+

Magazine of Civil Engineering

ISSN 2712-8172

journal homepage: http://engstroy.spbstu.ru/

DOI: 10.18720/MCE.100.1

# Frictional contact problem in building constructions analysis

***A.N. Popov*, A.D. Lovtsov***
*Pacific National University, Khabarovsk, Russia*
** E-mail: sanyapov@mail.ru*

**Keywords:** building constructions, contact nonlinearity, frictional contact, unilateral constraints, linear complementarity problem, numerical models, finite element method

**Abstract.** The article discusses the contact interaction of deformable building constructions or their parts. Such interaction is realized for example at: hydraulic structures; suspension pile foundations, girder, raft, sheet piling; friction bearings and kinematic bearings of seismically insulated buildings, etc. The subject of the study is the formulation of the contact interaction problem as a linear complementarity problem. Such formulation of the problem allows the use of effective step-by-step algorithms and provides minimum qualification requirements to user. Expansion of existing formulations of the problems of frictionless contact and contact with the known friction bound in the form of linear complementarity problem to the formulation of the frictional contact is offered. Eventually, a heuristic formulation of the contact problem with friction is obtained in the form of a linear complementarity problem. The problem is solved by the Lemke step-type algorithm in the form of a displacement method. The results of the solutions obtained on test problems and on the Ansys software almost coincide with the results obtained by the proposed algorithm.

## *1. Introduction*

The object of this study is the contact interaction of deformable building constructions or their parts (suspension pile foundations; girder; raft; sheet piling; friction bearings and kinematic bearings of seismically insulated buildings; pipelines with frictional contact on supports). One of the important problems of strength analysis in the calculation of buildings is the problem of determining the contact interaction forces between deformable bodies [1–9]. The new formulation of the frictional contact problem is proposed in the form of a linear complementarity problem. In the case of solving contact problems with Coulomb friction, three main statements can be distinguished:

1. Frictionless contact when contact zone and separation zone of the contacting parts of the structure are unknown. The problem is formulated as a conditional minimization of a differentiable functional [10–13].

2. A problem with known friction bound, when the contact zone is known, but the cohering zone and the slippage zone are unknown. In this case, the problem is posed as a variational inequality, which can be reduced to unconditional minimization of a non-differentiable functional [1, 2, 7, 14–18].

3. The general case is when the contact zone, and inside it is the zone of cohering and slip are unknown in advance. In this case, the problem is also put in the form of a variational inequality, but there is no formulation of the problem as a minimum of the energy functional.

In civil engineering the most commonly used numerical methods for solving contact problems in the form of variational inequalities are method of Lagrange Multipliers, the Penalty Function Method and their modifications [10, 11, 19–23]. These methods are implemented in most software tools for calculating buildings that can solve contact problems [24–26]. It is also important to mention alternative methods for solving contact problems: 1) group of methods using contact finite elements [27–30]; 2) group using algorithms [14, 15, 31–34] of linear complementarily problem (LCP); 3) other methods [8, 35, 36].

The purpose of this work is to set new formulation of the frictional contact problem of deformable bodies in the LCP form. The tasks can be divided:

Popov, A.N., Lovtsov, A.D. Frictional contact problem in building constructions analysis. Magazine of Civil Engineering. 2020. 100(8). Article No. 10001. DOI: 10.18720/MCE.100.1

1. To expand the formulations of frictionless contact and contract with friction bound obtained by the authors of this article for the general case of frictional contact;
2. To propose a method (algorithm) for calculating such problems;
3. Test Algorithm;
4. To compare solutions obtained using known and proposed algorithms.

## 2. Methods

In the following formulation of the calculation of contact problems with friction – contact "node to node" is used. These nodes will be referred to as the contact pair. It is assumed that in each contact pair the points are connected by unilateral joints. Joint, normal to the contact zone is turned on when these points are in contact and turned off otherwise. It is supposed that unilateral joints work only in compression in the normal direction. Joint, tangential to the contact zone is turned on if the interaction forces are less than the friction bound and turned off if the interaction forces are equal to the friction bound. That is, when the joint is turned on, the slipping of the points of the contact pair is impossible, and when it is off, it is possible.

The following signs` rule is accepted:

– For forces and displacements normal to the contact surface: the compressive force of interaction of points of a contact pair $x_{\mathrm{n}i} > 0$; mutual removal of points of a contact pair $z_{\mathrm{n}i} > 0$ (Fig. 1, *a*).

– For forces and displacements tangentially to the contact surface: if the points of the contact pair conditionally spread apart by normal to the contact zone, then interaction forces $x_{\tau i} > 0$ will create a couple of forces with the moment in the clockwise direction; mutual displacement $z_{\tau i} > 0$, if it coincides in direction with $x_{\tau i} > 0$ (Fig. 1, *b*).

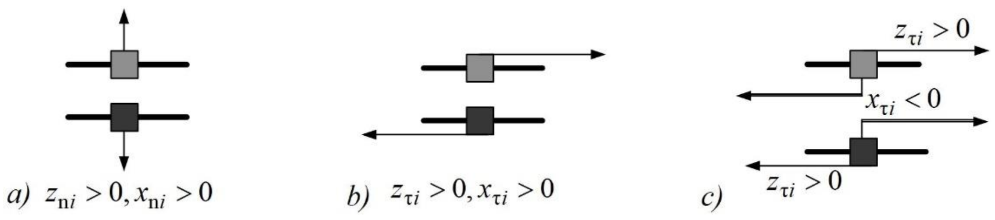


**Figure 1. Unilateral constraints. The signs` rule for the interaction forces $x$ and mutual displacements $z$.**

In [14], cases of frictionless contact and contact with known friction bound are reduced to linear complementarity problems, and the displacement method (stiffness method) of their solution have been developed.

To implement the displacement method, it is necessary to form a contact stiffness matrix (CSM) and a contact load vector (CLV) in the main system of mentioned method. The main system of the displacement method is obtained from a given system by transformation of all unilateral joints (along the normal and tangential to the contact zone in each contact pair) into bilateral joints (Fig. 2, nodes are spaced for visibility). The component $R_{i,j}$ of the CSM is the effort in the main system of the displacement method in the entered joint $i$ from a unit dislocation in the direction of the entered joint $j$. Component $R_{Fi}$ of the CSM is an effort in the entered constraint $i$ from external load.

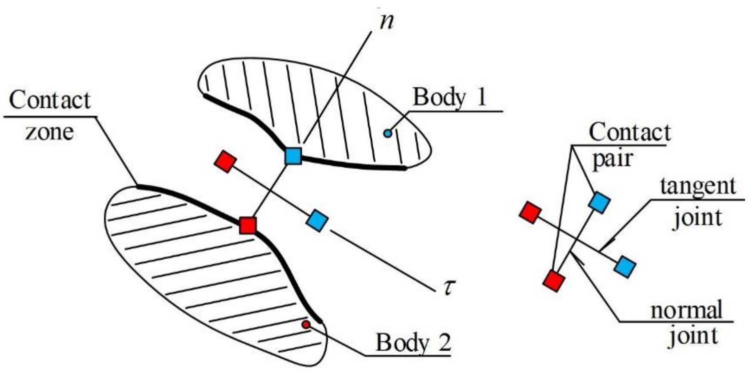


**Figure 2. Constraints in the main system of the displacement method.**

**Frictionless contact.** In this case, it is not necessary to enter joints tangentially to the contact zone. Thus, to form the main system, you should enter bilateral joints only normal to the contact zone in each contact pair. Therefore, to describe the stress-strain state (SSS) of the contact pair $i$ of a given system, two non-negative variables are enough $x_{ni}$, $z_{ni}$. LCP for the unilateral frictionless contact has the form:

$$\mathbf{x_n} = \mathbf{R_{nn}} \cdot \mathbf{z_n} + \mathbf{R_{Fn}};$$
$$\mathbf{z_n} \geq 0; \quad \mathbf{x_n} \geq 0; \quad \mathbf{z_n}^T \cdot \mathbf{x_n} = 0, \tag{1}$$

where $\mathbf{R_{nn}}$ is CSM $[m \times m]$ for joints in contact pairs along the normal to the contact zone from a unit dislocation of the nodes of contact pairs along the normal to the expected contact zone $L$ (Fig. 3); $\mathbf{x_n}$, $\mathbf{z_n}$ are vectors $[m \times 1]$ of interaction forces and mutual displacements of contact pairs along the normal to the contact zone; $\mathbf{R_{Fn}}$ is CLV $[m \times 1]$ for constraints normal to the contact zone; m is total number of contact pairs.

Frictional contact with known friction bound. In the works [16–18] the cases are considered for which contact with the known friction bound is realized. The calculation of such systems is reduced to the problem of unconditional optimization of a non-differentiable functional. In [14, 37], such problems, in turn, are reduced to the conditional optimization of a differentiable functional, conditions of Kuhn-Tucker which lead to LCP of the form:

$$\begin{bmatrix} \mathbf{x_\tau^+} \\ \mathbf{x_\tau^-} \end{bmatrix} = \begin{bmatrix} \mathbf{R_{\tau\tau}} & -\mathbf{R_{\tau\tau}} \\ -\mathbf{R_{\tau\tau}} & \mathbf{R_{\tau\tau}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{z_\tau^+} \\ \mathbf{z_\tau^-} \end{bmatrix} + \begin{bmatrix} \mathbf{R_{F\tau}} \\ -\mathbf{R_{F\tau}} \end{bmatrix} + \begin{bmatrix} \mathbf{r_f} \\ \mathbf{r_f} \end{bmatrix};$$
$$\mathbf{x_\tau^+} \geq 0; \quad \mathbf{x_\tau^-} \geq 0; \quad \mathbf{z_\tau^+} \geq 0; \quad \mathbf{z_\tau^-} \geq 0; \quad \mathbf{z_\tau^{+T}} \cdot \mathbf{x_\tau^+} = 0; \quad \mathbf{z_\tau^{-T}} \cdot \mathbf{x_\tau^-} = 0; \tag{2}$$

where $\mathbf{R_{\tau\tau}}$ is CSM $[m \times m]$ for joints entered in contact pairs *tangentially* to the contact zone from a unit dislocation of contact pairs *tangential* to the contact zone. $\mathbf{R_{F\tau}}$ is CLV for the joints tangential to the contact zone; $\mathbf{x_\tau} = \left(\mathbf{x_\tau^+} - \mathbf{x_\tau^-}\right)/2$ is vector $[m \times 1]$ of efforts of interaction of contact pairs tangentially to the contact zone (Fig. 3, where $q'', q'$ are external loads); $\mathbf{z_\tau} = \mathbf{z_\tau^+} - \mathbf{z_\tau^-}$ is vector $[m \times 1]$ of mutual displacements of contact pairs tangentially to the contact zone; $\mathbf{r_f} = f \cdot \mathbf{x_n}$ is vector of friction bound; $f$ is coefficient of friction between contact pairs; $\mathbf{x_n}$ is pressing force vector.

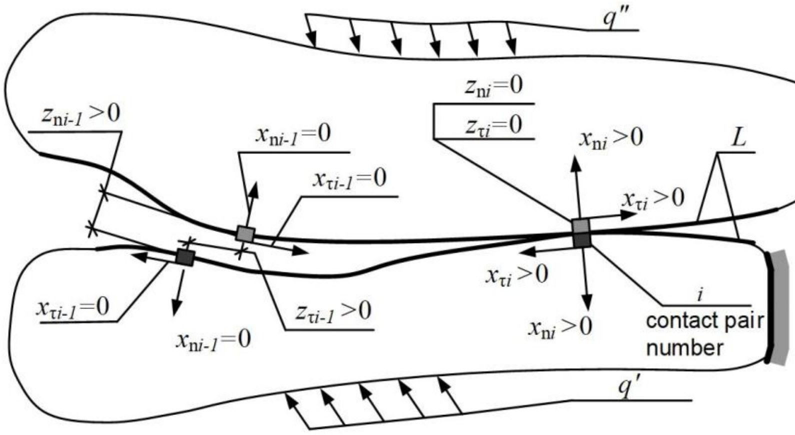


**Figure 3. Contact of deformable bodies after deformation.**
**Separation of *i*-1 contact pair and touch of *i* contact pair.**

Note that to describe the SSS of a contact pair $i$ of a given system, four non-negative variables are needed: $x_{\tau i}^+$, $x_{\tau i}^-$, $z_{\tau i}^+$, $z_{\tau i}^-$, since mutual displacements $z_{\tau i}$ and forces $x_{\tau i}$ have no sign limitation.

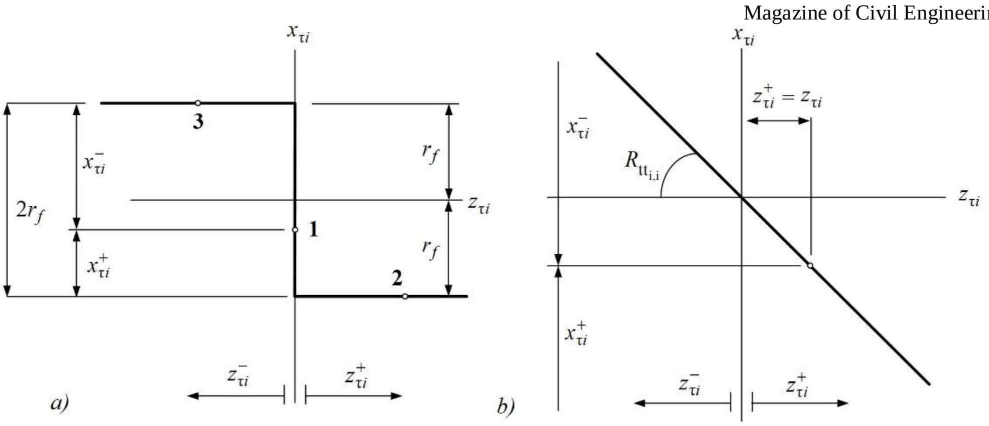

**Figure 4. The dependence of the interaction forces and mutual displacements of the contact pair i: a) in a given system; b) in the main system.**

The variable $x_{\tau i}^{+}$ can be understood as the “cohering reserve” of the contact pair $i$ with the tendency (attempt) to shift from the catenation state in the direction $z_{\tau i} > 0$ (see sign` rule in Fig. 1). This case is shown in Fig. 4.

Fig. 4, a point 1. The variable $x_{\tau i}^{-}$ is the “cohering reserve” of the contact pair $i$ when attempting to shift from the catenation state towards $z_{\tau i} < 0$. The variables $z_{\tau i}^{+}$ and $z_{\tau i}^{-}$ in turn are the mutual displacements of the points of the contact pair $i$ in the positive and negative directions, respectively. That is, with a positive mutual displacement of $z_{\tau i} > 0$, we have $z_{\tau i}^{+} > 0$, $z_{\tau i}^{-} = 0$ and $z_{\tau i} = z_{\tau i}^{+}$, and with a negative $z_{\tau i} < 0$ we have $z_{\tau i}^{-} > 0$, $z_{\tau i}^{+} = 0$ and $z_{\tau i} = -z_{\tau i}^{-}$.

In this way, with exhaustion of the reserve $x_{\tau i}^{+} = 0$ friction force in the contact pair is equal to friction bound $x_{\tau i}^{+} = r_{\mathrm{fi}}$ and slippage occurs in the direction $z_{\tau i} > 0$, i.e. variable $z_{\tau i}^{+} > 0$ (point 2 on Fig. 4).

Fig. 4, a: when reserve $x_{\tau i}^{-}$ is exhausted the situation is similar (point 3 on Fig. 4).

Let us turn further to the main system of the displacement method for the problem under consideration. It is not required to enter constraints normal to the contact zone. The main displacement method system is obtained by converting all unilateral friction constraints into bilateral. Thus, the slipping of points of a contact pair in the main system is excluded.

Vectors $\mathbf{x}_{\boldsymbol{\tau}}, \mathbf{z}_{\boldsymbol{\tau}}$ in the main system are connected by the ratio $\mathbf{x}_{\boldsymbol{\tau}} = -\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}$ (taking into account accepted signs` rule Fig. 1, b and Fig. 1, c). Similar ratios are connecting the vectors $\mathbf{x}_{\boldsymbol{\tau}}^{+}, \mathbf{x}_{\boldsymbol{\tau}}^{-}, \mathbf{z}_{\boldsymbol{\tau}}^{+}, \mathbf{z}_{\boldsymbol{\tau}}^{-}$.

Figure 4, b, when $z_{\tau i}^{+}$ is increasing: $x_{\tau i}^{+}$ is decreasing and $x_{\tau i}^{-}$ is increasing; when $z_{\tau i}^{-}$ is increasing: $x_{\tau i}^{+}$ is increasing and $x_{\tau i}^{-}$ is decreasing):

$$\mathbf{x}_{\boldsymbol{\tau}}^{+} = -(-\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{+}) + (-\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{-}) = +\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{+} - \mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{-};$$
$$\mathbf{x}_{\boldsymbol{\tau}}^{+} = +(-\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{+}) - (-\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{-}) = -\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{+} + \mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{-}.$$

If we also take into account external load and friction bound, then:

$$\begin{aligned} \mathbf{x}_{\boldsymbol{\tau}}^{+} &= \mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{+} - \mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{-} + \mathbf{R}_{\mathbf{F}\boldsymbol{\tau}} + \mathbf{r}_{\mathbf{f}}, \\ \mathbf{x}_{\boldsymbol{\tau}}^{-} &= -\mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{+} + \mathbf{R}_{\boldsymbol{\tau\tau}} \cdot \mathbf{z}_{\boldsymbol{\tau}}^{-} - \mathbf{R}_{\mathbf{F}\boldsymbol{\tau}} + \mathbf{r}_{\mathbf{f}}, \end{aligned} \tag{3}$$

which coincides with the system of equations in (2).

Thus, in both cases considered above, the LCP can be formed directly using the main system of the displacement method. It is further proposed to use this approach.

**General case of frictional contact.** In the general case of friction contact, the formulation of the problem in form of conditional extremum of some functional is not gotten. Therefore, based on “plausible” reasoning, we propose further setting up a general case of friction contact in the form of a LCP, written for a discretized system. In this case, for the formation of the main system, you should enter bilateral joints along the normal and tangential to the contact zone for each contact pair.

In this case, for the formation of the main system, it is needed to enter bilateral joints along the normal and tangential to the contact zone for each contact pair. To describe the SSS of the contact pair $i$ of a given system, we use six non-negative variables $x_{\mathrm{n}i}, z_{\mathrm{n}i}, x_{\tau i}^{+}, x_{\tau i}^{-}, z_{\tau i}^{+}, z_{\tau i}^{-}$. It is clear that in such a main system, the force $\mathbf{x_n}$ depends not only on $\mathbf{z_n}$, but also on $\mathbf{z_\tau}$:

$$\mathbf{x_n} = \mathbf{R_{nn}} \cdot \mathbf{z_n} + \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau} + \mathbf{R_{Fn}},$$

where $\mathbf{R_{n\tau}}$ is CSM $[m \times m]$ for normal joints from unit dislocations tangential to the contact zone.

Or, given that: $\mathbf{z_\tau} = \mathbf{z_\tau^+} - \mathbf{z_\tau^-}$ :

$$\mathbf{x_n} = \mathbf{R_{nn}} \cdot \mathbf{z_n} + \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^+} - \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^-} + \mathbf{R_{Fn}}, \tag{4}$$

In addition to (3), we take into account the dependence on $\mathbf{x_\tau^+}, \mathbf{x_\tau^-}$ and on $\mathbf{z_n}$:

$$\begin{aligned} \mathbf{x_\tau^+} &= \mathbf{R_{\tau n}} \cdot \mathbf{z_n} + \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^+} - \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^-} + \mathbf{R_{F\tau}} + \mathbf{r_f}, \\ \mathbf{x_\tau^-} &= -\mathbf{R_{\tau n}} \cdot \mathbf{z_n} - \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^+} + \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^-} - \mathbf{R_{F\tau}} + \mathbf{r_f}, \end{aligned} \tag{5}$$

where $\mathbf{R_{\tau n}}$ is CSM $[m \times m]$ for tangential joints from unit dislocations normal to the contact zone (note that $\mathbf{R_{\tau n}} = \mathbf{R_{n\tau}^T}$), $\mathbf{r_f}$ is friction bound depends on contact compressive interaction forces (4):

$$\mathbf{r_f} = f \cdot \mathbf{x_n} = f \cdot (\mathbf{R_{nn}} \cdot \mathbf{z_n} + \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^+} - \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^-} + \mathbf{R_{Fn}}). \tag{6}$$

As a result of combining (4) and (5), taking into account (6), we have:

$$\begin{aligned} \mathbf{x_n} &= \mathbf{R_{nn}} \cdot \mathbf{z_n} + \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^+} - \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^-} + \mathbf{R_{Fn}}; \\ \mathbf{x_\tau^+} &= \mathbf{R_{\tau n}} \cdot \mathbf{z_n} + \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^+} - \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^-} + \mathbf{R_{F\tau}} + f \cdot (\mathbf{R_{nn}} \cdot \mathbf{z_n} + \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^+} - \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^-} + \mathbf{R_{Fn}}); \\ \mathbf{x_\tau^-} &= -\mathbf{R_{\tau n}} \cdot \mathbf{z_n} - \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^+} + \mathbf{R_{\tau\tau}} \cdot \mathbf{z_\tau^-} - \mathbf{R_{F\tau}} + f \cdot (\mathbf{R_{nn}} \cdot \mathbf{z_n} + \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^+} - \mathbf{R_{n\tau}} \cdot \mathbf{z_\tau^-} + \mathbf{R_{Fn}}). \end{aligned} \tag{7}$$

Adding to the system of equations (7) the conditions of non-negativity and complementarity non-rigidity from (1) and (2) we come to the LCP for frictional contact:

$$\begin{gathered} \begin{bmatrix} \mathbf{x_n} \\ \mathbf{x_\tau^+} \\ \mathbf{x_\tau^-} \end{bmatrix} = \begin{bmatrix} \mathbf{R_{nn}} & \mathbf{R_{n\tau}} & -\mathbf{R_{n\tau}} \\ \mathbf{R_{\tau n}} + f \cdot \mathbf{R_{nn}} & \mathbf{R_{\tau\tau}} + f \cdot \mathbf{R_{n\tau}} & -\mathbf{R_{\tau\tau}} - f \cdot \mathbf{R_{n\tau}} \\ -\mathbf{R_{\tau n}} + f \cdot \mathbf{R_{nn}} & -\mathbf{R_{\tau\tau}} + f \cdot \mathbf{R_{n\tau}} & \mathbf{R_{\tau\tau}} - f \cdot \mathbf{R_{n\tau}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{z_n} \\ \mathbf{z_\tau^+} \\ \mathbf{z_\tau^-} \end{bmatrix} + \begin{bmatrix} \mathbf{R_{Fn}} \\ \mathbf{R_{F\tau}} + f \cdot \mathbf{R_{Fn}} \\ -\mathbf{R_{F\tau}} + f \cdot \mathbf{R_{Fn}} \end{bmatrix} \\ \mathbf{z_n} \ge 0; \quad \mathbf{x_n} \ge 0; \quad \mathbf{z_n^T} \cdot \mathbf{x_n} = 0; \\ \mathbf{x_\tau^+} \ge 0; \quad \mathbf{x_\tau^-} \ge 0; \quad \mathbf{z_\tau^+} \ge 0; \quad \mathbf{z_\tau^-} \ge 0; \quad \mathbf{z_\tau^{+T}} \cdot \mathbf{x_\tau^+} = 0; \quad \mathbf{z_\tau^{-T}} \cdot \mathbf{x_\tau^-} = 0, \end{gathered} \tag{8}$$

There are many methods for solving LCP. They can be divided into 2 groups: iterative methods (including using machine learning [38, 39]); step methods, that can obtain a solution in a finite number of steps. Lemke's algorithm and its modifications are step methods and allow you to physically interpret each step of the algorithm. And this interpretation does not contradict the physical nature of the problem.

To solve the LCP (7), we use the Lemke method with the extension parameter of the problem $p$ [40, 41], which in our case takes on the physical meaning of an additional “tightening weight” (force bringing together nodes) of the contact pair [14, 42]. The steps of the Lemke algorithm are a change of work schemes with a sequential decrease in the “tightening weight”. Three types of LCP solutions are possible:

1. a trivial solution is a case when separation and slipping do not occur along the supposed contact zone (the main system is the solution to the problem);

2. a normal solution is when the value of the artificially entered "tightening weight" $p$ becomes equal to zero;

3. a ray solution for which there are two interpretations:

- $p > 0$ means that the given system is substatic structure (when the "tightening weight" is fully removed, the displacements become uncertain).
- the $p$ value is close to zero and small in comparison with the friction bound. This can be interpreted as a normal solution.

## 3. Results and Discussion

There are plenty of problems in in the construction of buildings and structures that are connected with frictional unilateral contact, for example: contact seams [28]; kinematic foundations; constructions under the soil which interacting with that soil with separation and slippage [31, 33], multilayer structures; crack opening in concrete elements; slope stability [32], frictional supports of seismically insulated building. We show the features of the solution with a few simple examples. All results were rounded to 3 decimal places.

**Example 1**. We consider a freely supported multi-span beam on unilateral supports with friction (Fig. 5). The beam is approximated by frame rod elements with three degrees of freedom in the node. It is loaded with forces $F$ that do not change during deformation, which ensures the constant friction bound $f \cdot F$ on supports. The solution to this problem is obvious:

1. with $F_2 \le f \cdot F$ the cohering of the beam with all the supports is realized;

2. with $f \cdot F < F_2 < 6f \cdot F$, slippage is realized on the part of the supports, and on the other part – cohesion;

3. with $F_2 > 6f \cdot F$, slippage is realized on all six supports and the beam displacements become undefined;

4. when $F_2 = 6f \cdot F$, the friction bound is realized on all supports, while the displacements on the extreme left support are zero, and on all other supports – slippage. This case is the most difficult for the algorithms.

In the first case, according to the algorithm, we obtain a trivial solution (see (7)):

$$\mathbf{z_n} = 0,\ \mathbf{z}_\tau^+ = 0,\ \mathbf{z}_\tau^- = 0,\ \mathbf{x_n} = \mathbf{R_{Fn}},\ \mathbf{x}_\tau^+ = \mathbf{R}_{\mathbf{F}\tau} + f \cdot \mathbf{R_{Fn}},\ \mathbf{x}_\tau^- = -\mathbf{R}_{\mathbf{F}\tau} + f \cdot \mathbf{R_{Fn}},$$

$$\text{here: } \mathbf{x}_\tau = \left(\mathbf{x}_\tau^+ - \mathbf{x}_\tau^-\right)/2 = \mathbf{R}_{\mathbf{F}\tau} \text{ and } \mathbf{z}_\tau = \mathbf{z}_\tau^+ - \mathbf{z}_\tau^- = 0.$$

In the second case, using the algorithm, we obtain a normal solution in which at least one support is cohering with the beam, or the beam is slipping on at least one support. The results of solving the LCP, with $F_2 = F = 100H$, for tangential mutual displacements (increased scale) and interaction forces (friction forces) on unilateral supports are presented in Fig. 5, a (in a black frame the values for the interaction forces are shown, and in grey frames – mutual displacements).

In the third case, as a result of using the algorithm, we obtain a "ray solution" with the parameter $p > 0$. This is interpreted as the destruction of the system: a further decrease in the "tightening weight" $p$ will lead to undefined displacements. In the fourth case, the result of using the algorithm can be both specific interaction forces and mutual displacements in the contact area ( $p = 0$, "normal solution"), as well as the "ray solution", which is due to rounding errors, the "tightening weight" parameter is $p << f \cdot F$ (this situation should also be interpreted as a "normal solution"). The results of solving the LCP, with $F_2 = 6 \cdot f \cdot F = 100H$, for tangential mutual displacements (increased scale) and interaction forces (friction forces) on unilateral supports are presented in Fig. 5, b. In this case, the ratio of the applied external load to the "tightening weight" turned out: $F/p > 10^{13}$.

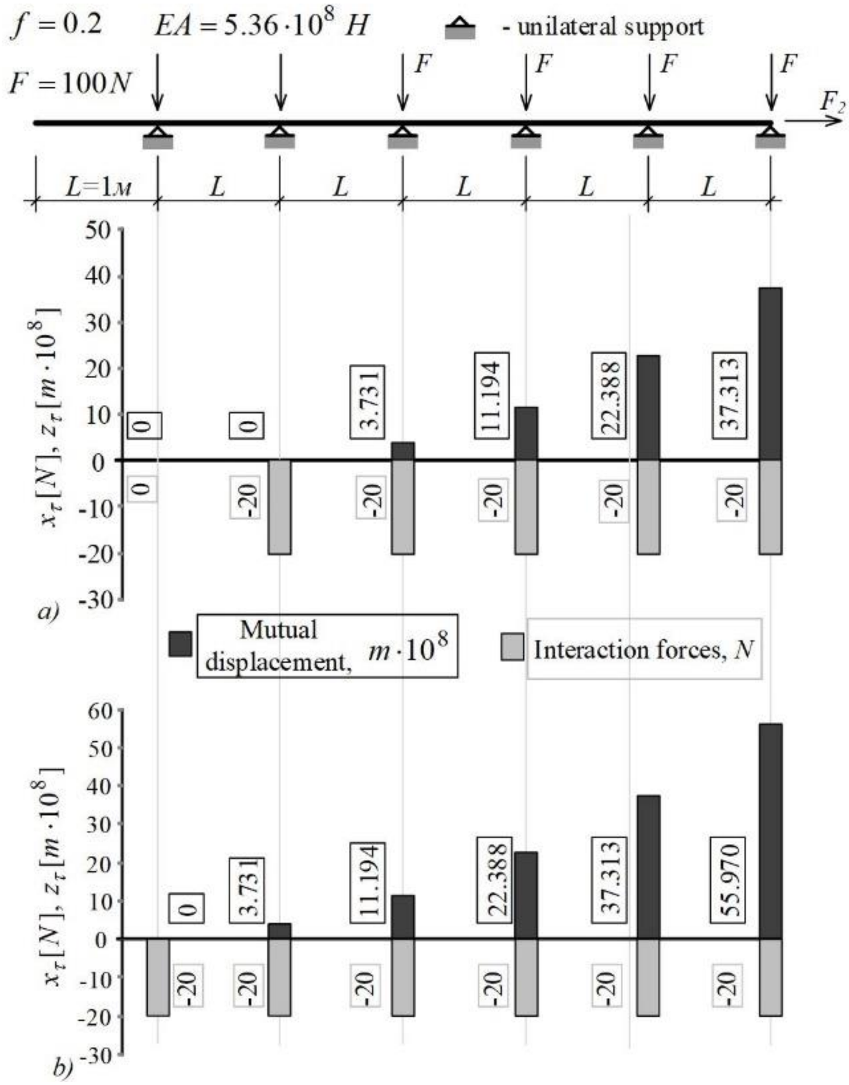


**Figure 5. Scheme of the beam and LCP calculation results.**

**Example 2**. The second problem is a frame rigidly pinched from the left end, supported on frictional unilateral supports (Fig. 6). The frame is approximated by frame rod elements with three degrees of freedom in the node. Two variants of this problem are considered: with frictional unilateral supports directed vertically (Fig. 6, a-c) and with third support rotated 45 degrees to the horizontal axis (Fig. 6, d-f).

For scheme with vertical supports:

- the results of solving the LCP for tangential mutual displacements (increased scale) and interaction forces (friction forces) on unilateral supports are presented in Fig. 6, a;
- the results of solving the LCP for normal mutual displacements (increased scale) and interaction forces on unilateral supports are presented in Fig. 5, b.

For scheme with third (from left) rotated support in Fig. 6, d and Fig. 6, e respectively. Using the results of the LCP we get a working scheme. Next, we solve the linear problem and get the SSS of the system with unilateral constraints. The corresponding deformable schemes for the frame (S40000:1) as a result of the final calculation of the frame, taking into account unilateral constraints, are shown in Fig. 6, c and Fig. 6, f.

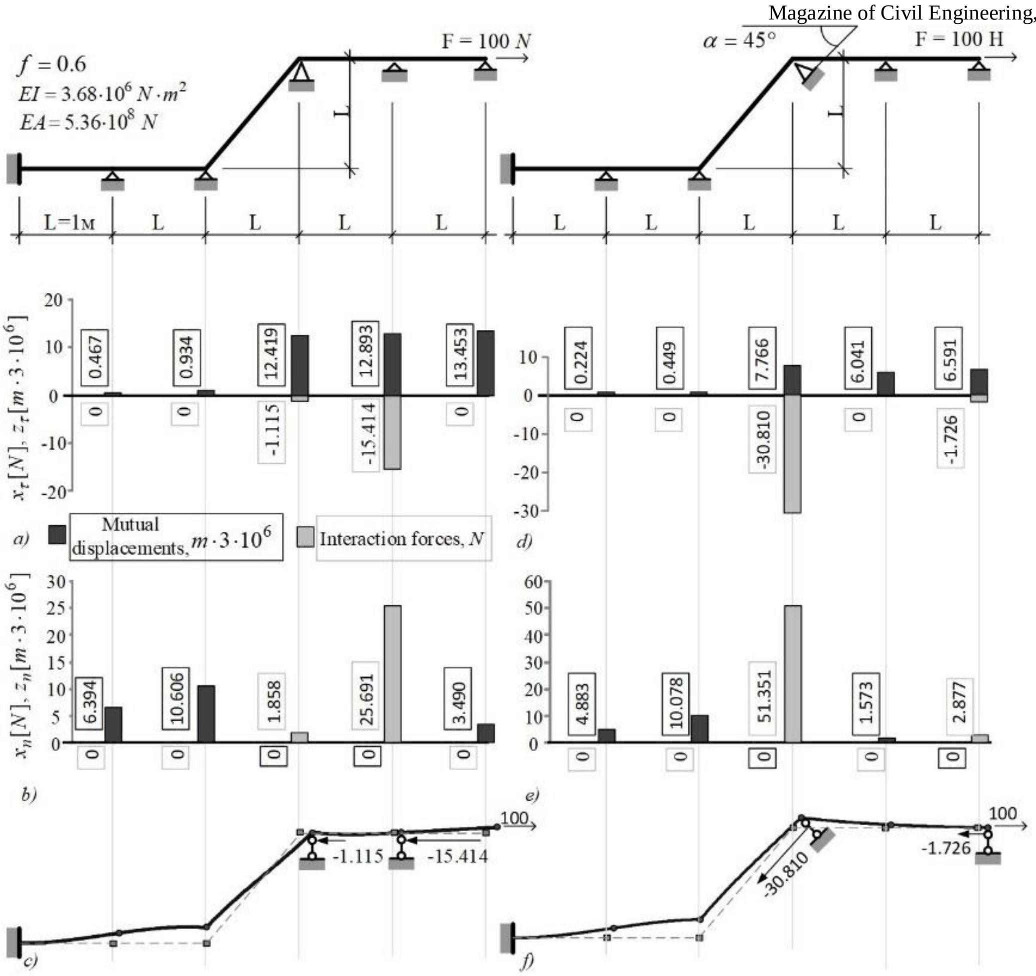

**Figure 6. Scheme of the frame and calculation results: a-c) the frame with vertical supports, d-f) the frame with the third rotated support.**

**Example 3.** The plane deformation problem for plates 0.1 m thick and 8m by 4m size (Fig. 7). For the plate approximation four-node finite element with two degrees of freedom in the node was used. Dimensions of square four-node finite elements 0.5×0.5 m. The expected contact zone considered to be fixed [26]. Two cases of support conditions of the left boundary with a rigidly pinched lower plate at the base are considered.

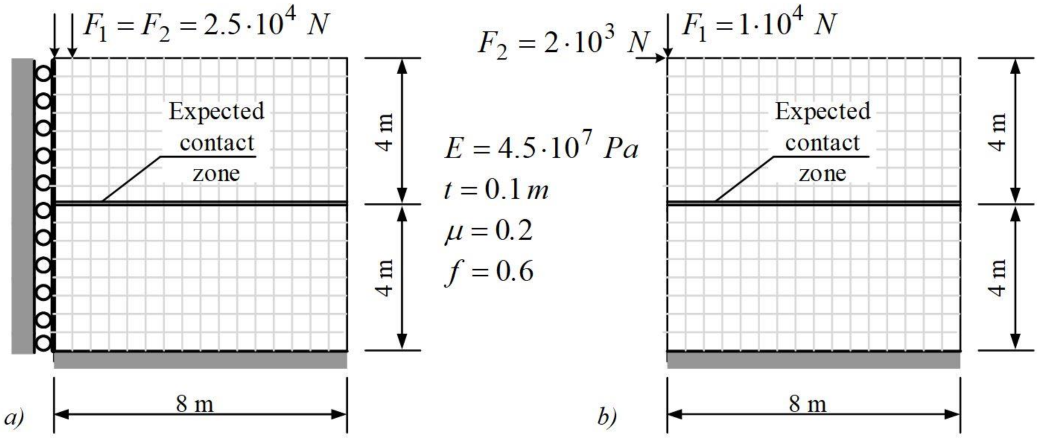

**Figure 7. The design scheme of the plates for the calculation of the contact problem: a) with hinge support on the left, b) without supports on the left.**

Use of CSM and CLV for a problem (at Fig. 7, *a*) with the total number of unknowns equal to 612, the contact problem (8) is formed using only 48 unknowns, and in [12], [26–29] and others the size of the problem is determined by the total number of degrees of freedom. The results of solving LCP for frictional unilateral contact of the plates with hinge supports on the left and deformation of the contact zone (S50:1) are shown in Fig. 8 (note that results are shown from the second node from left, because first one is fixed in horizontal direction and it is connecting plates). The results of solving LCP for plates without supports on the left (free boundary) with additional horizontal load and frictional unilateral contact along all bottom of the upper body (it can move as a rigid whole) are shown in Fig. 9. Note, that friction bound occurred on the leftmost contact pair, this case similar with situation 4 in example 1.

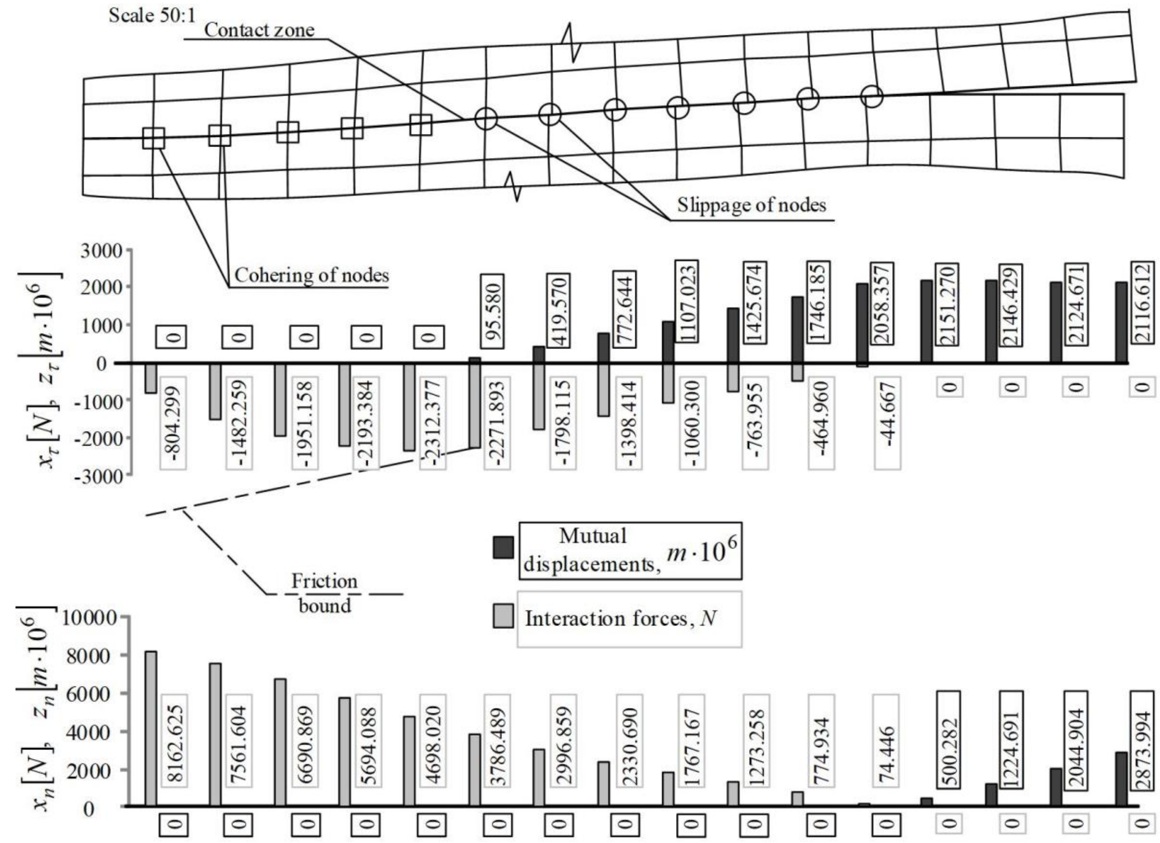


**Figure 8. Plates calculation results of LCP with hinge supports on the left.**

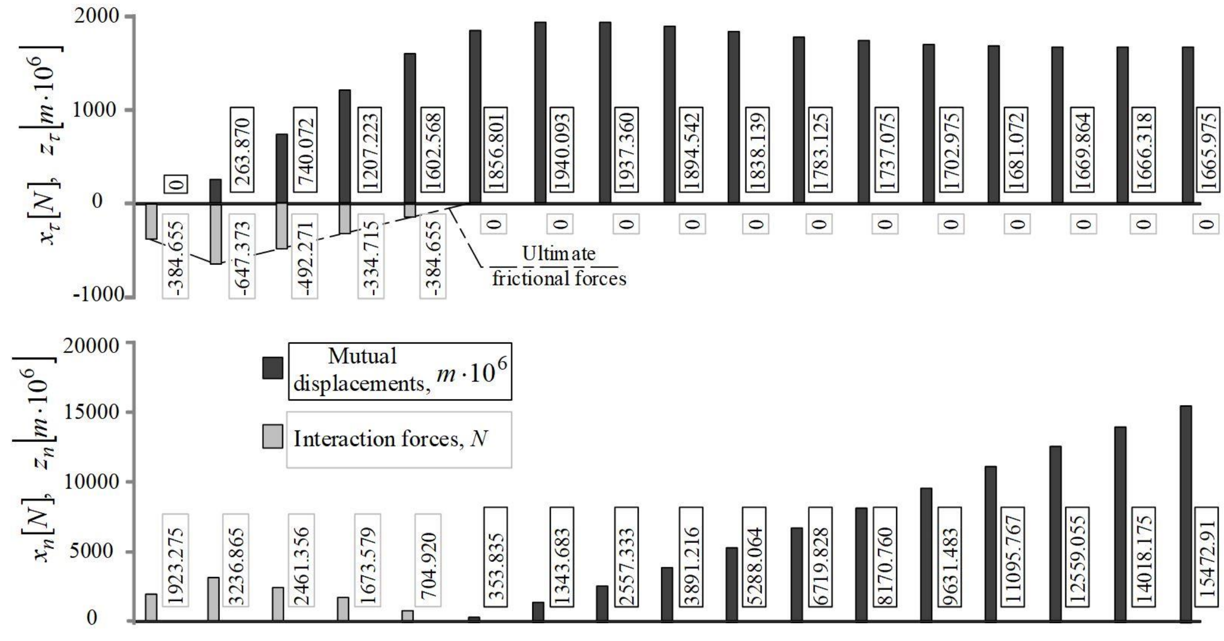


**Figure 9. Plates calculation results of LCP with the free left boundary.**

The problem (case with hinged left side Fig. 7, *a*) was calculated on Ansys 18.2 Academic. It's turned out that contact parameters that most strongly affecting the result are: Penetration Tolerance and Elastic Slip Tolerance. A decrease in the values of these parameters brought the Ansys solution closer to the solution obtained using the algorithm described in this paper. Getting a more precise solution in Ansys requires (thorough) fine-tuning the program. The second case showed on Fig. 7, *b* can't be solved strictly without adding boundary conditions (it have to be fixed from moving as a rigid whole) to the upper body in Ansys. Allowance of the penetration [26], [30] and requiring of fine-turning [25], [29], [36] is normal to the existing most widespread unilateral contact solvers.

# 4. Conclusions

1. A heuristic formulation of the contact problem with friction is obtained in the form of a linear complementarity problem

2. To solve this problem, the displacement method has been developed as a modification of the Lemke step by step algorithm. For discretized problems, this method ensures the fulfillment of the non-interpenetration condition; does not use the artificial concept of "stiffness of contact joints"; is step by step, which removes the question of the user assigning parameters to exit the iterative process.

3. The algorithm has been tested on a large number of problems. The algorithm has proven its effectiveness by coinciding with exact solutions or solutions obtained by other algorithms.

4. Comparison of the results of solutions obtained using the proposed algorithm and the Ansys 18.2 Academic software showed their almost complete match. Adjustment of the contact parameters in Ansys brought the solution closer to the solution obtained using the proposed displacement method. Getting a more precise solution in Ansys requires (thorough) fine-tuning the program.

## References


1. Glowinski, R., Lions, J.-L., Trémolières, R. Numerical Analysis of Variational Inequalities Elsevier Science, 1981. DOI: 10.1016/S0168-2024(08)70201-7
2. Duvaut, G., Lions, J.-L. Inequalities in Mechanics and Physics. (219). Berlin, Heidelberg: Springer Berlin Heidelberg, 1976. DOI: 10.1007/978-3-642-66165-5
3. Kravchuk A.S., Neittaanmäki P.J. Variational and Quasi-Variational Inequalities in Mechanics. (147). Dordrecht: Springer Netherlands, 2007. DOI: 10.1007/978-3-642-66165-5
4. Panagiotopoulos, P.D. Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions. Birkhäuser: Boston-Base-Stuttgart, 1985. DOI: 10.1007/978-1-4612-5152-1
5. Eck, C., Jarusek, J., Krbec, M. Unilateral Contact Problems. New York: CRC Press, 2005. DOI: 10.1201/9781420027365
6. Sofonea, M., Matei, A. Mathematical Models in Contact Mechanics. Cambridge: Cambridge University Press, 2012. DOI: 10.1201/9781420027365
7. Aleksandrov, V.M., Kalker, J.J., Pozharsky, D.A. The spatial contact problem for two-layered of elastic base with pre-unknown contact area. Izv. RAN, Mekhanika tverdogo tela. 1999. (4). Pp. 51–55. URL: http://mtt.ipmnet.ru/ru/Issues.php?y=1999&n=4
8. Vorovich, I.I., Aleksandrov, V.M. Mekhanika kontaktnykh vzaimodeistvii [Contact Mechanics Interaction]. Moscow: Fizmatlit, 2001. URL: https://www.rfbr.ru/rffi/ru/books/o_460?FILTER_ID=23@2
9. Raous, M. Art of Modeling in Contact Mechanics. The Art of Modeling Mechanical Systems. (570). Springer, 2016. Pp. 203–276. DOI: 10.1007/978-3-319-40256-7_4
10. Kravchuk, A.S. The variational method in contact problems. The present state of the problem and trends in its development. Journal of Applied Mathematics and Mechanics. 2009. № 3(73). Pp. 351–357. DOI: 10.1016/j.jappmathmech.2009.07.004
11. Azbelev, N., Bravyi, E., Gusarenko, S. On a New Approach to Some Problems of Classical Calculus of Variations. Nonlinear Dynamics and Systems Theory. 2002. № 2(2). Pp. 117–124. URL: https://www.researchgate.net/publication/2664169-80_On_a_new_approach_to_some_problems_of_classical_calculus_of_variations
12. Ignatyev, A.V., Ignatyev, V.A., Onischenko, E.V. Analysis of Systems with Unilateral Constraints through the Finite Element Method in the Form of a Classical Mixed Method. Procedia Engineering. 2016. (150). Pp. 1754–1759. DOI: 10.1016/j.proeng.2016.07.166
13. Morozov, N.F., Tovstik, P.Y. Bending of a two-layer beam with non-rigid contact between the layers. Journal of Applied Mathematics and Mechanics. 2011. № 1(75). Pp. 77–84. DOI: 10.1016/j.jappmathmech.2011.04.012
14. Lovtsov, A.D. Algoritmy metoda peremeshchenij dlya raschyota sistem s odnostoronnimi svyazyami [Algorithms of the displacements method for the calculation of systems with unilateral constraints]. Scientific and technical bulletin of St. Petersburg State Polytechnic University. 2004. 2(36). Pp. 220–227. URL: https://engtech.spbstu.ru/issue/36/
15. Lovtsov, A.D. Algoritmy rascheta sistem s odnostoronnimi svyazyami metodom sil [Algorithms of the force method for the calculation of systems with unilateral constraints]. Scientific and technical bulletin of St. Petersburg State Polytechnic University. 2003. 3(33). Pp. 186–192. URL: https://engtech.spbstu.ru/issue/33/
16. Rozin, L.A., Lovtsov, A.D. Izgib balki-truboprovoda, vzaimodejstvuyushchej s uprugim osnovaniem pri dejstvii treniya Kulona [Bending of a beam-pipeline, interacting with an elastic base under the action of Coulomb friction]. Izvestie Vusov. Stroitelstvo. 2005. 7(559). Pp. 17–22. URL: http://izvuzstr.sibstrin.ru/uploads/publication/fulltext/N7-2005.pdf
17. Rozin, L.A., Lovtsov, A.D. Izgib balki-truboprovoda, vzaimodejstvuyushchej s nelinejno-uprugim osnovaniem, pri uchete treniya Kulona [Beam-Pipeline Interacting with Nonlinear Elastic Foundation considering Coulomb Friction Bending Analysis]. St.-Petersburg State Polytechnical University Scientific and Technical Bulletin. 2005. 3(41). Pp. 132–142. URL: https://engtech.spbstu.ru/issue/41/
18. Rozin, L.A., Lovtsov, A.D., Smirnov, M.S. Prodol'naya deformaciya mnogoprolyotnoj balki s treniem i podatlivost'yu opor [Longitudinal deformation of a multi-span beam with friction and compliance of supports]. Izvestie Vusov. Stroitelstvo. 2004. 8(548). Pp. 17–22. URL: http://izvuzstr.sibstrin.ru/uploads/publication/fulltext/8-2004.pdf
19. Kikuchi, N., Oden, J.T. Contact Problems in Elasticity. Philadelphia: Society for Industrial and Applied Mathematics, 1988. DOI: 10.1137/1.9781611970845
20. Bathe, K.J., Chaudhary, A.A. A solution method for planar and axisymmetric contact problems. International Journal for Numerical Methods in Engineering. 1985. 21. Pp. 65–88. DOI: 10.1002/nme.1620210107
21. Klarbring, A. A mathematical programming approach to three-dimensional contact problems with friction. Computer Methods in Applied Mechanics and Engineering. 1986. 58. Pp. 175–200. DOI: 10.1016/0045-7825(86)90095-2
22. Landenberger, A., El-Zafrany, A. Boundary element analysis of elastic contact problems using gap finite elements. Computers and Structures. 1999. No. 6(71). Pp. 651–661. DOI: 10.1016/S0045-7949(98)00303-4
23. Parisch, H. A consistent tangent stiffness matrix for three-dimensional non-linear contact analysis. International Journal for Numerical Methods in Engineering. 1989. No. 8(28). Pp. 1803–1812. DOI: 10.1002/nme.1620280807

24. Zhiltsov, A.V. Modified duality scheme for numerical simulation of the contact between elastic bodies. Mathematical notes of NEFU. 2016. No. 4(23). Pp. 99–114. URL: http://www.mzsvfu.ru/index.php/mz/article/view/modified-duality-scheme-for-numerical-simulation-of-the-contact-between-elastic-bodies
25. Semenov, A.S., Semenov, S.G., Melnikov, B.E., Tikhonov, V.S. Modeling of indentation and slip of wedge punch. Magazine of Civil Engineering. 2017. 74(6). Pp. 78–101. DOI: 10.18720/MCE.74.8
26. Barboteu, M., Ouakik, Y., Sofonea, M. Numerical nodelling of a dynamic contact problem with normal damped response and unilateral constraint. Journal of Theoretical and Applied Mechanics. 2018. No. 2(56). Pp. 483–496. DOI: 10.15632/jtam-pl.56.2.483
27. Lukashevich, A.A., Rozin, L.A. On the decision of contact problems of structural mechanics with unilateral constraints and friction by step-by-step analysis. Magazine of Civil Engineering. 2013. 36(1). Pp. 75–81. DOI: 10.5862/MCE.36.9
28. Lukashevich, A.A. Computational modelling of stiffness and strength properties of the contact seam. Magazine of Civil Engineering. 2018. 81(5). Pp. 149–159. DOI: 10.18720/MCE.81.15
29. Bokiy, I.B. CHislennyj podhod k resheniyu kontaktnoj zadachi vzaimodejstviya dvuh uprugih tel s uchyotom treniya i istorii prilozheniya vneshnego nagruzheniya [Numerical approach to the solution of the contact problem of the interaction of two elastic bodies, taking. Vestnik YSU. 2006. No. 3(3). Pp. 42–46. URL: https://cyberleninka.ru/article/n/chislennyy-podhod-k-resheniyu-kontaktnoy-zadachi-vzaimodeystviya-dvuh-uprugih-tel-s-uchetom-treniya-i-istorii-prilozheniya-vneshnego
30. Berezhnoi, D.V., Sagdatullin, M.K. Calculation of interaction of deformable designs taking into account friction in the contact zone by finite element method. Contemporary Engineering Sciences. 2015. No. 23(8). Pp. 1091–1098. DOI: 10.12988/ces.2015.58237
31. Fedorova, V.S., Lovtsov, A.D. The calculation of corrugated culvert as arch, which interacts with an elastic medium through coulomb friction. New ideas of the new century: proceedings of the Thirteenth International Scientific Conference: in 3 vol. 2013. 3. Pp. 469–473. URL: https://elibrary.ru/item.asp?id=19138399
32. Fedorova, V.S., Stepanova, N.O., Lovtsov, A.D. Lemke algorithm for analyzing of slope stability. New ideas of the new century: materials of the Eleventh International Scientific Conference of the IAS TOGU: in 2 vol. 2011. 2. Pp. 144–149. URL: https://elibrary.ru/item.asp?id=18964360
33. Popov, A.N., Lovtsov, A.D. The calculation of a culvert in a high mound of a road as a contact problem with friction. IOP Conference Series: Materials Science and Engineering. 2018. No. 1(456). DOI: 10.1088/1757-899X/456/1/012085
34. Negrut, D., Serban, R., Tasora, A. Posing multibody dynamics with friction and contact as a differential complementarity problem. Journal of Computational and Nonlinear Dynamics. 2018. No. 1(13). Pp. 6. DOI: 10.1115/1.4037415
35. Stankevich, I. Mathematical Modeling of Contact Problems of Elasticity Theory with Continuous Unilateral Contact. Mathematics and Mathematical Modeling. 2015. 4. Pp. 93–110. DOI: 10.7463/mathm.0515.0812348
36. Streliaiev, Y.M. A nonlinear boundary integral equations method for the solving of quasistatic elastic contact problem with Coulomb friction. Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences. 2016. No. 2(20). Pp. 306–327. DOI: https://doi.org/10.14498/vsgtu1471
37. Mitra, G., Cottle, R.W., Giannessi, F., Lions, J.L. Variational Inequalities and Complementarity Problems – Theory and Application. The Journal of the Operational Research Society. 1981. No. 9(32). Pp. 848. DOI: 10.2307/2581414
38. Liao, L.Z., Qi, H.D. A neural network for the linear complementarity problem. Mathematical and Computer Modelling. 1999. No. 3(29). Pp. 9–18. DOI: 10.1016/S0895-7177(99)00026-6
39. Liao, L.Z., Qi, H.D., Qi, L. Solving nonlinear complementarity problems with neural networks: A reformulation method approach. Journal of Computational and Applied Mathematics. 2001. 131. Pp. 343–359. DOI: 10.1016/S0377-0427(00)00262-4
40. Lemke, C.E. Bimatrix Equilibrium Points and Mathematical Programming. Management Science. 1965. No. 7(11). Pp. 681–689. DOI: 10.1287/mnsc.11.7.681
41. Lemke, C.E. Some pivot schemes for the linear complementarity problem. Complementarity and Fixed Point Problems. 7. 1978. Pp. 15–35. DOI: 10.1007/bfb0120779
42. Lovtsov, A.D. Primenenie algoritma Lemke k zadache izgiba balki, vzaimodejstvuyushchej s uprugim osnovaniem posredstvom treniya Kulona [Application of the Lemke algorithm to the problem of bending a beam, interacting with an elastic base by means of Coulomb friction]. Bulletin of Civil Engineers. 2006. No. 6(1). Pp. 110–114. URL: https://elibrary.ru/item.asp?id=12859487

***Contacts:***

*Aleksandr Popov, sanyapov@mail.ru*
*Alexander Lovtsov, Lovtsov@bk.ru*