacetimes [3]. But the problem is that, in the general case, it has been ill-defined [4] out of the scope of the semiclassical approximation, since the Euclidean action of General Relativity (GR) is not positive-definite. A negative contribution to the action is related to the conformal scale factor of geometry [5].

In the present paper we propose an adaptation (by integral regularization) of the Hartle-Hawking no-boundary wave function that allows to avoid the mentioned difficulty. Moreover, we put forward another physical interpretation of this regularization, namely, this selected state of the universe will be considered as initial, without any dynamical subject-matter. The dynamics can be formulated separately using the ordinary GR Hamiltonian with the Lorentzian signature. This is due to the fact that the proposed adaptation violates the initial covariance of the Hartle-Hawking formulation, and the obtained wave function will not generally be a solution of the Wheeler-DeWitt equation.

In order to determine this selected state within the Hartle-Hawking no-boundary formulation we propose to make (at our will) the change of the sign of the negative term in the Euclidean action of GR (subsequently as "Euclidean GR").

Our first comment is that the sign will be restored afterwards. But this change is fraught with consequence: classical constraints of the Euclidean GR become unsolvable in the real range of variables' values.

This means that integrating over the lapse and shift functions in the continual-integral representation of the no-boundary wave function becomes meaningless. Because of this, we simply fix these variables up to the next stage of our regularization procedure. In this way the Euclidean no-boundary wave function appears to be determined in a relativistic canonical calibration with a fixed Euclidean interval of time (it is arbitrary so far).

After integrating over all physical degrees of freedom, it is necessary to restore at once both the initial negative sign of the Euclidean action related to the conformal scale factor and the Lorentzian signature of the whole action by the Wick rotation of the Euclidean time in the opposite direction at the complex plane. As a result, the Euclidean no-boundary wave function will become complex. The final step of our regularization is fixation of the time parameter governing the wave function in addition.

For this purpose we propose to use the additional condition of the wave-function's phase stationary state relative to variations of the time parameter. The condition of the phase stationary state is a quantum equivalent of the classical principal of the least action in the GR. The equations resulting from this condition fix the lapse and shift functions. Solving the stationary equations, we determine the no-boundary wave function of the universe up to a constant multiplier.

In the present paper, we consider this regularization procedure in the case of a simplest minisuperspace model of the universe with a conformal invariant scalar field. Although this example is far from an appropriate description of reality, it is suitable for its simple model [6]. In our framework the resulting regularized no-boundary wave function will be a non-trivial solution of the Wheeler-DeWitt (WDW) equation of the model, and so it will be stationary.

Minisuperspace with the conformal invariant scalar field

In the case of the homogeneous Robertson-Walker metric (Euclidean signature) which has the form

$$ds^2 = \sigma^2[N^2(\tau)d\tau^2 + a^2(\tau)d\Omega_3^2], \quad (1)$$

where $\sigma^2 \equiv (2/3\pi)m_p^2$, and $d\Omega_3^2$ is the metric of the 3D sphere with the unit radius, and the conformal invariant scalar field $\phi(\tau)$, the classical action of GR may be written in the form of Ref. [6] as follows:

$$I = I_a + I_\phi, \quad (2)$$

$$I_a = -\frac{\xi}{2}\int_0^1 d\tau\tilde{N}\left[\left(\frac{1}{\tilde{N}}\frac{da}{d\tau}\right)^2 + a^2\right], \quad (3)$$

$$I_\phi = \frac{1}{2}\int_0^1 d\tau\tilde{N}\left[\left(\frac{1}{\tilde{N}}\frac{d\psi}{d\tau}\right)^2 + \psi^2\right], \quad (4)$$

where $\tilde{N} = Na$, $\psi = (\sqrt{2/\pi})a\sigma\phi$.

The Wheeler-DeWitt equation of the model has the form

$$\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left(a^p\frac{\partial}{\partial a}\right) - a^2 - \frac{\partial^2}{\partial\chi^2} + \chi^2\right]\Psi(a,\chi) = 0. \quad (5)$$

We have introduced a regularization parameter ξ in Eq. (3) whose "normal" value is +1. Further, for simplicity we will take the parameter of operator ordering $p = 0$.

Following Ref. [3], let us consider the configurations of the scale factor a on a disc with boundary conditions: $a(0) = 0$ (the South Pole) and $a(1) = b$ at the final spatial section.

For the initial configurations of the conformal scalar field ϕ at the South Pole let us consider two cases:

(*i*) $\phi(0) = 0$ (ψ is smooth in the South Pole);

(*ii*) $\phi(0) = 0$ (ϕ is smooth in the South Pole, but $\psi(0) = 0$).

Indeed,

$$\dot{\psi}(\tau) \propto \dot{a}\varphi + a\dot{\phi} = \frac{\pi}{\sqrt{2}\sigma}(\ln a)\psi + a\dot{\phi}. \quad (6)$$

It follows from here for the second case: $\psi \propto a$ in the limit $\tau \to 0$.

In both cases we take $\psi(1) = \chi$ at the final spatial section. Since the integration over the (renormalized) lapse function $\tilde{N}(\tau)$ will not be performed from this point on, we obtain the dependence of the universe's state at the final spatial section on an additional real Euclidean time parameter C:

$$C \equiv \int_0^1 d\tau\tilde{N}(\tau). \quad (7)$$

In the ordinary covariant quantum theory, the integration over the interval $C \in [0,\infty)$ with corresponding measure is supposed [7]. We have chosen another possibility: this parameter will be fixed by the QAP at the final stage of our definition of the no-boundary wave function of the universe.

Regularization of the Hartle-Hawking no-boundary wave function

Let us consider a functional integral over the field configurations $(a(\tau),\psi(\tau))$ with the given boundary values at the final spatial section (b,χ) and the corresponding smoothness conditions at the South Pole (setting here $\hbar = 1$):

$$\Psi(b,\chi,C) = \int DaD\psi\exp(-I). \quad (8)$$

For the integral (8) to be finite we set the regularization parameter ξ equal to −1 at this stage. Then the Gauss integral (8) can be calculated without effort. The following simple example illustrates the regularization procedure proposed here (for $\xi = 1$):

$$\int_{-\infty}^{+\infty} dx\exp(\xi x^2) \equiv \frac{\sqrt{\pi}}{\sqrt{-\xi}} = -i\sqrt{\pi}. \quad (9)$$

The only irritant in our regularization procedure is an occurrence of the constant multiplier $(-i)^\infty$, which arises in Eq. (8) after integration over $a(\tau)$. But this multiplier does not depend on the dynamical variables, so it can be omitted.

Notice that the regularization parameter ξ in the integral (3) can be inserted into the lapse function $\tilde{N}(\tau)$, so that we derive two independent variables $\tilde{N}_a(\tau), \tilde{N}_\phi(\tau)$ and, correspondingly, two parameters C_a, C_ϕ which should be identified at the final stage.

Let us consider the results of integration in two cases mentioned above.

In the former case (i) the Gauss integral (8) equals

$$(-i)^\infty \exp(-I_i), \quad (10)$$

where the action in the exponent is calculated on the classical trajectory $(a_i(t), \psi_i(t))$, $t \in [0, C]$ with the corresponding boundary condition:

$$a_i(t) = \frac{b}{\mathrm{sh} C}\mathrm{sh} t,\ \psi_i(t) = \frac{\chi}{\mathrm{ch} C}\mathrm{ch} t. \quad (11)$$

As a result, the action is equal to

$$I_i = \frac{1}{2}(-\xi b^2 \mathrm{th} C + \chi^2 \mathrm{cth} C). \quad (12)$$

In the latter case (ii),

$$a_{ii}(t) = \frac{b}{\mathrm{sh} C}\mathrm{sh} t,\ \psi_{ii}(t) = \frac{\chi}{\mathrm{sh} C}\mathrm{sh} t, \quad (13)$$

and

$$I_{ii} = \frac{1}{2}(-\xi b^2 + \chi^2)\mathrm{th} C. \quad (14)$$

Let us now restore the “normal” value of the regularization parameter $\xi = +1$, and return to the real time $C = iT$. As the result, the Euclidean action in the exponent of formula (10) becomes an imaginary phase function, which defines a real phase which we consider as a quantum action corresponding to the birth of the universe. In the former case (ψ is smooth in the South Pole) the quantum action is

$$S_i = \frac{1}{2}(b^2 \mathrm{tg} T + \chi^2 \mathrm{ctg} T). \quad (15)$$

At the last step in our definition of the wave function, we fix the time of birth T using the additional condition of the extreme value of the quantum action:

$$\frac{\partial S_i}{\partial T} = \frac{b^2}{\cos^2 T} - \frac{\chi^2}{\sin^2 T} = 0, \quad (16)$$

from which

$$\mathrm{tg} T = \frac{\chi}{b}, \quad (17)$$

The solution of Eq. (17) can be interpreted as the time of the universe’s birth in the intimated state from “nothing”. The corresponding stationary value of the quantum action is

$$S_{i0} = b\chi. \quad (18)$$

It is easy to check that the stationary wave function

$$\Psi_{i0} = A \exp(ib\chi) \quad (19)$$

is one of the solutions of the WDW equation (5) with $p = 1$.

In the latter case (ϕ is smooth in the South Pole) the corresponding quantum action is

$$S_{ii} = \frac{1}{2}(-b^2 + \chi^2)\mathrm{tg} T. \quad (20)$$

The condition for it to be stationary with respect to the variation of the time of birth T implies $-b^2 + \chi^2 = 0$.

Therefore, the stationary wave function would be taken as

$$\Psi_{ii0} = A\delta(b^2 - \chi^2). \quad (21)$$

It is also a solution of the WDW Eq. (5). The time of this state’s birth is not defined.

Conclusion

In the present paper a regularized definition of the universe’s Hartle-Hawking no-boundary wave function being divergence-free has been proposed. The regularization was achieved by abandoning integration over the lapse and shift functions, the wave function being in the functional-integral representation. This adaptation violates the covariance of the initial theory, so in general, the obtained wave function is not a solution of the Wheeler-DeWitt equation.

This procedure can be interpreted as a complex amplitude of the universe’s birth from “nothing” with the time parameter not defined yet. Considering the phase of the complex amplitude as a quantum equivalent of the classical action, at the last step in our definition of the wave function we proposed to fix the time of birth T using the additional condition of the parameter extremum of the quantum action.

In the present paper, two variants of initial conditions (conditions of smooth) for the scalar field were considered for the uniform model of the universe with the conformally invariant scalar field. The no-boundary wave functions of the universe were obtained in the both cases. These functions were solutions of the WDW equations.

Hence, both solutions have turned to be stationary for the simple model of the universe considered here.

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REFERENCES

[1] **J.B. Hartle, S.W. Hawking,** Wave function of the universe, Phys. Rev. D. D28 (12) (1983) 2960–2975.

[2] **A.D. Linde,** Quantum creation of the inflationary universe, Lettere al Nuovo Cimento. 39 (17) (1984) 401–405.

[3] **J.B. Hartle, S.W. Hawking, T. Hertog,** The no-boundary measure of the universe, arXiv :0711.4630v4[hep-th] (2008).

[4] **A. Vilenkin,** Quantum cosmology and eternal inflation, arXiv:gr-qc/0204061v1 (2002).

[5] **S.W. Hawking, W. Israil,** General relativity: An Einstein centenary survey, Cambridge Univ. Press (1979).

[6] **G.W. Gibbons, S.W. Hawking,** Euclidean quantum gravity, World Scientific (1993).

[7] **J. Govaerts,** Quantum action principle in relativistic mechanics, CERN-TH. 5010/88.

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СПИСОК ЛИТЕРАТУРЫ

1. **Hartle J.B., Hawking S.W.** Wave function of the universe // Phys. Rev. D. 1983. Vol. D28. No. 12. Pp. 2960–2975.

2. **Linde A.D.** Quantum creation of the inflationary universe // Lettere al Nuovo Cimento. 1984. Vol. 39. No. 17. Pp. 401–405.

3. **Hartle J.B., Hawking S.W., Hertog T.** The no-boundary measure of the universe // arXiv :0711.4630v4[hep-th]8 Jun., 2008.

4. **Vilenkin A.** Quantum cosmology and eternal inflation // arXiv:gr-qc/0204061v1 18 Apr., 2002.

5. **Hawking S.W., Israil W.** General relativity: An Einstein centenary survey. Cambridge: Cambridge Univ. Press, 1979.

6. **Gibbons G.W., Hawking S.W.** Euclidean quantum gravity. World Scientific, 1993.

7. **Govaerts J.** Quantum action principle in Relativistic Mechanics. CERN-TH. 5010/88.

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