DOI: 10.18721/JPM.10310
UDC 517.923

OPERATIONAL MATRICES TO SOLVE NONLINEAR RICCATI DIFFERENTIAL EQUATIONS OF AN ARBITRARY ORDER

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In this paper, an effective numerical method to achieve the numerical solution of nonlinear Riccati differential equations of an arbitrary (integer and fractional) order has been developed. For this purpose, the fractional order of the Chebyshev functions (FCFs) based on the classical Chebyshev polynomials of the first kind have been introduced, that can be used to obtain the solution of these equations. Also, the operational matrices of fractional derivative and product for the FCFs have been constructed. The obtained results illustrated demonstrate that the suggested approaches are applicable and valid.

Key words: fractional order of the Chebyshev functions; operational matrix; Riccati differential equations; Galerkin method; differential equation of arbitrary order.

Citation: K. Parand, M. Delkhosh, Operational matrices to solve nonlinear Riccati differential equations of an arbitrary order, St. Petersburg Polytechnical State University Journal. Physics and Mathematics. 10 (3) (2017) 100–115. DOI: 10.18721/JPM.10310

ОПЕРАЦИОННЫЕ МАТРИЦЫ ДЛЯ РЕШЕНИЯ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ РИККАТИ ПРОИЗВОЛЬНОГО ПОРЯДКА

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В статье предложен эффективный численный метод численного решения нелинейных дифференциальных уравнений Риккати произвольного порядка (как целого, так и дробного). Для этого вводится дробный порядок функций Чебышёва на основе классических полиномов Чебышёва первого рода. Такая мера позволяет получать решение этих уравнений Риккати. Построены также операционная матрица дробных производных от функций и операционная матрица произведений ортогональных функций Чебышёва дробного порядка. Результаты применения метода на ряде примеров доказывают, что предлагаемый подход справедлив и достоин применения.

Ключевые слова: дробный порядок функций Чёбышева; операционная матрица; дифференциальные уравнения Риккати; метод Галёркина; дифференциальное уравнение произвольного порядка.

Ссылка при цитировании: Паранд К., Делхош М. Операционные матрицы для решения нелинейных дифференциальных уравнений Риккати произвольного порядка // Научно-технические ведомости СПБГПУ. Физико-математические науки. 2017. Т. 10. № 3. С. 100–115. DOI: 10.18721/JPM.10310

1. Introduction

The Chebyshev polynomials have frequently been used in the numerical analysis including polynomial approximation, Gauss-quadrature integration, integral and differential equations and spectral methods. Chebyshev polynomials have many properties, for example, orthogonal, recursive, simple real roots, complete in the space of polynomials. For these reasons, many researchers have employed these polynomials in their studies [1 – 3]. One of the attractive concepts in the initial and boundary value problems is the differentiation and integration of a fractional order [4, 5]. Many researchers extend classical methods in the studies of differential and integral equations of an integer order to fractional type of these problems [6, 7].

Using some transformations, a number of researchers extended Chebyshev polynomials to a semi-infinite or an infinite domain, for example, by taking

$$x = \frac{t - L}{t + L}, L > 0,$$

the rational Chebyshev functions on the semi-infinite domain [8 – 11], by taking

$$x = \frac{t}{\sqrt{t^2 + L}}, L > 0,$$

the rational Chebyshev functions on the infinite domain [12] being introduced.

In this study, by transformation

$$x = 1 - 2t^{\alpha}, \ \ \alpha > 0,$$

on the Chebyshev polynomials of the first kind, the fractional order of the Chebyshev orthogonal functions in the interval [0, 1] has been introduced. This can be used to solve differential equations of an arbitrary order.

Fractional calculus has a long mathematical history (since 1695 by Hopital [13]), but, for many reasons, it was not used in sciences for many years, for example, the various definitions of the fractional derivative have existed [14] and they have no exact geometrical interpretation [15]. A review of some definition and applications of fractional derivatives are given in Refs. [16] and [17]. In recent years, many physicists and mathematicians have undertaken studies on this subject, and fractional calculus has been employed in various investigations [18, 19]. During the last decades, several methods have been used to solve fractional ordinary/partial differential equations, and fractional integral/integro-differential equations, such as Adomian's decomposition method [20], a fractional order of Legendre functions [21], a fractional order of the Chebyshev functions of the second kind [22], homotopy analysis method [23], the Bessel functions and spectral methods [24], the Legendre and Bernstein polynomials [25], and other methods [26, 27].

One of the most popular differential equations that has been considered mostly in the literature is the Riccati differential equation. There are several applications of this equation in algebraic geometry, theory of conformal mapping, physics and applied problems (see, for example, Ref. [28]). Some researchers have used different methods to solve this type of equations, for examples, Abbasbandy [29] by using homotopy perturbation method, Ranjbar et al. [30] by using enhanced homotopy perturbation method, Cang et al. [31] by using homotopy analysis method, Balaji [32] by using the Legendre wavelet operational matrix method, Parand et al. [33] by using operational matrices method based on the Bernstein polynomials, Li et al. [34] by using the Haar wavelet operational matrix method, Ghomanjani and Khorram [35] by using the Bezier curves method, and Merdan [36] by using the fractional variational iteration method.

The goal of this paper is to present a numerical method (FCF Galerkin method; FCF is the Chebyshev function of a fractional order) to approximate the solution of the nonlinear Riccati differential equation of an arbitrary (integer and fractional) order as follows:

$$D^{\alpha} y(t) + p_1(t) y^2(t) + p_2(t) y(t) = g(t), \quad (1)$$

with n initial conditions:

$$y^{(i)}(t_0) = y_i, i = 0, 1, \ldots, n - 1, \quad (2)$$

where $\alpha = n$; $p_1(t), \ p_2(t), \ g(t) \in L^2([0,1))$ are known functions; $y(t)$ is the unknown function, and D^{α} is the Caputo fractional differentiation operator.

The organization of our paper is as follows: in section 2, some basic definitions and theorems of fractional calculus are presented. In section 3, the FCFs and their properties are obtained. Section 4 is devoted to applying the FCFs operational matrices of fractional derivative and product for obtaining the solution of a fractional differential equation. In section 5, the method of the work is explained. Examples of the applications of the proposed method are given in section 6. Finally, a conclusion is provided.

2. Basic definitions

In this section, some basic definitions and theorems which are useful for our method have been introduced.

Definition 1. *For any real function* $f(t)$, $t > 0$, *if there exists a real number* $p > \mu$, *such that* $f(t) = t^p f_1(t)$, *where* $f_1(t) \in C(0,\infty)$, *is said to be in space* C_μ, $\mu \in \Re$, *and it is in the space* C_μ^n *if and only if* $f^{(n)} \in C_\mu, n \in N$.

Definition 2. *The fractional derivative of* $f(t)$ *in the Caputo sense by the Riemann – Liouville fractional integral operator of an order* $\alpha > 0$ *is defined as follows [37]:*

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D^m f(s) ds,$$

for $m-1 \langle \alpha \le m, m \in N, t \rangle 0$ *and* $f \in C_{-1}^m$.

Some properties of the operator D^a are as follows. For

$$f \in C_\mu, \ \mu \ge -1, \ \alpha,\beta \ge 0, \ \gamma \ge -1,$$

$N_0 = \{0,1,2,\ldots\}$, $c_i \in R$, and constant C:

$$(i) D^\alpha C = 0,$$

$$(ii) D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t), \tag{3}$$

$$(iii) D^\alpha t^\gamma = \begin{cases} 0, \ \gamma \in N_0 \text{ and } \gamma < \alpha; \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \text{ Otherwise.} \end{cases} \tag{4}$$

$$(iv) D^\alpha \left(\sum_{i=1}^n c_i f_i(t) \right) = \sum_{i=1}^n c_i D^\alpha f_i(t). \tag{5}$$

Definition 3. *Suppose that* $f, g \in C(0,1]$ *and* $w(t)$ *is a weight function, then*

$$\|f(t)\|_w^2 = \int_0^1 f^2(t) w(t) dt,$$

$$\langle f(t), g(t) \rangle_w = \int_0^1 f(t) g(t) w(t) dt.$$

Theorem 1. (*Generalized Taylor's formula) Suppose that* $f(t) \in C[0,1]$ *and* $D^{k\alpha} f(t) \in C[0,1]$, *where* $k = 0,1, \ldots, m$, $0 < \alpha \le 1$. *Then we have*

$$f(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha} f(\xi), \tag{6}$$

with $0 < \xi \le t, \ \forall t \in [0,1]$.

And thus

$$\left| f(t) - \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) \right| \le \le M_\alpha \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{7}$$

where $M_\alpha \ge \left| D^{m\alpha} f(\xi) \right|$.

Proof: See Ref. [38].

In the case of $\alpha = 1$, the generalized Taylor's formula (6) is reduced to the classical Taylor's formula.

3. Fractional order of the Chebyshev functions (FCFs)

In this section, first, the fractional order of the Chebyshev functions has been defined, and then some properties and convergence of them for our method have been introduced.

3.1. The FCFs definition. By transformation

$$z = 1 - 2t^\alpha, \ \alpha > 0,$$

on the classical Chebyshev polynomials, the FCFs in the interval [0, 1] are defined, that will be denoted by

$$FT_n^\alpha(t) = T_n(1-2t^\alpha).$$

By this definition, the singular Sturm – Liouville differential equation of the classical Chebyshev polynomials becomes:

$$\frac{\sqrt{1-t^\alpha}}{t^{\frac{\alpha}{2}-1}} \frac{d}{dt} \left[\frac{\sqrt{1-t^\alpha}}{t^{\frac{\alpha}{2}-1}} \frac{d}{dt} FT_n^\alpha(t) \right] + n^2 \alpha^2 FT_n^\alpha(t) = 0, \tag{8}$$

where $t \in [0,1]$ and the FCFs are the eigenfunctions of Eq. (8).

The $FT_n^\alpha(t)$ can be obtained using the recursive relation, as follows $(n \ge 1)$:

$$FT_0^\alpha(t) = 1, FT_1^\alpha(t) = 1 - 2t^\alpha,$$

$$FT_{n+1}^\alpha(t) = (2 - 4t^\alpha) FT_n^\alpha(t) - FT_{n-1}^\alpha(t).$$

Fig. 1 shows graphs of FCFs for various values of n and α.

The analytical form of $FT_n^\alpha(t)$ of the degree $n\alpha$ is given by

$$FT_n^\alpha(t) = \sum_{k=0}^{n}(-1)^k \frac{n2^{2k}(n+k-1)!}{(n-k)!(2k)!} t^{\alpha k} = \\ = \sum_{k=0}^{n} \beta_{n,k}\ t^{\alpha k}, \quad (9)$$

where

$$\beta_{n,k} = (-1)^k \frac{n2^{2k}(n+k-1)!}{(n-k)!(2k)!} \text{ and } \beta_{0,k} = 1.$$

Note that $FT_n^\alpha(0) = 1$ and $FT_n^\alpha(1) = (-1)^n$.

The weight function for the FCFs is

$$w(t) = \frac{t^{\frac{\alpha}{2}-1}}{\sqrt{1-t^\alpha}},$$

and the FCFs with this weight function are orthogonal in the interval [0, 1] that are satisfied in a following relation:

$$\int_0^1 FT_n^\alpha(t) FT_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn}, \quad (10)$$

where δ_{mn} is the Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \ge 1$.

Eq. (10) is provable using the property of orthogonality in the Chebyshev polynomials.

3.2. Approximation of functions. Any function $y(t) \in C[0,1]$ can be expanded as follows:

$$y(t) = \sum_{n=0}^{\infty} a_n FT_n^\alpha(t),$$

where the coefficients a_n are obtained by the inner product:

$$\langle y(t), FT_n^\alpha(t)\rangle_w = \langle \sum_{n=0}^{\infty} a_n FT_n^\alpha(t), FT_n^\alpha(t)\rangle_w,$$

and using the property of orthogonality of the FCFs we have

$$a_n = \frac{2\alpha}{\pi c_n} \int_0^1 FT_n^\alpha(t) y(t) w(t) dt, n \ge 0.$$

In practice, we have to use the first *m* terms of FCFs and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m-1} a_n FT_n^\alpha(t) = A^T \Phi(t), \quad (11)$$

with

$$A = [a_0, a_1, \ldots, a_{m-1}]^T, \quad (12)$$

$$\Phi(t) = [FT_0^\alpha(t), FT_1^\alpha(t), \ldots, FT_{m-1}^\alpha(t)]^T. \quad (13)$$

3.3. Convergence of method. The following theorem shows that by increasing *m*, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 2. *Suppose that*

$$D^{k\alpha} f(t) \in C[0,1] \text{ for } k = 0, 1, \ldots, m,$$

and E_m^α *is the subspace being generated by*

$$\{FT_0^\alpha(t), FT_1^\alpha(t), \ldots, FT_{m-1}^\alpha(t)\}.$$

If $f_m = A^T\Phi$ *is the best approximation to f*

a) *b*)

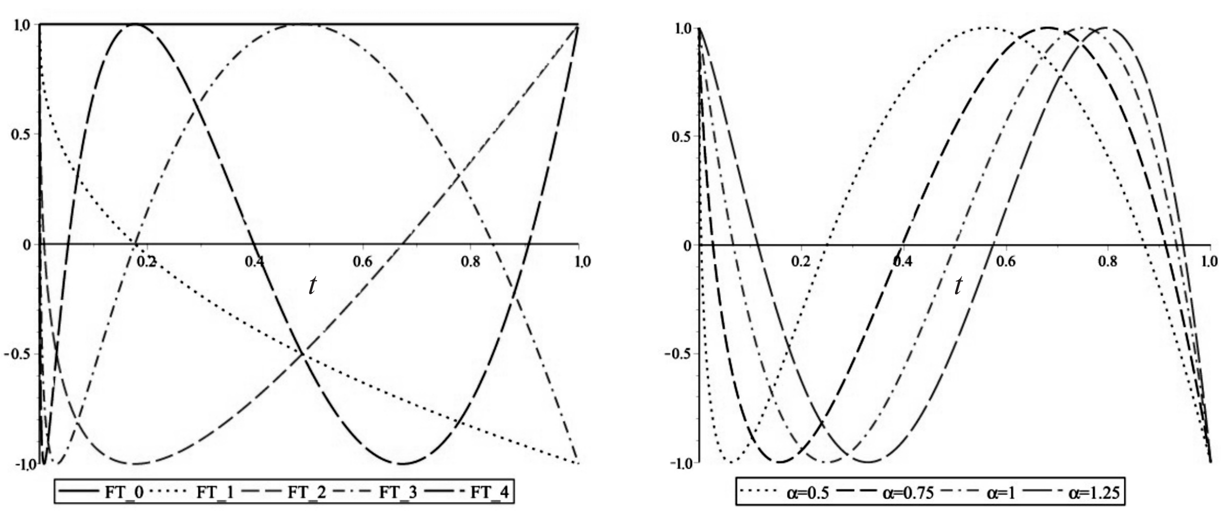

Fig. 1. Graphs of the FCFs with $\alpha = 0.40$ and various values of *n* (*a*), and with $n = 4$ and various values of α (*b*)

from E_m^α, *then the error bound is presented as follows*:

$$\| f(t) - f_m(t) \|_w \leq \frac{M_\alpha}{2^m \Gamma(m\alpha + 1)} \sqrt{\frac{\pi}{\alpha m!}},$$

where $M_\alpha \geq \left| D^{m\alpha} f(t) \right|, t \in [0,1]$.

Proof. By Theorem 1, we have

$$y = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} D^{i\alpha} f(0^+)$$

and

$$\left| f(t) - y(t) \right| \leq M_\alpha \frac{t^{m\alpha}}{\Gamma(m\alpha + 1)}.$$

Since the best approximation to f from E_m^α is $A^T \Phi(t)$, and $y \in E_m^\alpha$, thus

$$\left\| f(t) - f_m(t) \right\|_w^2 \leq \left\| f(t) - y(t) \right\|_w^2 \leq$$

$$\leq \frac{M_\alpha^2}{\Gamma(m\alpha + 1)^2} \int_0^1 \frac{t^{\frac{\alpha}{2} + 2m\alpha - 1}}{\sqrt{1 - t^\alpha}} dt =$$

$$= \frac{M_\alpha^2}{\Gamma(m\alpha + 1)^2} \frac{\pi}{2^{2m} \alpha m!}.$$

The theorem is proved.

4. Operational matrices of FCFs

In this section, operational matrices of fractional derivatives and the product for the FCFs are constructed. These matrices can be used to solve the linear and nonlinear differential equations of an arbitrary order.

4.1. The fractional derivative operational matrix of FCFs. The Caputo fractional derivative operator of an order $\alpha > 0$ of the vector $\Phi(t)$ in the Eq. (13) can be expressed by

$$D^\alpha \Phi(t) = D^{(\alpha)} \Phi(t). \qquad (14)$$

In the following theorem, the operational matrix of fractional derivatives of the FCFs is generalized.

Theorem 3. *Let* $\Phi(t)$ *be FCFs vector in the Eq. (13), and* $D^{(\alpha)}$ *be an* $m \times m$ *operational matrix of Caputo fractional derivatives of an order* $\alpha > 0$, *then:*

$$D_{i,j}^{(\alpha)} = \frac{2}{\sqrt{\pi} c_j} \sum_{k=1}^{i} \sum_{s=0}^{j} \beta_{i,k} \beta_{j,s} \times \frac{\Gamma(\alpha k + 1) \Gamma\left(s + k - \frac{1}{2} \right)}{\Gamma(\alpha k - \alpha + 1) \Gamma(s + k)}, i \neq 0, \qquad (15)$$

$$D_{0,j}^{(\alpha)} = 0, \qquad (16)$$

for $i, j = 0, 1, ..., m - 1$.

Proof. Using Eq. (14), by orthogonality property of FCFs, for $i = 1, 2, ..., m - 1$ and $j = 0, 1, ..., m - 1$, we have

$$D_{i,j}^{(\alpha)} = \frac{2\alpha}{\pi c_j} \int_0^1 D^\alpha FT_i^\alpha(t) FT_j^\alpha(t) w(t) dt = \\ = \frac{2\alpha}{\pi c_j} \int_0^1 \sum_{k=1}^{i} \beta_{i,k} \frac{\Gamma(\alpha k + 1) t^{\alpha k - \alpha}}{\Gamma(\alpha k - \alpha + 1)} \times \\ \times \sum_{s=0}^{j} \beta_{j,s} t^{\alpha s} \frac{t^{\frac{\alpha}{2} - 1}}{\sqrt{1 - t^\alpha}} dt = \\ = \frac{2\alpha}{\pi c_j} \sum_{k=1}^{i} \sum_{s=0}^{j} \beta_{i,k} \beta_{j,s} \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k - \alpha + 1)} \times \\ \times \int_0^1 \frac{t^{\alpha\left(k + s - \frac{1}{2}\right) - 1}}{\sqrt{1 - t^\alpha}} dt. \qquad (17)$$

Now, by integration of the above equation, Eq. (15) can be proved.

And since $D^\alpha FT_0^\alpha(t) = 0$, therefore

$$\int_0^1 D^\alpha FT_0^\alpha(t) FT_j^\alpha(t) w(t) dt = 0,$$

and Eq. (16) can be proved.

The theorem is proved.

Remark 1. The fractional derivative operational matrix of FCFs for $\alpha = 1$ is the same functions as the shifted Chebyshev polynomials [39].

4.2. The product operational matrix of FCFs. The following property of the product of two FCFs vectors will also be applied.

$$\Phi(t) \Phi(t)^T A \approx \hat{A} \Phi(t), \qquad (18)$$

where $\hat{A}$ is an $m \times m$ product operational matrix for the vector $A = \{a_i\}_{i=0}^{m-1}$.

Theorem 4. *Let* $\Phi(t)$ *be FCFs vector in Eq. (13) and* A *be a vector, then the elements of* $\hat{A}$ *are obtained as*

$$\hat{A}_{ij} = \sum_{k=0}^{m-1} a_k \hat{g}_{ijk}, \qquad (19)$$

where

$$\hat{g}_{ijk}=\begin{cases}\dfrac{c_k}{2c_j},\ i\neq 0\,and\ j\neq 0,\\ and\ \ (k=i+j\,or\,k=|i-j|);\\ \dfrac{c_k}{c_j}\ (j=0\,and\,k=i)\\ or\ \ (i=0\,and\,k=j);\\ 0,\ otherwise.\end{cases}$$

Proof. Using Eq. (18), by the orthogonal property Eq. (10) the elements $\{\widehat{A}_{ij}\}_{i,j=0}^{m-1}$ can be calculated from

$$\widehat{A}_{ij}=\frac{2\alpha}{\pi c_j}\sum_{k=0}^{m-1}a_k g_{ijk}, \tag{20}$$

where g_{ijk} is given by

$$g_{ijk}=\int_0^1 FT_i^{\alpha}(t)FT_j^{\alpha}(t)FT_k^{\alpha}(t)w(t)dt.$$

To simplify the g_{ijk}, the following property is used:

$$FT_i^{\alpha}(t)FT_j^{\alpha}(t)=\frac{1}{2}(FT_{i+j}^{\alpha}(t)+FT_{|i-j|}^{\alpha}(t)). \tag{21}$$

By substituting Eq. (21) in g_{ijk}, we have

$$g_{ijk}=\begin{cases}\dfrac{\pi c_k}{4\alpha},\ i\neq 0\ \ and\ j\neq 0,\\ and\ \ (k=i+j\,or\,k=|i-j|);\\ \dfrac{\pi c_k}{2\alpha}\ (j=0\,and\,k=i)\\ or\ \ (i=0\ \ and\ \ k=j);\\ 0,\ otherwise.\end{cases}$$

Now by using Eq. (20), the theorem can be proved.

The theorem is proved.

Remark 2. The product operational matrix of FCFs is the same function as the shifted Chebyshev polynomials [39]. As a whole, it can be said that the components of $\widehat{A}$ are independent of α values.

5. Application of the method

We expand unknown functions $y(t)$, $D^{\alpha}y(t)$ and known functions $p_1(t), p_2(t), g(t)$ as follows:

$$y(t)\approx y_m(t)=\sum_{n=0}^{m-1}a_n FT_n^{\alpha}(t)=A^T\Phi(t), \tag{22}$$

$$D^{\alpha}y(t)\approx\sum_{n=0}^{m-1}a_n D^{\alpha}FT_n^{\alpha}(t)=A^T D^{(\alpha)}\Phi(t), \tag{23}$$

$$p_1(t)\approx\sum_{n=0}^{m-1}p_{1n}FT_n^{\alpha}(t)=B_1^T\Phi(t),$$

$$p_2(t)\approx\sum_{n=0}^{m-1}p_{2n}FT_n^{\alpha}(t)=B_2^T\Phi(t),$$

$$g(t)\approx\sum_{n=0}^{m-1}g_n FT_n^{\alpha}(t)=G^T\Phi(t),$$

and

$$y^2(t)\approx A^T\widehat{A}\Phi(t),$$

$$p_1(t)y^2(t)\approx B_1^T\widehat{A_1}\Phi(t),$$

$$p_2(t)y(t)\approx B_2^T\widehat{A}\Phi(t),$$

where $\widehat{A}_1$ is the product operational matrix of vector $\widehat{A}^T A$.

By substituting the approximations presented above into Eq. (1) we obtain:

$$A^T D^{(\alpha)}\Phi(t)+B_1^T\widehat{A}_1\Phi(t)+ + B_2^T\widehat{A}\Phi(t)=G^T\Phi(t). \tag{24}$$

Now, by multiplying the two sides of Eq. (24) in $\Phi^T(t)$, then integrating in the interval [0, 1], according to orthogonality of FCFs, we get (the Galerkin method):

$$A^T D^{(\alpha)}+B_1^T\widehat{A}_1+B_2^T\widehat{A}=G^T, \tag{25}$$

which is a linear or a nonlinear system of algebraic equations.

Now, for satisfying the initial conditions, we replace *n* equations of these equations (25) with *n* initial conditions (2), and obtain a linear or a nonlinear system with *m* equations and *m* unknowns. By solving this system, the approximate solution of Eq. (1) according to Eq. (22) is obtained.

The residual error function has been defined according to Eqs. (1), (22), and (23) as follows:

$$\mathrm{Res}(t)=A^T D^{(\alpha)}\Phi(t)+p_1(t)y_m^2(t)+ + p_2(t)y_m(t)-g(t). \tag{26}$$

6. Illustrative examples

In this section, by using the proposed method, several nonlinear fractional Riccati differential equations are solved to show the efficiency and applicability of the FCFs method based on the spectral method.

Example 1. Consider the following nonlinear Riccati differential equation [21, 40, 41]:

$$D^{\alpha} y(t) + y^2(t) = 1,\ 0 < \alpha, t \le 1, \tag{27}$$

with the initial condition

$$y(0) = 0. \tag{28}$$

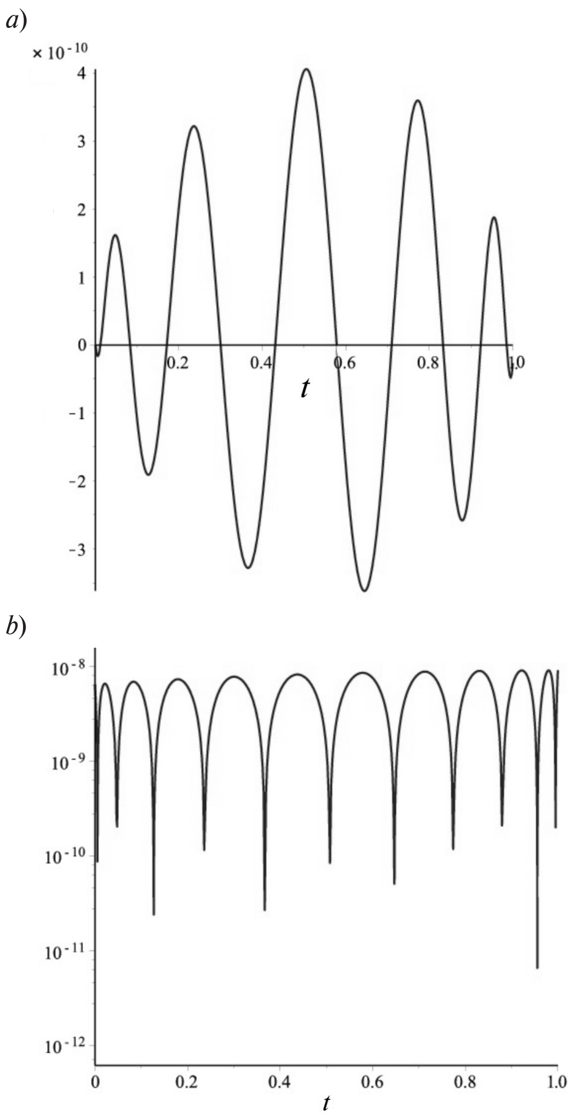

Fig. 2. Obtained graphs of the absolute (a) and the residual (b) error functions with m = 12 and α = 1 (for Example 1)

The exact solution, when $\alpha = 1$, is

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{29}$$

By applying the technique described in the last section, the problem can be converted to the following:

$$(A^T D^{(\alpha)} + A^T \widehat{A})\Phi(t) = G^T \Phi(t),$$

where $\widehat{A}$ is obtained from Eq. (19) and $G^T = [1, 0, 0, \ ..., \ 0]$.

Now, with the replacement of the m-th equation of these equations with the initial condition (28), a set of m nonlinear algebraic equations can be generated, as follows:

$$A^T (D^{(\alpha)} + \widehat{A}) = G^T,$$

$$A^T \Phi(0) = 0.$$

Fig. 2 shows the absolute error of the approximate solution with the exact solution and the residual error for $\alpha = 1$ and m = 12.

Fig. 3 shows the approximate solutions for various values α and m = 10. Definitely, in Fig. 3, a, when α tends to 1, the approximate solutions of $y(t)$ will converge to the exact solution in Eq. (29), and, in Fig. 3, b, when α tends to 0, the approximate solutions of $y(t)$ will converge to the exact solution

$$y(t) = \frac{-1 + \sqrt{5}}{2}.$$

Table 1 shows the residual errors and the obtained values of $y(t)$ by the present method for various values α and m = 12.

Table 2 shows a comparison of obtained values of $y(t)$ by the present method and HPM (see Ref. [41]) for $\alpha = 1$ and m = 12.

In the case with $\alpha = 0.50$ and m = 12 in the Riccati differential equation (27), the approximate solution in a series expansion is obtained as:

$$\begin{aligned} y(t) = {} & 1.1283789766\sqrt{t} + 0.0000436003t - \\ & - 0.9595868217t^{3/2} + 0.0298952318t^2 + \\ & + 1.0378491665t^{5/2} + 1.3663547362t^3 - \\ & - 6.3882854589t^{7/2} + 8.7043955759t^4 - \\ & - 6.1900399882t^{9/2} + 2.3468978237t^5 - \\ & - 0.3771636132t^{11/2}. \end{aligned}$$

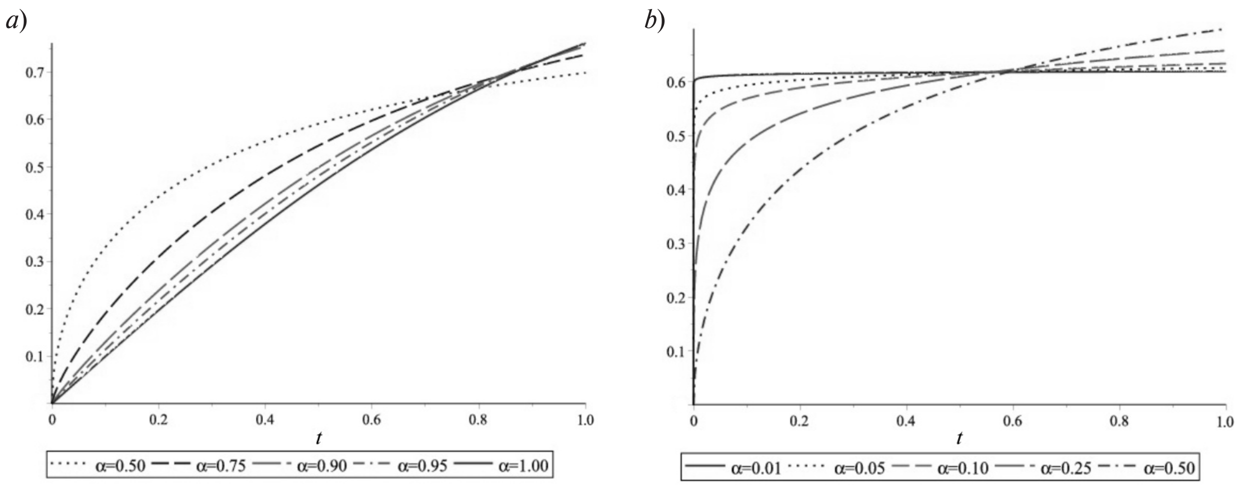

Fig. 3. Obtained graphs of the approximate solutions with $m = 10$ and the various values of α: when α tends to 1 (*a*) and to 0 (*b*) (for Example 1)

Table 1

Values of $y(t)$ obtained by the present method with $m = 12$ (for Example 1)

t	α = 0.50		α = 0.90		α = 1.00		
	Approximate solution	Residual error	Approximate solution	Residual error	Approximate solution	Absolute error	Residual error
0.0	0.00000000	0.00e−0	0.00000000	0.00e−0	0.00000000	0.00e−00	0.00e−0
0.1	0.33010841	4.52e−8	0.13003745	2.44e−9	0.09966799	1.11e−10	5.60e−9
0.2	0.43683875	5.94e−8	0.23878913	2.77e−9	0.19737532	2.04e−10	6.16e−9
0.3	0.50488936	4.06e−8	0.33596217	1.72e−8	0.29131261	2.10e−12	7.85e−9
0.4	0.55378188	1.30e−7	0.42258308	3.40e−8	0.37994896	2.23e−10	5.59e−9
0.5	0.59119411	6.50e−8	0.49913519	2.39e−8	0.46211715	4.03e−10	1.34e−9
0.6	0.62101362	8.59e−8	0.56617156	8.20e−9	0.53704956	1.79e−10	7.61e−9
0.7	0.64548540	1.07e−7	0.62439622	3.18e−8	0.60436777	8.59e−11	8.46e−9
0.8	0.66601875	7.7e−10	0.67462699	3.34e−8	0.66403677	2.70e−10	5.82e−9
0.9	0.68355221	7.44e−8	0.71773475	3.13e−8	0.71629787	1.89e−10	5.96e−9
1.0	0.69873922	1.11e−7	0.75458880	3.44e−8	0.76159415	2.66e−11	9.21e− 9

Table 2

Comparison of obtained values of $y(t)$ with α = 1(for Example 1)

t	HPM [41]	Present method	Exact solution	Absolute error	Residual error
0.1	0.099668	0.0996679945	0.0996679946	1.11e−10	5.60e−9
0.2	0.197375	0.1973753204	0.1973753202	2.04e−10	6.16e−9
0.3	0.291312	0.2913126124	0.2913126124	2.10e−12	7.85e−9
0.4	0.379944	0.3799489620	0.3799489622	2.23e−10	5.59e−9
0.5	0.462078	0.4621171576	0.4621171572	4.03e−10	1.34e−9
0.6	0.536857	0.5370495668	0.5370495669	1.79e−10	7.61e−9
0.7	0.603631	0.6043677770	0.6043677771	8.59e−11	8.46e−9
0.8	0.661706	0.6640367705	0.6640367702	2.70e−10	5.82e−9
0.9	0.709919	0.7162978700	0.7162978701	1.89e−10	5.96e−9
1.0	0.746032	0.7615941559	0.7615941559	2.66e−11	9.21e−9

Note. HPM – the Homotopy Pertubation Method.

Example 2. Consider the following nonlinear Riccati differential equation [21, 40, 41] that has the form

$$D^{\alpha} y(t) + y^2(t) - 2y(t) = 1, \quad 0 < \alpha, t \le 1, \tag{30}$$

with the initial condition

$$y(0) = 0. \tag{31}$$

The exact solution, when $\alpha = 1$, is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{32}$$

By applying the technique described in the last section, the problem can be converted to

$$(A^T D^{(\alpha)} + A^T \widehat{A} - 2A^T)\Phi(t) = G^T \Phi(t),$$

where $\widehat{A}$ is obtained from Eq. (19), and

$$G^T = [1, 0, 0, \ldots, 0].$$

Now, with the replacement of the m-th equation of these equations with the initial condition (31), a set of m nonlinear algebraic equations can be generated as follows:

$$A^T (D^{(\alpha)} + \widehat{A} - 2I) = G^T,$$

$$A^T \Phi(0) = 0.$$

Fig. 4 shows the absolute error of the approximate solution with respect to the exact one and the residual error for $\alpha = 1$ and $m = 30$.

Fig. 5 shows the approximate solutions for various values of α and m = 12. Definitely, in Fig. 5, *a*, when α tends to 1, the approximate solutions of $y(t)$ will converge to the exact solution of Eq. (32), and, in Fig. 5, *b*, when α tends to 0, the approximate solutions of $y(t)$ will converge to the exact solution

$$y(t) = \frac{1+\sqrt{5}}{2}.$$

Table 3 shows the residual errors and the obtained values of $y(t)$ by the present method for various α values.

Table 4 shows a comparison of obtained values of $y(t)$ by the present method and by HPM (see Ref. [41]) for $\alpha = 1$ and $m = 30$.

Example 3. Consider the following nonlinear Riccati differential equation that has the form

$$D^{\alpha} y(t) - y^2(t) + e^t y(t) = e^t,$$

$$0 < \alpha \le 2, \; 0 \le t \le 1, \tag{33}$$

with initial conditions

$$y(0) = 1, \; y'(0) = 1 \; (\text{if } \alpha > 1). \tag{34}$$

The exact solution, when $\alpha = 2$ and $\alpha = 1$, is

$$y(t) = e^t. \tag{35}$$

By applying the technique described in the last section, the problem can be converted to

$$(A^T D^{(\alpha)} - A^T \widehat{A} + B_2^T \widehat{A})\Phi(t) = G^T \Phi(t),$$

where $\widehat{A}$ is obtained from Eq. (19).

Now, with the replacement of the two last

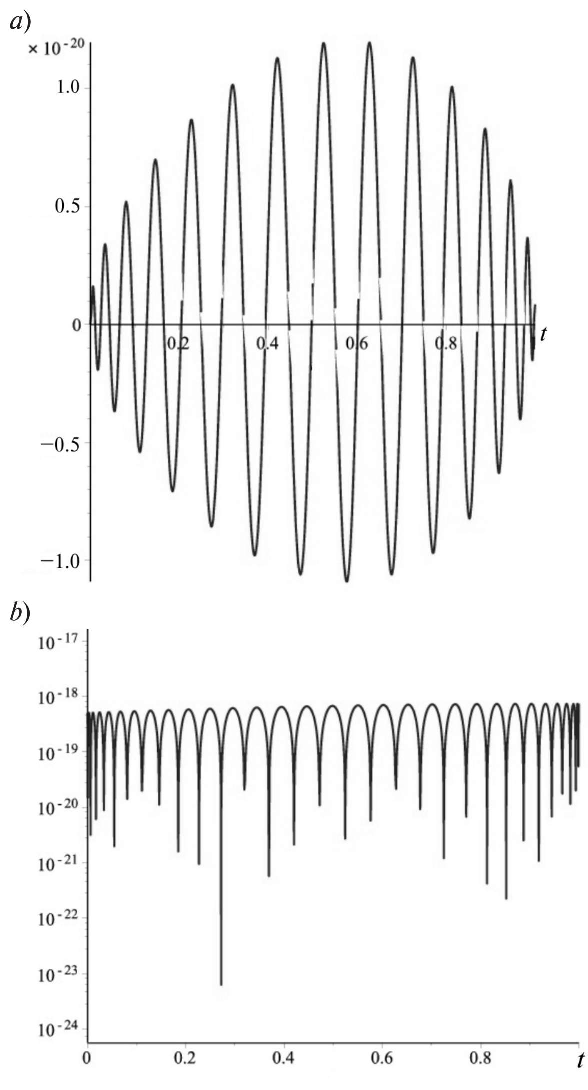

Fig. 4. Obtained graphs of the absolute (*a*) and the residual (*b*) error functions with $m = 30$ and $\alpha = 1$ (for Example 2)

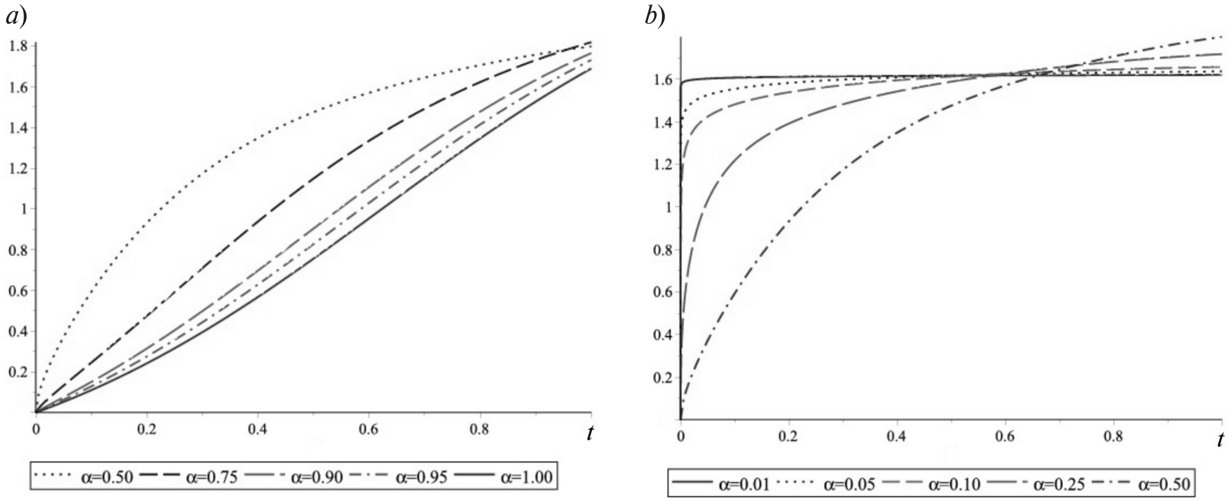

Fig. 5. Obtained graphs of the approximate solutions (*a*) and the residual error functions (*b*) with $m = 12$ and the various values of α: when α tends to 1 (*a*) and to 0 (*b*) (for Example 2)

Table 3

Values of $y(t)$ obtained by the present method (for Example 2)

t	$\alpha = 0.75$ ($m = 15$)		$\alpha = 0.90$ ($m = 15$)		$\alpha = 1.00$ ($m = 30$)		
	Approximate solution	Residual error	Approximate solution	Residual error	Approximate solution	Absolute error	Residual error
0.0	0.00000000	1.82e−5	0.00000000	5.95e−8	0.00000000	0.00e−00	5.07e−19
0.1	0.24543249	9.80e−6	0.15070989	5.93e−8	0.11029519	2.40e−21	4.79e−19
0.2	0.47509479	4.56e−6	0.31486440	1.74e−8	0.24197679	2.51e−21	5.36e−19
0.3	0.71002417	1.20e−5	0.49866532	1.31e−8	0.39510484	3.23e−21	5.90e−19
0.4	0.93853496	1.83e−5	0.69753897	3.40e−8	0.56781216	3.96e−21	6.14e−19
0.5	1.14906032	1.21e−5	0.90366760	6.32e−8	0.75601439	1.69e−21	6.74e−19
0.6	1.33433341	4.40e−6	1.10786162	8.52e−8	0.95356621	9.35e−21	6.95e−19
0.7	1.49192213	1.66e−5	1.30143258	9.38e−8	1.15294896	6.26e−21	7.15e−19
0.8	1.62298951	1.76e−5	1.47770301	9.52e−8	1.34636365	5.69e−21	6.15e−19
0.9	1.73060956	1.67e−5	1.63273978	6.72e−8	1.52691131	3.33e−21	6.89e−19
1.0	1.81851003	1.86e−5	1.76527518	9.64e−8	1.68949839	8.45e−21	7.38e−19

Table 4

Comparison of obtained values of $y(t)$ with $\alpha = 1$ (for Example 2)

t	HPM [41]	Present method	Exact solution	Absolute error	Residual error
0.1	0.110294	0.11029519691696228095	0.11029519691696228096	2.40e−21	4.79e−19
0.2	0.241965	0.24197679962110923224	0.24197679962110923224	2.51e−21	5.36e−19
0.3	0.395106	0.39510484866037839343	0.39510484866037839343	3.23e−21	5.90e−19
0.4	0.568115	0.56781216629293854988	0.56781216629293854987	3.96e−21	6.14e−19
0.5	0.757564	0.75601439343137566624	0.75601439343137566624	1.69e−21	6.74e−19
0.6	0.958259	0.95356621647192273865	0.95356621647192273865	9.35e−21	6.95e−19
0.7	1.163459	1.15294896697962321762	1.15294896697962321762	6.26e−21	7.15e−19
0.8	1.365240	1.34636365536837509274	1.34636365536837509274	5.69e−21	6.15e−19
0.9	1.554960	1.52691131328062418721	1.52691131328062418721	3.33e−21	6.89e−19
1.0	1.723810	1.68949839159438298686	1.68949839159438298686	8.45e−21	7.38e−19

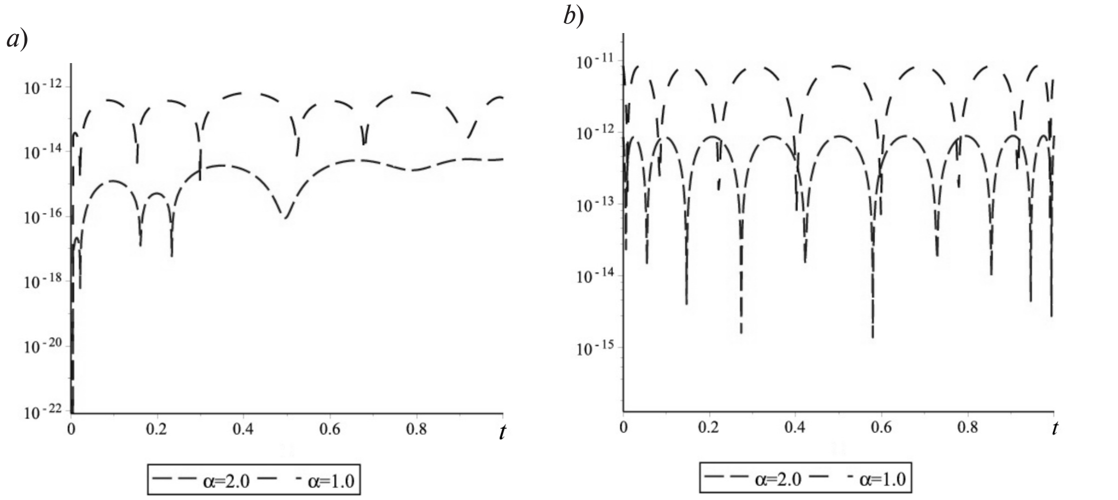

Fig. 6. Obtained graphs of the absolute (*a*) and the residual (*b*) errors with $m = 12$, $\alpha = 1$ and $\alpha = 2$ (for Example 3)

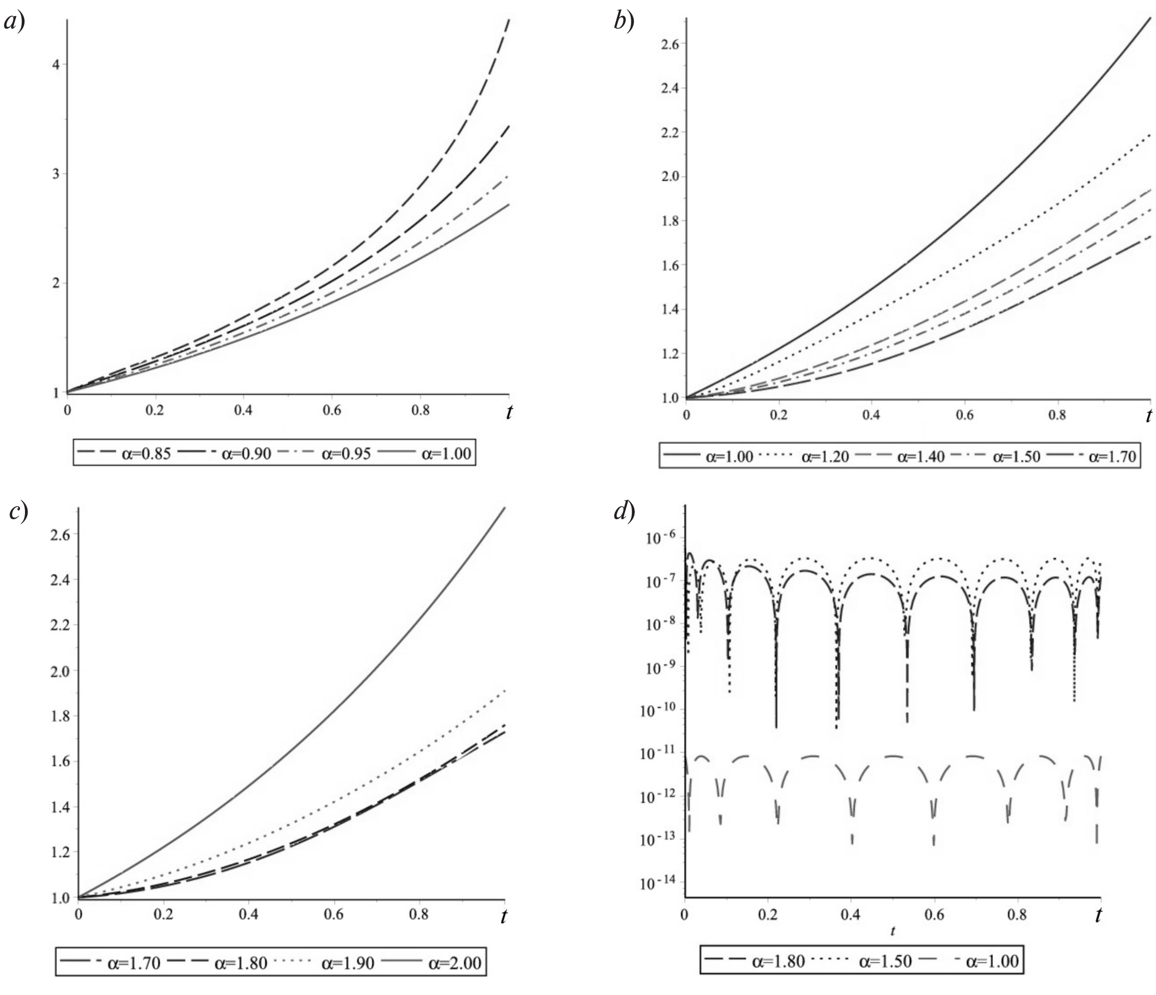

Fig. 7. Obtained graphs of the approximate solutions with $m = 10$ ($a - c$) and the residual errors with $m = 12$ (*d*) for various values of α: $0 \le \alpha \le 1.0$ (*a*), $1.0 \le \alpha \le 1.7$ (*b*), $1.7 \le \alpha \le 2.0$ (*c*), $1.00 \le \alpha \le 1.80$ (*d*) (for Example 3)

Table 5

Values of $y(t)$ with m = 12 obtained by the present method (for Example 3)

t	α = 1.80		α = 1.50		α = 1.00	
	Approximate solution	Residual error	Approximate solution	Residual error	Approximate solution	Residual error
0.0	1.0000000000	0.00e−0	1.0000000000	0.00e−0	1.0000000000	0.00e−00
0.1	1.0235085766	2.33e−8	1.0247048727	6.95e−8	1.1051709180	3.51e−12
0.2	1.0595333960	8.59e−8	1.0697960272	1.34e−7	1.2214027581	3.51e−12
0.3	1.1076128039	1.63e−7	1.1293315559	3.17e−7	1.3498588075	8.22e−12
0.4	1.1674004066	8.30e−8	1.2014933888	2.15e−7	1.4918246976	3.31e−12
0.5	1.2387187663	8.14e−8	1.2853141729	1.67e−7	1.6487212707	8.31e−12
0.6	1.3214261255	1.19e−7	1.3800725660	3.14e−7	1.8221188003	3.31e−12
0.7	1.4153208915	1.19e−8	1.4850282438	5.83e−8	2.0137527074	8.22e−12
0.8	1.5200543734	8.75e−8	1.5992421055	2.20e−7	2.2255409284	3.51e−12
0.9	1.6350374400	1.13e−7	1.7214121635	3.04e−7	2.4596031111	3.51e−12
1.0	1.7593322223	1.21e−7	1.8496977803	3.39e−7	2.7182818284	8.31e−12

equations of these equations with the initial conditions (34), a set of m nonlinear algebraic equations can be generated as follows:

$$(A^T D^{(\alpha)} - A^T \widehat{A} + B_2^T \widehat{A}) = G^T,$$

$$A^T \Phi(0) = 1,$$

$$A^T D^{(1)} \Phi(0) = 1, \ \mathit{if}\ \alpha > 1.$$

Fig. 6 shows the absolute errors of the approximate solutions with respect to the exact solution and the residual errors for $\alpha = 1$ and $\alpha = 2$ with $m = 12$.

Fig. 7 shows the approximate solutions for the various values

$$0 \le \alpha \le 1.0, \ 1 \le \alpha \le 1.7, \ \text{and} \ 1.7 \le \alpha \le 2.0$$

with $m = 10$.

Definitely, when α tends to 1, from the left-hand side (Fig. 7, *a*), the approximate solutions of $y(t)$ will converge to the exact one in Eq. (35), and when α tends to 1, from the right-hand side (Fig. 7, *b*), the approximate solutions of $y(t)$ will converge to the exact solution in Eq. (35), and when α tends to 2, from the left-hand side (Fig. 7, *c*), the approximate solutions of $y(t)$ will converge to the exact solution in Eq. (35). As can be seen, for α from 1.0 to about 1.7, the graph of the function is moving from $\alpha = 1.0$ to $\alpha = 1.7$ (Fig. 7, *b*), and then the graph of the function is returning to $\alpha = 2.0$ (Fig. 7, *c*). Fig. 7, *d* shows the residual errors for various values α with $m = 12$.

Table 5 shows the residual errors and the obtained values of $y(t)$ by the present method for various values α and $m = 12$.

7. Conclusion

In this paper, we have introduced the fractional order of the Chebyshev functions of the first kind. Then the operational matrices of fractional derivatives and the product of these orthogonal functions have been obtained. These matrices can be used to solve the linear and nonlinear differential equations, as well as the nonlinear Riccati differential equations of an arbitrary (integer and fractional) order. As it has been shown, the method is converging and has an approximate accuracy and stability. Illustrative examples have shown that this method has good results and suitable accuracy in comparison to other methods.

Acknowledgments

The authors are very grateful to the reviewers and the editor for careful reading of the paper and for their comments and suggestions which have improved it.

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Received 05.03.2017, accepted 20.07.2017.

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Статья поступила в редакцию 05.03.2017, принята к публикации 20.07.2017.

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