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STRATEGIC DEVELOPMENT OF THE MULTI-LEVEL SOCIO-ECONOMIC SYSTEM OF THE STATE IN THE DIGITAL ECONOMY

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The relevance of the study is due to the development of the digital economy in the Russian Federation. Forecasting, strategic planning in this aspect is a promising direction of socio-economic development of the state, as well as its constituent parts – regions, municipalities, enterprises (firms) – the latter being on the lower level of the economy forming its foundation. This area of research provides a methodological basis for the development and implementation of management decisions focused on priority areas of economic development of the state. The purpose of the study is to analyze and develop theoretical and methodological provisions of strategic and innovative development of the multi-level economic system of the state in the digital economy. Other objectives include the research and development of mathematical models of the lower level of the state economy of large corporations (clusters). To achieve these goals, the first part of the work examines the structure of a multi-level hierarchical system of the state economy aimed at solving the problems of strategic planning and management at individual levels of the state within the digital economy. In the second part, based on the analysis and theoretical studies of previously proposed mathematical models of enterprise development, we developed a mathematical model of a corporation (cluster). The cluster model takes into account both extensive and intensive factors of production development. The input data of the cluster model is, first, statistical information and, second, technological information of pre-production. It is shown that taken together statistical, technological information, as well as the relationship with consumers of products and with the financial (banking) sphere is characterized as "digital economy". The direction of further research is related to the practical implementation of mathematical models and their use in the practice of forecasting the development of an industrial corporation (cluster).

Keywords: state, corporation, strategic development, statistical information, technological information, vector optimization

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СТРАТЕГИЧЕСКОЕ РАЗВИТИЕ МНОГОУРОВНЕВОЙ СОЦИАЛЬНО-ЭКОНОМИЧЕСКОЙ СИСТЕМЫ ГОСУДАРСТВА В УСЛОВИЯХ ЦИФРОВОЙ ЭКОНОМИКИ

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Актуальность исследования обусловлена развитием цифровой экономики в Российской Федерации. Прогнозирование, стратегическое планирование в этом аспекте является перспективным направлением социально-экономического развития государства, а также его составных частей: регионов, муниципальных образований и предприятий (фирм). Предприятия находятся на нижнем уровне экономики и которые являются ее основой. Это направление исследований создает методологическую основу разработки, реализации управленческих решений, ориентированных на приоритетные направления экономического развития государства. Цель исследования состоит в анализе и разработке теоретико-методологических положений стратегического и инновационного развития многоуровневой экономической системы государства в условиях цифровой экономики, а также в исследовании и формировании математических моделей нижнего уровня экономики государства крупных корпораций (кластеров). Для реализации этих целей в первой части работы исследованы структуры блоков иерархических многоуровневых систем экономики государства, направленная на решение задач стратегического планирования и управления на отдельных уровнях государства в условиях цифровой экономики. Во второй части на базе проведенного анализа и теоретических исследований раннее созданных математических моделей развития предприятия, мы разработали математическую модель корпорации (кластера). Математическая модель фирмы (кластера) учитывает экстенсивные и интенсивные производственные факторы развития. Входом математической модели кластера являются информация, представленная статистическими органами, а также технологическая информация подготовки производства. Показано, что в совокупности статистическая, технологическая информация, а также взаимосвязь с потребителями продукции и с финансовой (банковской) сферой характеризуется как «цифровая экономика». Дальнейшие исследования связаны с разработкой и внедрением математических моделей в практику прогнозирования развития промышленного кластера.

Ключевые слова: государство, корпорация, стратегическое развитие, статистическая информация, технологическая информация, векторная оптимизация

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Introduction

Forecasting, strategic planning is a promising direction of socio-economic development of the state and its constituent parts: firms (enterprises), municipalities, regions, which form the basis of the state's economy. This area of research is aimed at forming a methodological basis for the development and implementation of management decisions that are focused on priority areas of social and economic development of the state, which is reflected in related legal documents[1,2,3,4]. The theory of economic management, research objectives[5] as well as strategic development of industrial enterprises, corporations and the adoption of the optimal solutions are given much attention both in foreign countries [5, 7, 9, 12, 24, 28], and in the Russian Federation [1, 2, 3, 14–23, 29–34].

The social and economic processes of state development are quite complex and require a systematic [5, 7, 29], regional-balanced approach to management [6, 12]. Hence, state regulation to support such transformations is particularly important and relevant [30]. The forecast and dynamics of the development of regional and state economy are carried out by constructing mathematical models of both individual enterprises [20–22, 29], and municipalities [30], regions (subjects of the Russian Federation) [30], as well as the state as a whole. The forecast and development of the state economy requires legislative[6] support for decision-making, as well as mathematical [32, 33] software [30], information (including the digital economy [1, 2, 18, 19, 20]), and statistical [30]) support.

[1] The Constitution of the Russian Federation: Adopted by popular vote on December 12, 1993, with the amendments of 30.12.2008 No. 7-FKZ.
[2] Budget Code of the Russian Federation, Moscow: TK Velbi, Prospect Publishing House, 2010, 215 p.
[3] Tax Code of the Russian Federation. Part One [adopted by the State Duma on 16.07.1998, Approved by the Federation Council on 17.07.1998]. Access mode URL: http://www.consultant.ru/document/cons_doc_LAW_19671/
[4] Federal Law of the Russian Federation No. 172-FZ of June 28, 2014 "On Strategic Planning in the Russian Federation". http://www.rg.ru/2014/07/03/strategia-dok.html
[5] Putin's Message on January 15, 2020: all President Putin's proposals to the Federal Assembly http://kremlin.ru/events/president/news/62582
[6] Program "Digital Economy of the Russian Federation" Decree of the Government of the Russian Federation No. 1632-r of July 28, 2017, Moscow. http://static.government.ru/media/files/9gFM4FHj4PsB79I5v7yLVuPgu4bvR7M0.pdf

The object of the study is a multi-level management system of the state economy, including regions (subjects of the Russian Federation), municipal entities and enterprises (firms).

The subject of the research includes analytical, theoretical and methodological provisions for the strategic and innovative development of the multi-level economic system of the state in the digital economy.

The purpose of the study is to analyze and develop theoretical and methodological provisions for the strategic and innovative development of the multi-level economic system of the state in the digital economy. It also includes the research and formation of mathematical models of the lower level of the state economy of large corporations (clusters).

To achieve this goal, we have divided the work into two parts.

In the first part of the work, the hierarchical system (HS) of the multi-level economy of the state is investigated. In the study of the HS management of the state, four levels are considered: enterprises (firms), municipalities, subjects of the Russian Federation (regions), the state, and the world economy. The goals and objectives of strategic planning and management at the appropriate level are considered. We developed a modeling methodology to solve planning and management problems at individual levels of the state's HS within the digital economy.

In the second part, a study of the production activity of the company, the lower level of the state economy, is carried out. The functioning of the company is determined by a number of the following indicators: sales (revenue), profit, benefit, profitability. Modeling the development of the enterprise within the framework of modern (with one criterion) optimization methods allowed us to solve one directed development of the enterprise (firm), [21, 22, and 29]. Therefore, the development of economic theory is characterized by the study of the multiplicity of goals (criteria) of firm management [25, 30, and 31]. This class of problems includes multi-criteria (vector) problems of mathematical programming. The development of methods for solving vector problems is presented in [32, 33]. Vector problems and methods of their solution are used to create a high-quality management system, strategic development of industrial production, taking into account statistical, technological information and the digital economy.

The research methods are based on the theory of hierarchical systems research, as well as on the basis of economic and mathematical methods, the theory and methods of vector optimization. Methods of statistical analysis, systematization and generalization of the obtained results are used.

Results and discussion. The results are presented in three sections: "Management of a multi-level state system within the digital economy"; "Theoretical foundations of mathematical modeling of the development and management of an industrial enterprise: a cluster (the lower level of the hierarchical structure of the state)"; "Theory and methods of vector optimization as mathematical support for modeling in the digital economy of the state".

1. Managing the multi-level system of the state within the digital economy

The functioning of state authorities related to the management of the economic and social development of the state includes relations between the central governing bodies of the state, the subjects of the Russian Federation (regions), and municipalities, which are an integral part of the national policy of the state. In this aspect, we will adhere to V.I. Lenin's definition of politics as a "concentrated expression of the economy"[7]. This implies concentrated activity of the state's governing bodies in the conditions of the global environment of competing states.

The implementation of the state's economic policy is reflected in a certain system of indicators. These economic indicators are used as the goals of the socio-economic development of the state when solving problems in the implementation of these goals. The system of economic indicators and related goals is represented by the Types of Economic Activity (TEA). The objectives of the state's economic policy are reflected in the legislative, administrative, and economic laws and measures implemented by the central, constituent entities of the Russian Federation, and local authorities. The organizational structure of the

[7] Lenin V. I. Complete collection of works, 5 ed., vol. 42, p. 278.

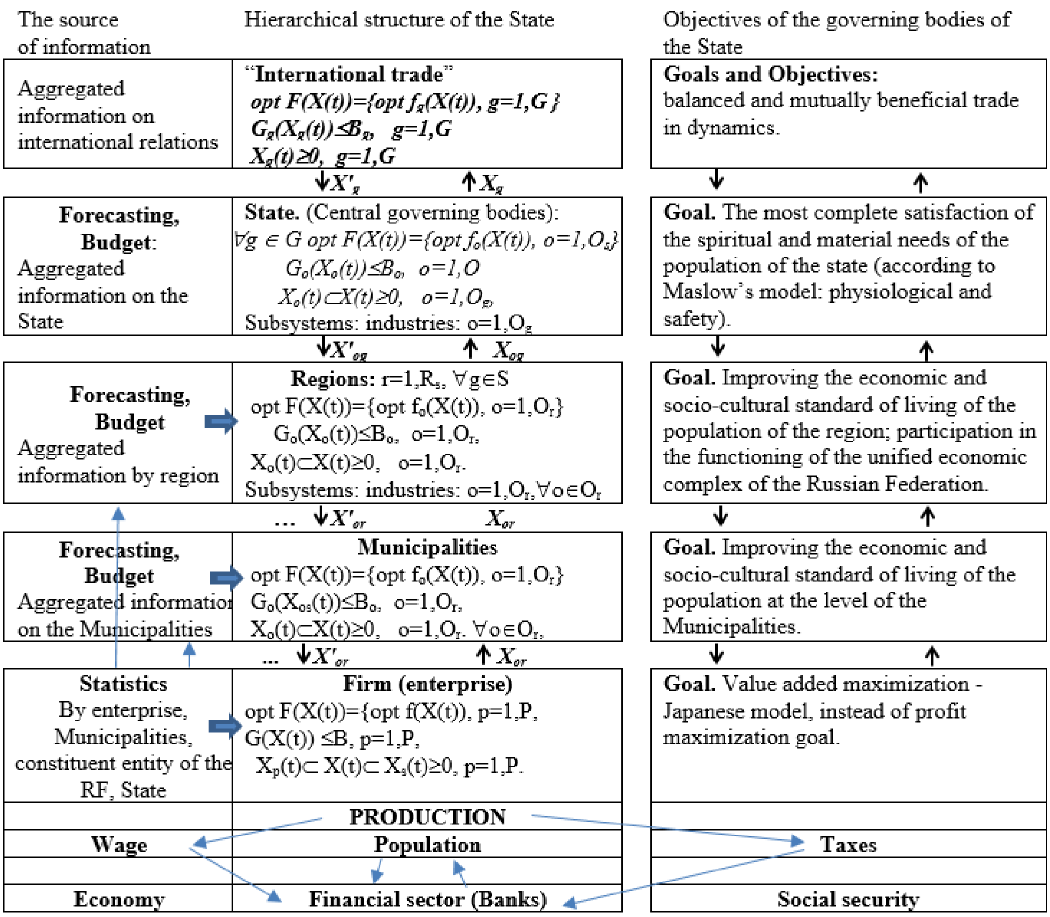

Fig. 1. Goals, tasks of management in a multilevel hierarchical system of the state in a digital economy

multi-level HS of the state economy is aimed at solving the problems of current, strategic planning and management at individual levels of the state in the digital economy (Fig. 1).

The goals of the current, strategic planning and management of the state's HS, presented in Fig. 1, include four levels: enterprises (firms), municipalities, regions (subjects of the Russian Federation), the state and the added level of the world economy.

The top level of the HS represents the highest management subsystem, which forms information for decision-making associated with international organizations, and coordinates the development of foreign trade operations of various countries. Coordination of foreign trade operations is carried out when solving the problems of balanced and mutually beneficial trade in dynamics. As a result of the decision, the amount of profit (income) is determined: $X_g, g=\overline{1,G}$, which the $g \in \boldsymbol{G}$ state will have when selling the corresponding goods to other countries.

- ***At the highest level of state HS*** management, we consider management subsystems, whose activities are related to the management of the economy of their countries. The target orientation of this level is determined by the most complete satisfaction of the material and spiritual needs of the population of the state, the growth of its standard of living (i.e., according to the Maslow model: the implementation of the first and second needs – physiological and safety). As a result, the public administration subsystem forms a vector of aggregated Types of Economic Activity (TEA) by industry: $o=\overline{1,O_g}$, including exports that could be mastered by this state: $X_{og}, o=\overline{1,O_g}$, $g \in G$. The received data is reported to the regional level.

• ***At the regional (second) level of government***, the mechanism of forecasting, planning and management is aimed, firstly, at improving the social and economic standard of living of the population of the region, and secondly, at participating in the all-Russian division of labor, as the functioning of a single economic complex of the Russian Federation with mutual benefits. The development of the region due to the intensification of the development of industry, the agricultural production should be carried out taking into account the rational use of natural resources and the development of resource-saving technologies. Regional current, strategic planning and management should be based on the production and economic activities of individual firms (enterprises) and their associations, which are the main suppliers of the region's products. The growth of the population's well-being is closely related to the growth of labor productivity and the increase in the volume of products produced in the region, quality improvement, competitiveness [6]. The above-stated global goals, assessments of the development of the region in the process of implementing production tasks, are divided into industry-level goals, which, in their essence, represent the solution of large sets of tasks.

• ***At the municipal (third) level of government***, the forecast, planning and management is aimed at improving the social and economic standard of living of the local government population.

In the Constitution of the Russian Federation, Article 12 states "Local self-government is recognized and guaranteed in the Russian Federation. Local self-government within the limits of its powers is independent. Local self-government bodies are not part of the system of state authorities".

It should be understood that it is in the municipalities, along with military facilities, that all industrial and agricultural enterprises are concentrated, which:

first, create the economic base for the development of the state's economy and the economic potential for the future development of the state;

secondly, it is these enterprises that determine the main tax revenues and form the revenue part of the budget of the Russian Federation, which determines the socio-economic development of the state, its protection.

With this in mind, we would like to see municipalities (MO) more actively participate in state activities. President V.V. Putin in his speech[8] said that currently only 1% of the total tax revenue remains in the Ministry of Defense. It is advisable to raise its level to 5%. Unfortunately, the progress on this issue has stalled. Each enterprise (firm), according to the legislation, pays taxes for the year, the size of which is approximately 20–25% of annual gross receipts. Thus, in an implicit form (de facto), the state owns 20–25% of fixed assets, and is obliged to supervise their safety, and better yet, their growth. If the local government were to transfer the state functions (de jure) related to the control of tax receipts (the Federal authorities will naturally collect taxes, as they do now), the level of tax receipts would increase, and as a result, the living (socio-economic) level of the population would increase. Moreover, if the level of tax collection exceeds some average level for the state, then part of the exceeded amount of taxes should remain at the disposal of the Ministry of Defense. And this is possible only when each enterprise of the Ministry of Defense increases the volume of production, as a rule, due to the growth of labor productivity.

At the level of (fourth) management of the firm (enterprise), the target orientation is determined by increasing the economic standard of living not only for shareholders, but also for all participants in production, who are represented by engineering, production, and management personnel. All participants in production equally form the profit of the firm (enterprise) and the maximum increase in labor productivity is only possible in their joint effort. The analysis of the company's production activity, theory, and software for the strategic development of the company (industrial cluster) using statistical information and the digital economy is the basis of this article [18–21, 29, 31].

Statistical information at the enterprise (firm) level is the source information, i.e. the base of the digital economy.

[8] Putin's Message on January 15, 2020: all President Putin's proposals to the Federal Assembly http://kremlin.ru/events/president/news/62582

Statistical information is generated from three sources: first, from consumers who are engaged in the material and technical support of production; second, from manufacturers who implement technological processes of production organization; and third, from manufacturers who sell manufactured goods and services. In accordance with the decision of the Board of the Eurasian Economic Commission of September 3, 2013. No. 185 “On Approval of the list of Statistical indicators of Official Statistical Information provided to the Eurasian Economic Commission by the Authorized Bodies of the Member States of the Customs Union and the Single Economic Space”, [34], all enterprises annually (quarterly) submit a number of macroeconomic indicators.

The main one is the Gross Domestic Product (in current prices), which includes information: quarterly, annual by type of economic activity (Appendix 1 in [34]), by institutional sectors (appendix 2 in [34]), by elements of end use (Appendix 3 in [34]), by elements of income (appendix 4 in [34]). These and other indicators, distributed across the entire state HS, are the digital economy. The information database is further used in the analysis of the development of the enterprise, the municipality, the region and the state as a whole.

2. Digital economy. Electronic (digital, web, Internet) economy is economic activity based on digital technologies (quote from Wikipedia) [1]. Within this definition, “it is not so much about the development and sale of software, but about electronic goods and services produced by e-business and e-commerce” [2, 18–21]. Hence, the digital economy cannot exist without the real and raw materials sectors, without production, which turns raw materials into products, without agriculture and without transport, delivering raw materials to the factory, products to the warehouse and goods from the warehouse to the store or to your home [1, 2]. Thus, the digital economy is not a complete economy, but its part, consisting of electronic goods and services. It is more correct to say not “digital economy”, but the digital sector of the economy. Let’s try to define the digital economy [30].

The “digital economy” is an information representation on paper and, above all, on electronic media: firstly, of all material resources (which include: land resources, manufacturing enterprises, firms, goods (including the process of creation and implementation), financial (including banks) and government organizations), and, secondly, of the population with its financial capabilities, which (material resources and population) together dynamically change in space and time.

To form the informational display of the “Digital Economy”, digital technologies or software have been developed, which represent technologies for collecting, storing (database), processing, searching, transmitting and presenting large-volume data. These information technologies for data collection and transmission are presented in general form (arrows) in the lower part of Fig. 1.

3. Theoretical foundations of mathematical modeling of the development and management of an industrial enterprise: cluster

3.1. Analysis of modern economic and mathematical models of the firm development

There are different mathematical models developed for an assessment of behavior of firms in society, formations of the purposes and tasks of the organization of management. The following models are currently used in the theory: maximizing profit, maximizing sales volume, maximizing growth, administrative behavior, the Japanese model directed on maximizing value added [29, 30].

The traditional firm theory recognizes that the behavior of a firm is defined by its only desire of maximizing profit (*model 1*).

The model of maximizing sales volume is the most widely known alternative of the model of maximizing profit. The model of maximizing sales is formed in two options: with criterion of maximizing sales under the condition of restricted resources (*model 2a*), and with criterion of minimizing prime cost while some volume of economic indicators is restricted (*model 2b*).

The model of maximizing growth (*model 3*) is characterized by a strategy with the cornerstone in continuous growth: long-term increase in production and sales [30].

The management theory of firm claims that the economic behavior of a firm is defined not by its owners, but the managers who have a purpose of maximizing sales volume and profit because of the income. The model of administrative behavior (*model 4*) includes the model of administrative benefit, the model of administrative prudence and the agency model [9].

The model of maximizing value added is a Japanese model (*model 5*) defined by that the value added is calculated as a difference between sales to the company for a certain time point and expenses (costs) of the goods of service acquired at external suppliers. Thereby the value added includes work, management, the capital, costs of profit. Such an approach implies that each worker and the shareholder of the firm maximizing value added knows that irrespective of economic conditions the priority has to be given to constant investments into capacities and the equipment, into researches and development, into development of the market. If there is a need to reduce remuneration to workers, the salary of the senior administrative personnel is reduced first. Japanese automakers and other companies, despite of economic conditions, seek to maximize value added in such a way year after year [30].

The emergence of such a variety of models indicates that none of them can adequately reflect the real life situations, which arise in practice of enterprise management. This results in new approaches (models) to the management of enterprises.

The economic theory of multiple goals stems from the fact that a firm has not one goal (sales, profit, growth), but a set of goals, which is a counterweight to single-criteria goals. Currently, a firm is a complex corporate system in which the organizational structure characterizes the hierarchy of subjects and objects of management. This management hierarchy corresponds to a hierarchy of interests and goals. First, the top management's interest is to increase the firm's prestige, improve its economic performance, and ensure its stability and sustainability. Secondly, it is in the interest of shareholders to raise higher dividends. Thirdly, managers' interest is aimed at improving their social status, making a good career, and ensuring income growth. Fourth, the interest of wage earners is to raise wages, get good working conditions, and improve their skills, professional growth, and so on. In the development of this direction, an approach based on a multi-purpose mathematical model is proposed. It is used in modeling the annual, strategic (long-term) development plan of the company.

3.2. Creation of mathematical model of the production plan of firm development

Currently, as it was stated above, there are some alternative mathematical behavior models of firms. We will unite the purposes of all models in the form of a vector of criteria. We will consider the restrictions of each model. We will present a vector of criteria and the restrictions in the form of a mathematical model, which represents a vector problem of mathematical programming.

Creation of mathematical model of the annual (strategic) plan of a firm assumes formation: a vector of variables, a vector of criteria (purposes) and restrictions imposed on the functioning of the firm [29, 30].

Vector of variables. Let us introduce a vector of variables:

$$X(t) = \left\{x_j, j = \overline{1, N}\right\},$$

where each component $j \in N$ determines the type $j = \overline{1, N}$ and the volume $x_j(t)$ of products, which included in production in the planned year of $t \in \boldsymbol{T}$. N is a set of indices of types (nomenclature) of products, works, and services. Restrictions of $u_j(t)$, $j \in N$ are imposed on the production variables $x_j(t)$, $j \in N$: they determine the probable volume of production of the j^{th} type. The values of $u_j(t)$, $j \in N$, as a rule, are obtained by the marketing department of the firm which conducts a research of the commodity market:

$$x_j(t) \le u_j(t), j = \overline{1, N}.$$

Vector of criteria which defines the functional objectives of the firm.

The firm production is characterized by a set of $\boldsymbol{K}$ technical-economic indicators. Let us represent the functional dependence of a $k \in \boldsymbol{K}$ economic indicator on the output of $X(t)$ using a $f_k(X(t))$ function, in the assumption that such functional dependence exists and is linear, i.e.:

$$\forall k \in \boldsymbol{K}, f_k\left(X\left(t\right)\right) = \sum\nolimits_{j=1}^{N} c_j^k x_j\left(t\right),$$

where c_j^k is a value of the k^{th} economic indicator (criterion) characterizing a unit of the j^{th} production type, $j \in \boldsymbol{N}$. In general, let us present a set of the indicators by a vector function:

$$F\left(X\left(t\right)\right) = \left\{f_k\left(X\left(t\right)\right) = \sum\nolimits_{j=1}^{N} c_j^k x_j\left(t\right),\ k = \overline{1,K}\right\}. \tag{3.1}$$

Let us divide the whole set of the $\boldsymbol{K}$ criteria into three subsets $\boldsymbol{K_1}$, $\boldsymbol{K_2}$, $\boldsymbol{K_3}$.

The first $\boldsymbol{K_1}$ *subset*: $F_1(X(t)) \subset F(X(t))$ depends on organizational structure of the enterprises (the neoinstitutional theory). It is supposed that the firm consists of $\boldsymbol{Q}$ – a set of local enterprises (divisions), at each of them the development purposes functionally depend on the $f_q(X(t))$, $q \in \boldsymbol{Q}$ output of this q^{th} enterprise. Each q^{th} enterprise is presented by a set of criteria of $\boldsymbol{K}_q$, $q \in \boldsymbol{Q}$:

$$f_q\left(X\left(t\right)\right) = \left\{f_{kq}\left(X\left(t\right)\right),\ k = \overline{1,K_q},\ q \in \boldsymbol{Q}\right\}.$$

If $\boldsymbol{Q} = 1$, then is a standard definition of the firm. A set of criteria (purposes) of all enterprises is presented by a vector function:

$$F_1\left(X\left(t\right)\right) = \left\{f_q\left(X\left(t\right)\right),\ q = \overline{1,K_1}\right\},\ \boldsymbol{K_1} = \boldsymbol{Q},\ \boldsymbol{K_1} \subset \boldsymbol{K}.$$

The second $\boldsymbol{K_2}$ *subset* defines the purposes of the firm in general: it is the highest managing subsystem. Each criterion $k = \overline{1,K_2}$ includes the corresponding indicators of all local subsystems and contains the nomenclature, volumes, technical and economic indicators of the products produced by the corporation as a whole:

$$F_2\left(X\left(t\right)\right) = \left\{f_k\left(X\left(t\right)\right),\ k = \overline{1,K_2}\right\},\ \boldsymbol{K_2} \subset \boldsymbol{K}.$$

The vector of $F_2(X(t))$ includes sales volumes of the made production, profits, value added, etc. It is desirable to maximize these indicators.

The third $\boldsymbol{K_3}$ *subset* includes economic indicators related to costs that are desirable to minimize:

$$F_3\left(X\left(t\right)\right) = \left\{f_k\left(X\left(t\right)\right),\ k = \overline{1,K_3}\right\},\ \boldsymbol{K_3} \subset \boldsymbol{K}.$$

Indicators (criteria) $F_3(X(t))$ characterize the minimization of material and labor costs, which together determine the cost of production. Economic indicators (criteria) $\boldsymbol{K_2}$, $\boldsymbol{K_3}$ represent the system characteristics of the corporation (firm). They determine the relationship between the firm and the society:

$$\boldsymbol{K_1} \cup \boldsymbol{K_2} \cup \boldsymbol{K_3} = \boldsymbol{K},$$

where $\boldsymbol{K}$ is the set of indicator (criteria) indices of the firm development in general.

The restrictions that are imposed when developing a plan include two groups. The first (main) group of restrictions is related to resource (technological) costs. They include restrictions on production capacity, labor, and material and technical resources. The second group of restrictions is determined by the planned indicators, which in the very least need to be obtained.

Resource restrictions. We assume that the dependence of resource costs on the volume of goods produced $X(t)=\{x_j,\ j=\overline{1,N}\}$ is linear:

$$\sum_{j=1}^{N} a_{ij}(t)x_j(t)\le b_i(t),\ i=\overline{1,M}, \tag{3.2}$$

where $a_{ij}(t),\ i=\overline{1,M},\ j=\overline{1,N}$ is a norm which characterizes the quantity of the i^{th} resource necessary for production of a unit of the j^{th} product type.

The set of indices of resources $\boldsymbol{M}$ includes:

a set of material resources $\boldsymbol{M}_{mat}\subset\boldsymbol{M}$, which characterizes the materials, semi-finished products, etc., used in production;

a set of labor resources $\boldsymbol{M}_{tr}\subset\boldsymbol{M}$ involved in production;

a set of funded resources (capacities) $\boldsymbol{M}_f\subset\boldsymbol{M}$ in the period $t\in\boldsymbol{T}$.

Similarly, (3.2) presents the costs of the i^{th} resource of the q^{th} division:

$$\sum_{j=1}^{N} a_{ij}^q(t)x_j(t)\le b_i^q(t),\ i=\overline{1,M_q},\ i=\overline{1,Q},$$

where $b_i^q(t),\ i=\overline{1,M_q},\ i=\overline{1,Q}$ is the value of the i^{th} resource which is available in the q^{th} division of the enterprise for the planned period. $\boldsymbol{M}_q$ is a set of resources types which are used in production in q^{th} division.

The restrictions connected with the planned indicators:

$$\sum_{j=1}^{N} c_i x_j(t)\ge b_k(t), \tag{3.3}$$

where c_i is the value of the j^{th} economic indicator characterizing the production unit, b_k is the minimum value of the planned indicator which the firm should achieve in the planned period of time, $t\in\boldsymbol{T}$.

Variable expenses depend on the output:

$a_{ij}(t),\ i=\overline{1,M},\ j=\overline{1,N}$ is the i^{th} resource costs needed to produce a unit of the j^{th} type of production (norm), i, $\boldsymbol{M}$ are an index and a set of all types of resources (material, labor, etc.), i.e. the variable expenses changing in proportion to output and used by production of all types;

$G_i(X)=\sum_{j=1}^{N} a_{ij}(t)x_j(t)\le b_i(t),\ i=\overline{1,M}$ is the costs of the i^{th} resource of all production types.

We will present the production prime cost of a production unit as a sums of variable expenses:

$$a_j^p=\sum_{i=1}^{M_{mat}} p_i a_{ij}^{mat}+\sum_{i=1}^{M_{tr}} p_i a_{ij}^{tr}+\sum_{i=1}^{M_f} p_i a_{ij}^{f},\ j=\overline{1,N}, \tag{3.4}$$

where a_{ij}^{mat}, a_{ij}^{tr}, a_{ij}^{f} are costs of a unit of production and p_i is the cost of the material, labor, and accumulated resources respectively;

$A^p(t)=\{a_j^p(t),\ j=\overline{1,N}\}$ is a vector of the planned production prime cost of a unit of all types of production.

Constant expenses don't depend on the output. They are calculated per product unit a_j^{nak} (depreciation charges, administrative and managerial expenses, maintenance costs of buildings and the equipment). In general, the planned full prime cost of a unit of production is defined as the sum of production prime cost and of overhead costs:

$$a_j=a_j^p+a_j^{nak},\ j=\overline{1,N}.$$

The full prime a_j is a basis for formation of market value of production. We can define (as far as possible) the costs per production unit of similar goods of competitor company and compare them with a_j in a similar way.

The price of the p_j production unit of the j^{th} type follows from market researches or on the basis of a calculation of level of the price taking into account the pricing policy. Important role is played by methods of calculation of the settlement price from the prime cost of a_j, $j=\overline{1,N}$.

Profit: the gross profit per unit is calculated as a difference between the cost of p_j and variable expenses $\pi_j^{val}=p_j-a_j^p$, $j=\overline{1,N}$.

The profit on product sales in the firm as a whole equals:

$$\pi=\sum_{j=1}^{N}\pi_j x_j(t). \tag{3.5}$$

The value added is determined per a unit of production as a difference between the p_j cost and material inputs a_j^{mat} for production of j^{th} type:

$$p_j^{dob}=p_j-a_j^{mat},\ j=\overline{1,N}, \tag{3.6}$$

where $a_j^{mat}=\sum_{i=1}^{M_{mat}}p_i a_{ij}$, $j=\overline{1,N}$ is the cost of material inputs per unit of the j^{th} production type arriving from external producers.

Using the calculated indicators (3.1)–(3.6), we construct the above described theoretical models.

Model 1: maximizing profit. Define:

$$max\ f(X(t))\equiv\sum_{j=1}^{N}\pi_j x_j(t) \tag{3.7}$$

at restrictions

$$\sum_{j=1}^{N}a_{ij}(t)x_j(t)\le b_i(t),\ i=\overline{1,M}, \tag{3.8}$$

$$x_j(t)\le u_j(t),\ j=\overline{1,N}. \tag{3.9}$$

Model 2a: maximizing sales volume. Define:

$$max\ f(X(t))\equiv\sum_{j=1}^{N}p_j x_j(t), \tag{3.10}$$

at restrictions

$$\sum_{j=1}^{N}a_{ij}(t)x_j(t)\le b_i(t),\ i=\overline{1,M}, \tag{3.11}$$

$$x_j(t)\le u_j(t),\ j=\overline{1,N}. \tag{3.12}$$

Model 2b: minimization of total expenses. Define:

$$min\ f(X(t))\equiv\sum_{i=1}^{M}c_i\sum_{j=1}^{N}a_{ij}(t)x_j(t), \tag{3.13}$$

at restrictions

$$\sum_{j=1}^{N}c_j^k x_j(t)\ge b_k(t),\ k\in K, \tag{3.14}$$

$$x_j(t) \le u_j(t),\ j = \overline{1,N}, \tag{3.15}$$

where c_j^k is the economic indicator characterizing a unit of the j^{th} type of production, b_k is a set economic indicator characterizing such volume of production at which $b_k(t)$ has to be reached ($\geq$).

Model 3: maximizing growth. Define:

$$max\ f(X(t)) \equiv \sum_{j=1}^{N} \pi_j x_j(t), \tag{3.16}$$

at restrictions

$$\sum_{j=1}^{N} a_{ij}(t) x_j(t) \le b_i(t) + \Delta b_i(t+\Delta t),\ i = \overline{1,M}, \tag{3.17}$$

$$x_j(t) \le u_j(t),\ j = \overline{1,N}, \tag{3.18}$$

where $\Delta b_i(t + \Delta t)$ is a planned gain of volume of resources as of the planned period Δt which is created due to depreciation charges, profit, or of loans.

Model 4: administrative behavior. Define:

$$max\ f(X(t)) \equiv \sum_{j=1}^{N} p_j^z x_j(t), \tag{3.19}$$

at restrictions

$$\sum_{j=1}^{N} a_{ij}(t) x_j(t) \le b_i(t),\ i = \overline{1,M}, \tag{3.20}$$

$$x_j(t) \le u_j(t),\ j = \overline{1,N}, \tag{3.21}$$

where p_j^z is the economic indicator determines the salary volume from production and sale of a unit of the j^{th} type of production.

Model 5: maximizing value added, Japanese model. Define:

$$max\ f(X(t)) \equiv \sum_{j=1}^{N} p_j^{dob} x_j(t), \tag{3.22}$$

at restrictions

$$\sum_{j=1}^{N} a_{ij}(t) x_j(t) \le b_i(t),\ i = \overline{1,M}, \tag{3.23}$$

$$x_j(t) \le u_j(t),\ j = \overline{1,N}, \tag{3.24}$$

where p_j^{dob} is the economic indicator (3.6) which determines the value added volume for the production and sale of a unit of the j^{th} type of production.

Economic and mathematical models: the model 1 (3.7)–(3.9), ..., model 5 (3.22)–(3.24) represent the current state of mathematical modeling of production systems.

3.3. Construction of a mathematical model of a manufacturing company in the form of a vector linear programming problem

The economic theory of plurality of the purposes assumes that the listed above purposes of all models actually exist and have to be considered simultaneously. We will present such objectives in a model of the annual plan of the enterprise in the form of a vector problem of linear programming:

$$opt\ F\left(X(t)\right)=\{max\ F_1\left(X(t)\right)=\left\{max\ f_q\left(X(t)\right)=\right.$$
$$=\left\{max\ f_{kq}\left(X(t)\right)\equiv \sum\nolimits_{j=1}^{N_q} c_j^k x_j(t),\ k=\overline{1,K_q}\right\},\ q=\overline{1,Q}\},\tag{3.25}$$

$$max\ F_2\left(X(t)\right)=\left\{max\ f_k\left(X(t)\right)\equiv \sum\nolimits_{j=1}^{N} c_j^k x_j(t),\ k=\overline{1,K_2}\right\},\tag{3.26}$$

$$min\ F_3\left(X(t)\right)=\left\{min\ f_k\left(X(t)\right)\equiv \sum\nolimits_{i=1}^{M} c_i \sum\nolimits_{j=1}^{N} a_{ij}(t)x_j(t),\ k=\overline{1,K_3}\right\},\tag{3.27}$$

at restrictions

$$\sum\nolimits_{j=1}^{N} a_{ij}(t)x_j(t)\le b_i(t),\ i=\overline{1,M},\tag{3.28}$$

$$\sum\nolimits_{j=1}^{N_q} a_{ij}^q x_j(t)\le b_i^q(t),\ i=\overline{1,M_q},\ q=\overline{1,Q},\tag{3.29}$$

$$\sum\nolimits_{j=1}^{N} c_j^k x_j(t)\ge b_k(t),\ k\in K,\tag{3.30}$$

$$0\le x_j(t)\le u_j(t),\ j=\overline{1,N},\tag{3.31}$$

where $F(X(t))$ is vector criterion (2.25), the set of criteria of which is subdivided into three subsets:

$\boldsymbol{K_1}$ is a subset of criteria of the divisions of the firm, $k=\overline{1,K_q}$, $q=\overline{1,Q}$, $\boldsymbol{K_1}$ = Q;

$\boldsymbol{K_2}$ is a subset of criteria each component of which should be maximized (sales volumes of the made production, profits, value added, etc.);

$\boldsymbol{K_3}$ is a subset of criteria aimed at minimizing the indicators connected with prime cost of products; $\boldsymbol{K_2}$, $\boldsymbol{K_3}$ are the system criteria characterizing the firm in general (3.26), (3.27):

$$\boldsymbol{K_1}\cup\boldsymbol{K_2}\cup\boldsymbol{K_3}=\boldsymbol{K}.$$

In (3.25), the vector of criteria of $F(X(t))$ reflects the purposes of all models in total: (3.7)–(3.9) is the model of profit, ..., (3.22)–(3.24) is the model of maximizing value added;

$X=\left\{x_j(t),\ j=\overline{1,N}\right\}$ is a vector of variables each component of which defines the quantity of the j^{th} type products included in the plan;

c_j^k, $j=\overline{1,N}$ is a set of economic indicators of the k^{th} $k=\overline{1,K}$ type characterizing a unit of the j^{th} type of production.

A set of restrictions (3.28)–(3.31) which in total reflect the restrictions of all models under consideration: profits (3.8) – (3.9), ..., models of maximizing value added (3.23) – (3.24).

We will notice that a problem of definition:

$$\forall q\in\boldsymbol{Q}, max\ f_q\left(X(t)\right)=\left\{max\ f_{kq}\left(X(t)\right)\equiv \sum\nolimits_{j=1}^{N_q} c_j^k x_j(t),\ k=\overline{1,K_q}\right\},$$

at restrictions (3.27)–(3.31) represents a model of a separate division of the firm in a form of a vector problem of linear programming.

For the solution of a vector problem of linear programming (3.25)–(3.31), the methods based on normalization of criteria and the principle of the guaranteed result are used [31, 32].

4. A vector problem of mathematical programming

A vector problem in mathematical programming (VPMP) is a standard mathematical-programming problem including a set of criteria, which, as a whole, represent a vector of criteria.

It is important to distinguish between uniform and non-uniform VPMP:

A uniform maximizing VPMP is a vector problem in which each criterion is directed towards maximizing;

A uniform minimizing VPMP is a vector problem in which each criterion is directed towards minimizing;

A non-uniform VPMP is a vector problem in which the set of criteria is shared between two subsets (vectors) of criteria (maximization and minimization respectively), e.g., inhomogeneous VPMP is a combination of two types of homogeneous problems.

According to these definitions, we will present a VPMP with non-uniform criteria in the following form:

$$Opt\ F(X) = \{\max F_1(X) = \{\max f_k(X), k = \overline{1, K_1}, \tag{4.1}$$

$$\min F_2(X) = \{\min f_k(X), k = \overline{1, K_2}\}, \tag{4.2}$$

$$G(X) \le B, \tag{4.3}$$

$$X \ge 0, \tag{4.4}$$

where $X = \{x_j, j = \overline{1, N}\}$ is a vector of material variables of an N-dimensional Euclidean space of R^N, (notation $j = j = \overline{1, N}$ is equivalent to $j = 1,..., N$);

$F(X)$ is a vector function (vector criterion) having K which is a function component (K is the power of the $\boldsymbol{K}$ set): $F(X) = \{f_k(X), k = \overline{1, K}\}$. The $\boldsymbol{K}$ set consists of the $\boldsymbol{K}_1$, subset of maximization criteria and the $\boldsymbol{K}_2$ subset of minimization; $\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2$, therefore we introduce an "*opt*" notation of the operation, which includes *max* and *min*;

$F_1(X) = \{f_k(X), k = \overline{1, K_1}\}$ is a maximizing vector criterion, K_1 is the number of criteria, and $K_1 \equiv \overline{1, K_1}$ is a set of maximizing criteria (problem (4.1), (4.3), (4.4) represents VPMP with the homogeneous maximizing criteria). Let's further assume that $f_k(X), k = \overline{1, K_1}$ are continuous concave functions (we will sometimes call them the maximizing criteria);

$F_2(X) = \{f_k(X), k = \overline{1, K_2}\}$ is a vector criterion in which each component is minimized, $K_2 \equiv$ $\equiv \overline{K_1 + 1, K} \equiv \overline{1, K_2}$ are a set of minimization criteria, $\boldsymbol{K}_2$ is the number of criteria (the problems (4.2)–(4.4) are VPMP with the homogeneous minimization criteria). We assume that $f_k(X), k = \overline{1, K_2}$ are continuous convex functions (we will sometimes call them the minimization criteria), i.e., $\boldsymbol{K}_1 \cup \boldsymbol{K}_2 = \boldsymbol{K}$, $\boldsymbol{K}_1 \subset \boldsymbol{K}, \boldsymbol{K}_2 \subset \boldsymbol{K}$.

$G(X) \le B, X \ge 0$ is standard restrictions, $g_i(X) \le b_i, i = 1,..., M$ where b_i is a set of real numbers, and the $g_i(X)$ functions are assumed to be continuous and convex.

$$\boldsymbol{S} = \{X \in \boldsymbol{R}^n \mid X \ge 0, G(X) \le B, X^{min} \le X \le X^{max}\} \ne \varnothing,$$

where the set of admissible points set by standard restrictions (4.3)–(4.4) is not empty and represents a compact.

The vector minimization function (criterion) $F_2(X)$ can be transformed to the vector maximization function (criterion) by the multiplication of each component of $F_2(X)$ by minus one. The vector criterion of $F_2(X)$ is introduced into the VPMP (4.1)–(4.4) to show that, in the problem, there are two subsets of criteria of $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ with essentially different directions of optimization.

We assume that the optimum points received by each criterion do not coincide for at least two criteria. If all points of an optimum coincide among themselves for all criteria, then we regard the decision as trivial.

5. Theory and methods of vector optimization as mathematical support for modeling in the digital economy of the state

The theory of vector optimization includes theoretical foundations (axioms) and methods of solving vector problems both with equivalent criteria and with the given criterion priority. The theory is a basis of mathematical apparatus of modeling of the "object for optimal decision-making".

As an "object of decision-making", we consider the socio-economic development of the state, including objects at the level of: firms (enterprises), municipalities, regions and the state as a whole. We presented the axioms and methodology for solving vector optimization problems (4.1)–(4.4) with equivalent criteria and a given criterion priority. [32, 33].

5.1. The axioms and the principle of optimality for vector optimization with the equivalent criteria

Definition 1. (*Definition of the relative estimate of the criterion*).

Let us introduce a notation:

$$\lambda_k\left(X\right)=\frac{f_k\left(X\right)-f_k^0}{f_k^*-f_k^0},\ \forall k\in \boldsymbol{K} \tag{5.1}$$

in vector problem (4.1)–(4.4), which is a relative estimate of the $k \in \boldsymbol{K}$ criterion at the point of $X \in \boldsymbol{S}$;

f_k^* is the value of the k^{th} criterion at the point of optimum X_k^*, obtained in vector problem (4.1)–(4.4) for the individual k^{th} criterion; f_k^0 is the worst value of the k^{th} criterion (anti optimum) at the X_k^0 point (superscript 0 for zero) on the admissible set $\boldsymbol{S}$ in vector problem (4.1)–(4.4);

The value of f_k^0 is the lowest value of the k^{th} criterion in the *max* problem (4.1), (4.3), (4.4): $f_k^0 = min_{X\in S} f_k\left(X\right) \forall k \in K_1$, and in the *min* problem, f_k^0 is the greatest: $f_k^0 = max_{X\in S} f_k\left(X\right) \forall k \in K_2$.

First, the relative estimate of the $\lambda_k(X)$ $\forall$ $k \in K$, is performed in relative units; secondly, the relative estimate of the $\lambda_k(X)$ $\forall$ $k \in K$ in the admissible set changes from zero in the X_k^0 point $\left(\forall k \in \boldsymbol{K}\ \lim_{X_\to \to X_k^0} \lambda_k\left(X\right)=0\right)$ to one at the optimum X_k^* point of an optimum of $X_k^*\left(\forall k \in \boldsymbol{K}\ \lim_{X_\to \to X_k^*} \lambda_k\left(X\right)=1\right)$, therefore

$$\forall k\in \boldsymbol{K}\ 0\le \lambda_k\left(X\right)\le 1, X\in \boldsymbol{S}. \tag{5.2}$$

This allows a comparison of the criteria, measured in relative units, and their use in joint optimization.

Axiom 1. (About equality and equivalence of criteria in an admissible point of vector problems).

In the VPMP, two criteria with the indices $k \in K$, $q \in K$ shall be considered as equal in the $X \in S$ point, if relative estimates of the k^{th} and q^{th} criterion are equal in this point, i.e. $\lambda_k\left(X\right)=\lambda_q\left(X\right), k,q\in K$. We will consider criteria equivalent in the VPMP, if in the $X \in S$ point when numerically comparing the relative estimates of $\lambda_k\left(X\right), k=\overline{1,K}$, there are no conditions about priorities of criteria imposed on each criterion of $f_k\left(X\right), k=\overline{1,K}$, and, respectively, relative estimates of $\lambda_k(X)$.

Definition 2. (*Definition of a minimum level among all relative estimates of criteria*).

The relative level λ in the vector problem represents the lower estimate of a point of $X \in S$ among all relative estimates of $\lambda_k\left(X\right), k=\overline{1,K}$:

$$\forall X\in \boldsymbol{S}\ \lambda\le \lambda_k\left(X\right), k=\overline{1,K}, \tag{5.3}$$

the lower level for performance of a condition (5.3) in the admissible point of $X \in S$ is determined by a formula

$$\forall X\in \boldsymbol{S}\ \lambda = min_{k\in K}\lambda_k\left(X\right). \tag{5.4}$$

Ratios (5.3) and (5.4) are interconnected. They serve as transition from operation (5.4) of definition of *min* to restrictions (5.3) and vice versa.

The lower relative λ level allows uniting all criteria in the vector problem by one numerical characteristic of λ and to perform certain operations with it, thereby, at the same applying these operations to all the criteria measured in relative units. The λ level functionally depends on the $X \in \boldsymbol{S}$ variable, therefore by changing X, we can change the lower λ level.

Further, we formulate the rule of searching of the optimum solution.

Definition 3. (*The principle of an optimality with equivalent criteria*).

The VPMP with equivalent criteria is solved, if we find such a point of $X^o \in \boldsymbol{S}$ and such a maximum level of λ^o (superscript o for optimum) among all relative estimates that

$$\lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}} \boldsymbol{min}_{\boldsymbol{k} \in \boldsymbol{K}} \lambda_k \left(X \right). \tag{5.5}$$

Using interrelation of Eqs. (5.3) and (5.4), we transform a maximin problem (5.5) into an extreme problem

$$\lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}} \lambda, \tag{5.6}$$

at restrictions

$$\lambda \le \lambda_k \left(X \right), k = \overline{1, K}. \tag{5.7}$$

Let's call optimization problem (5.6)–(5.7) the λ-problem.

λ-problem (5.6)–(5.7) has (N+1) dimension, and consequently the result of solving λ-problem (5.6)–(5.7) consists in an optimum vector of $\boldsymbol{X^o} \in R^{N+1}$, in which the $(N + 1)^{\text{th}}$ component is essentially the λ^o value. Thus in the optimum point of $\boldsymbol{X^o} = \left\{ x_1^o, x_2^o, ..., x_N^o, x_{N+1}^o \right\}$, the component $x_{N+1}^o = \lambda^o$, and the $(N + 1)$ component of the $\boldsymbol{X}^o$ vector is selected in view of its specificity.

The received pair of $\{\lambda^o, X^o\} = \boldsymbol{X}^o$ characterizes the optimum solution of λ-problem (5.6)–(5.7) and at the same time is the solution to VPMP (4.1)–(4.4) with the equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result. In the optimum solution of $\boldsymbol{X}^o = \{\lambda^o, X^o\}$, we will denote X^o as an optimal point, and λ^o as a maximum level.

The following theorem is an important result of the algorithm for solving vector problems (4.1)–(4.4) with equivalent criteria [32].

Theorem 1. (*The theorem of the two inconsistent criteria in the vector problem of mathematical programming with equivalent criteria*).

In the convex VPMP with the equivalent criteria which is solved on the basis of normalization of criteria and the principle of the guaranteed result, in an optimum point of $\boldsymbol{X}^o = \{\lambda^o, X^o\}$, there are always two criteria: let us denote them with indices $q \in \boldsymbol{K}, p \in \boldsymbol{K}$ (which in a sense are the most inconsistent in the $k = \overline{1, K}$ criteria set), for which the following equality holds:

$$\lambda^o = \lambda_q \left(X^o \right) = \lambda_p \left(X^o \right), q, p \in \boldsymbol{K}, X \in \boldsymbol{S}, \tag{5.8}$$

and other criteria are determined by inequalities:

$$\lambda^o \le \lambda_k \left(X^o \right), \forall k \in \boldsymbol{K}, q \ne p \ne k. \tag{5.9}$$

5.2. Axioms and the principle of optimality of vector optimization with a criterion priority

To develop the methods of solving the vector optimization problems with a priority of criterion, we use the following definitions:

Priority of one criterion of vector problems, with a criterion priority over other criteria;

Numerical expression of a priority;

Set priority of a criterion;

Lower (minimum) level among all criteria with a priority of one of them;

Subset of points with priority by criterion (Axiom 2);

The principle of optimality of the solution of problems of vector optimization with the set priority of one of the criteria, and related theorems.

For more details, see [32, 33].

Definition 4. (*About the priority of one criterion over the other*).

The criterion of $q \in \boldsymbol{K}$ in the vector problem of Eqs. (4.1)–(4.4) in a point of $X \in \boldsymbol{S}$ has priority over other criteria of $k = \overline{1, K}$, and the relative estimate of $\lambda_q(X)$ by this criterion is greater than or equal to relative estimates of $\lambda_k(X)$ of other criteria, i.e.:

$$\lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, \tag{5.10}$$

and a strict priority for at least one criterion of $t \in \boldsymbol{K}$, $\lambda_q(X) > \lambda_k(X)$, $t \neq q$, and for other criteria of $\lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, k \neq t \neq q$.

Introduction of the definition of a priority of criterion $q \in \boldsymbol{K}$ in the vector problem of Eqs. (4.1)–(4.4) executed the redefinition of the early concept of a priority. Earlier the intuitive concept of the importance of this criterion was outlined, now this "importance" is defined as a mathematical concept: the higher the relative estimate of the q^{th} criterion compared to others, the more important it is (i.e., has higher priority), and the highest priority at a point of an optimum is $X_k^*, \forall q \in \boldsymbol{K}$.

From the definition of a priority of criterion of $q \in \boldsymbol{K}$ in the vector problem of Eqs. (4.1)–(4.4), it follows that it is possible to reveal a set of points $\boldsymbol{S}_q \subset \boldsymbol{S}$ that is characterized by $\lambda_q(X) \geq \lambda_k(X), \forall k \neq q$, $\forall X \in \boldsymbol{S}_q$. However, the answer to whether a criterion of $q \in \boldsymbol{K}$ at a point of the set $\boldsymbol{S}_q$ has more priority than others do remains open. For clarification of this question, we define a communication coefficient between a couple of relative estimates of q and k that, in total, represent a vector:

$$P^q(X) = \left\{ p_k^q(X) \middle| k = \overline{1, K} \right\}, q \in \boldsymbol{K}\ \forall X \in \boldsymbol{S}_q.$$

Definition 5. (*About numerical expression of a priority of one criterion over another*).

In the vector problem of Eqs. (4.1) and (4.4), with priority of the q^{th} criterion over other criteria of $k = \overline{1, K}$, for $\forall X \in \boldsymbol{S}_q$, and a vector of $P^q(X)$ which shows by how many times a relative estimate of $\lambda_q(X)$, $q \in \boldsymbol{K}$, is greater than other relative estimates of $\lambda_k(X), k = \overline{1, K}$, we define a numerical expression of the priority of the q^{th} criterion over other criteria of $k = \overline{1, K}$ as:

$$P^q(X) = \left\{ p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K} \right\},$$

$$p_k^q(X) \geq 1, \forall X \in \boldsymbol{S}_q \subset S, k = \overline{1, K}, \forall q \in \boldsymbol{K}. \tag{5.11}$$

This definition of priority in the form of the ratio $p_k^q(X) = \dfrac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}, \forall q \in \boldsymbol{K}$ is called a *numerical expression of the priority* of the q^{th} criterion over the other criteria $k = \overline{1, K}$.

Definition 6. (*About the set numerical expression of a priority of one criterion over another*).

In the vector problem of Eqs. (4.1)–(4.4) with a priority of criterion of $q \in \boldsymbol{K}$ for $\forall X \in \boldsymbol{S}$, vector $P^q = \left\{p_k^q, k = \overline{1,K}\right\}$ is considered to be set by the person making decisions (i.e., the decision-maker) if every component of this vector is set. Set by the decision-maker, the p_k^q component, from the point of view of the decision-maker, shows by how many times a relative estimate of $\lambda_q(X)$, $q \in \boldsymbol{K}$ is greater than other relative estimates of $\lambda_k\left(X\right), k = \overline{1,K}$. The vector of $p_k^q, k = \overline{1,K}$, is the numerical expression of the priority of the q^{th} criterion over other criteria of $k = \overline{1,K}$:

$$P^q\left(X\right) = \left\{p_k^q\left(X\right), k = \overline{1,K}\right\}, \; p_k^q\left(X\right) \geq 1, \; \forall X \in \boldsymbol{S}_q \subset S, \quad k = \overline{1,K}, \; \forall q \in \boldsymbol{K}. \tag{5.12}$$

The vector problem of Eqs. (4.1)–(4.4), in which the priority of any criteria is set, is called a vector problem with the set priority of criterion. The problem of a priorities vector arises when it is necessary to determine the point $X^o \in \boldsymbol{S}$ by the set vector of priorities. In the comparison of relative estimates with a priority of criterion of $q \in \boldsymbol{K}$, as well as in a problem with equivalent criteria, we define the additional numerical characteristic of λ which we will refer to as *the level*.

Definition 7. (*About the lower level among all relative estimates with a criterion priority*).

The λ level is the lowest among all relative estimates with a priority of criterion $q \in \boldsymbol{K}$ such that:

$$\lambda \leq p_k^q \lambda_k\left(X\right), k = \overline{1,K}, q \in \boldsymbol{K}, \forall X \in \boldsymbol{S}_q \subset S. \tag{5.13}$$

The lower level for the performance of the condition in Eq. (5.13) is defined as:

$$\lambda = min_{k \in K} p_k^q \lambda_k\left(X\right), q \in \boldsymbol{K}, \forall X \in \boldsymbol{S}_q \subset S. \tag{5.14}$$

Eqs. (5.13) and (5.14) are interconnected and serve as a further transition from the operation of determining the *min* to restrictions, and vice versa. In Section 3.1, we gave the definition of a Pareto optimal point $X^o \in \boldsymbol{S}$ with equivalent criteria. Considering this definition as an initial one, we will construct a number of axioms dividing an admissible set of $\boldsymbol{S}$ into, first, a subset of Pareto optimal points S^o, and, secondly, a subset of points $\boldsymbol{S}_q \subset \boldsymbol{S}$, $q \in \boldsymbol{K}$, with priority for the q^{th} criterion.

Axiom 2. (About a subset of points, priority by criterion in the VPMP).

In the vector problem of Eqs. (4.1)–(4.4), the subset of points $S_q \subset S$ is called the area of priority of criterion of $q \in K$ over other criteria, if

$$\forall X \in S_q \; \forall k \in K \; \lambda_q\left(X\right) \geq \lambda_k\left(X\right), q \neq k.$$

This definition extends to a set of Pareto optimal points S^o that is given by the following definition.

Axiom 2a. (About a subset of points, priority by criterion, on Pareto's great number in a vector problem).

In the VPMP, a subset of points $\boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset S$ is called the area of a priority of the $q \in \boldsymbol{K}$ criterion over other criteria, if $\forall X \in S_q^o \; \forall k \in K \; \lambda_q\left(X\right) \geq \lambda_k\left(X\right), q \neq k$.

In the following we provide explanations.

Axiom 2 and 2a allow breaking the vector problem in Eqs. (4.1)–(4.4) into an admissible set of points $\boldsymbol{S}$, including a subset of Pareto optimal points, $\boldsymbol{S}^o \subset \boldsymbol{S}$, and subsets:

one subset of points $\boldsymbol{S}' \subset \boldsymbol{S}$ where criteria are equivalent, and a subset of points of $\boldsymbol{S}'$ crossing with a subset of points $\boldsymbol{S}^o$, allocated to a subset of Pareto optimal points at equivalent criteria $\boldsymbol{S}^{oo} = \boldsymbol{S}' \cap \boldsymbol{S}^o$. As will be shown further, this consists of one point of $X^o \in \boldsymbol{S}$, i.e.

$$X^o = \boldsymbol{S}^{oo} = \boldsymbol{S}' \cap \boldsymbol{S}^o, \boldsymbol{S}' \in \boldsymbol{S}, \boldsymbol{S}^o \subset \boldsymbol{S}.$$

"$\boldsymbol{K}$" subsets of points where each criterion of $q=\overline{1,K}$ has a priority over other criteria of $k=\overline{1,K}$, $q \neq k$, and thus breaks, first, sets of all admissible points $\boldsymbol{S}$, into subsets $\boldsymbol{S}_q \subset \boldsymbol{S}$, $q=\overline{1,K}$ and, second, a set of Pareto optimal points, $\boldsymbol{S}^o$, into subsets $\boldsymbol{S}_q^o \subset \boldsymbol{S}_q \subset \boldsymbol{S}$, $q=\overline{1,K}$. This yields:

$$\boldsymbol{S}'U\left(U_{q\in K}\boldsymbol{S}_q^o\right) \equiv \boldsymbol{S}^o, \boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S}, q=\overline{1,K}.$$

We note that the subset of points $\boldsymbol{S}_q^o$, on the one hand, is included in the area (a subset of points) of priority of criterion of $q \in \boldsymbol{K}$ over other criteria: $\boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S}$, and, on the other, in a subset of Pareto optimal points $\boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S}$.

Axiom 2 and the numerical expression of priority of criterion (Definition 5) allow the identification of each admissible point of $X \in \boldsymbol{S}$ (by means of vector):

$P^q(X)=\left\{p_k^q(X)=\frac{\lambda_q(X)}{\lambda_k(X)}, k=\overline{1,K}\right\}$, to form and choose:

a subset of points by priority criterion $\boldsymbol{S}_q$, which is included in a set of points $\boldsymbol{S}$, $\forall q \in \boldsymbol{K}\ X \in \boldsymbol{S}_q \subset \boldsymbol{S}$, (such a subset of points can be used in problems of clustering, but is beyond this article);

a subset of points by priority criterion $\boldsymbol{S}_q^o$, which is included in a set of Pareto optimal points $\boldsymbol{S}^o$, $\forall q \in K, X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}^o$.

Thus, full identification of all points in the vector problem of Eqs. (4.1)–(4.4) is executed in sequence as:

Set of admissible points $X \in \boldsymbol{S} \rightarrow$	Subset of Pareto optimal points $X \in \boldsymbol{S}^o \subset \boldsymbol{S} \rightarrow$	Subset of Pareto optimal points, $X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S} \rightarrow$	Separate point $\forall X \in \boldsymbol{S}\ X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S}$

This is the most important result which allows constructing the principle of optimality and methods of choosing any point from a set of Pareto optimal points.

Definition 8. (*Principle of optimality 2. The solution of a vector problem with the set criterion priority*).

The vector problem of Eqs. (4.1)–(4.4) with the set priority of the q^{th} criterion of $p_k^q\lambda_k(X)$, $k=\overline{1,K}$, is considered solved if there is such a point X^o and such a maximum level λ^o found among all relative estimates that:

$$\lambda^o = \boldsymbol{max}_{X\in S}\boldsymbol{min}_{k\in K}\, p_k^q\lambda_k(X), q \in \boldsymbol{K}. \tag{5.15}$$

Using the interrelation of Eqs. (5.13) and (5.14), we can transform the maximin problem of Eq. (5.15) into an extreme problem of the form:

$$\lambda^o = \boldsymbol{max}_{X\in S}\lambda, \tag{5.16}$$

at restriction

$$\lambda \leq p_k^q\lambda_k(X), k=\overline{1,K}. \tag{5.17}$$

We call Eqs. (5.16) and (5.17) the λ-problem with a priority of the q^{th} criterion.

The solution of the λ-problem is the point $\boldsymbol{X}^o = \{X^o, \lambda^o\}$. This is also the result of the solution of the vector problem of Eqs. (4.1)–(4.4) with the set priority of the criterion, solved on the basis of normalization of criteria and the principle of the guaranteed result.

In the optimum solution $\boldsymbol{X}^o = \{X^o, \lambda^o\}$, X^o, an optimum point, and λ^o, the maximum lower level, the point of X^o and the λ^o level correspond to restrictions of Eq. (5.9), which can be written as:

$$\lambda^o \le p_k^q \lambda_k\left(X^o\right), k = \overline{1, K}.$$

These restrictions are the basis of an assessment of the correctness of the solution in the practical vector problems of optimization.

From Definitions 1 and 2, "Principles of optimality", follows the opportunity to formulate the concept of the operation "opt".

Definition 9. (*Mathematical operation "opt"*).

In the vector problem of Eqs. (4.1)–(4.4), in which "max" and "min" are part of the criteria, the mathematical operation "opt" consists in determining such a point X^o and such a maximum lower level λ^o in which all criteria are measured in relative units:

$$\lambda^o \le \lambda_k\left(X^o\right) = \frac{f_k\left(X\right) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}, \tag{5.18}$$

i.e., all criteria of $\lambda_k\left(X^o\right), k = \overline{1, K},$ are equal to or greater than the maximum level of λ^o (therefore λ^o is also called the guaranteed result).

Theorem 2. (The theorem of the most inconsistent criteria in a vector problem with the set priority).

If in the convex VPMP of Eqs. (4.1)–(4.4) the priority of the q^{th} criterion of $p_k^q, k = \overline{1, K}, \forall q \in \boldsymbol{K}$ over other criteria is set, at a point of an optimum $X^o \in \boldsymbol{S}$ obtained on the basis of normalization of criteria and the principle of guaranteed result, there will always be two criteria with the indices $r \in \boldsymbol{K}$, $t \in \boldsymbol{K}$, for which the following strict equality holds:

$$\lambda^o = p_k^r \lambda_r\left(X^o\right) = p_k^t \lambda_t\left(X^o\right), r, t \in \boldsymbol{K}, \tag{5.19}$$

and other criteria are defined by inequalities:

$$\lambda^o \le p_k^q\left(X^o\right), k = \overline{1, K}, \forall q \in \boldsymbol{K}, q \ne r \ne t. \tag{5.20}$$

Criteria with the indices $r \in \boldsymbol{K}$, $t \in \boldsymbol{K}$ for which the equality of Eq. (5.20) holds are called the most inconsistent.

Proof. Similar to Theorem 2 [7].

We note that in Eqs. (5.19) and (5.20), the indices of criteria r, $t \in \boldsymbol{K}$ can coincide with the $q \in \boldsymbol{K}$ index.

Consequence of Theorem 1, about equality of an optimum level and relative estimates in a vector problem with two criteria with a priority of one of them.

In a convex VPMP with two equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result, at an optimum point X^o equality is always carried out at a priority of the first criterion over the second:

$$\lambda^o = \lambda_1\left(X^o\right) = p_2^1\left(X^o\right)\lambda_2\left(X^o\right), X^o \in \boldsymbol{S}, \tag{5.21}$$

where $p_2^1\left(X^o\right) = \lambda_1\left(X^o\right) / \lambda_2\left(X^o\right)$, and at a priority of the second criterion over the first:

$$\lambda^o = p_1^2\left(X^o\right)\lambda_1\left(X^o\right) = \lambda_2\left(X^o\right), X^o \in \boldsymbol{S},$$

where $p_1^2\left(X^o\right)=\lambda_2\left(X^o\right).\lambda_1\left(X^o\right)$.

5.3. Mathematical algorithm of the solution of a vector problem with equivalent criteria

To solve VPMP (4.1)–(4.4), we proposed the methods based on axioms of the normalization of criteria and the principle of the guaranteed result which follow from Axiom 1 and Principle of Optimality 1. We will present it in a number of steps:

Step 1. VPMP (3.1)–(3.4) is solved for each criterion separately, i.e. at the maximum for $\forall k \in \boldsymbol{K}_1$, and at the minimum for $\forall k \in \boldsymbol{K}_2$. As a result, we obtain:

X_k^* is an optimum point by the corresponding criterion, $k=\overline{1,K}$;

$f_k^*=f_k\left(X_k^*\right)$ is the value of the k^{th} criterion in this point, $k=\overline{1,K}$.

Step 2. We find the worst value of each criterion on $\boldsymbol{S}$: f_k^0, $k=\overline{1,K}$. For this purpose, we solve problem (4.1)–(4.4) for each criterion of $k=\overline{1,\boldsymbol{K}_1}$ at the minimum:

$$f_k^0=\min f_k\left(X\right), G\left(X\right)\le B, X\ge 0, k=\overline{1,\boldsymbol{K}_1}.$$

Problem (4.1)–(4.4) is solved for each criterion $k=\overline{1,\boldsymbol{K}_2}$ at the minimum:

$$f_k^0=\max f_k\left(X\right), G\left(X\right)\le B, X\ge 0, k=\overline{1,\boldsymbol{K}_2}.$$

As a result, we obtain: $X_k^0=\left\{x_j, j=\overline{1,N}\right\}$, which is an optimum point by the corresponding criterion, $k=\overline{1,K}$; $f_k^0=f_k\left(X_k^0\right)$, which the value of the k^{th} criterion in the point of X_k^0, $k=\overline{1,K}$.

Step 3. We perform a system analysis of a set of Pareto optimal points. For this purpose, in optimum points of $X^*=\left\{X_k^*, k=\overline{1,K}\right\}$ we determine the values of criterion functions of $F(X^*)$ and relative estimates $\lambda\left(X^*\right), \lambda_k\left(X\right)=\dfrac{f_k\left(X\right)-f_k^0}{f_k^*-f_k^0}, \forall k\in\boldsymbol{K}$:

$$\lambda_k\left(X\right)=\frac{f_k\left(X\right)-f_k^0}{f_k^*-f_k^0}, \forall k\in\boldsymbol{K}.$$

$$F\left(X^*\right)=\left\{f_k\left(X_k^*\right), q=\overline{1,K}, k=\overline{1,K}\right\}=\begin{vmatrix} f_1\left(X_1^*\right),\ldots,f_K\left(X_1^*\right)\\ \ldots\\ f_1\left(X_K^*\right),\ldots,f_K\left(X_K^*\right)\end{vmatrix},$$

$$\lambda\left(X^*\right)=\left\{\lambda_q\left(X_k^*\right), q=\overline{1,K}, k=\overline{1,K}\right\}=\begin{vmatrix} \lambda_1\left(X_1^*\right),\ldots,\lambda_K\left(X_1^*\right)\\ \ldots\\ \lambda_1\left(X_K^*\right),\ldots,\lambda_K\left(X_K^*\right)\end{vmatrix}. \tag{5.22}$$

In general, in VPMP, the relative $\forall k\in\boldsymbol{K}$ estimate of $\lambda_k\left(X\right), k=\overline{1,K}$ lies within $0\le\lambda_k\left(X\right)\le 1$, $k=\overline{1,K}$.

Step 4. Construction of the λ-problem.

The construction of the λ-problem is carried out in two stages:

Initially, we construct a maximin problem of optimization with the normalized criteria which at the second stage is transformed to the standard problem of mathematical programming called the λ-problem. For the construction of a maximin problem of optimization, we use definition 2:

$$\forall X \in \boldsymbol{S} \;\; \lambda = \boldsymbol{min}_{\boldsymbol{k} \in \boldsymbol{K}} \lambda_k\left(X\right).$$

The lower λ level is maximized with respect to $X \in S$, as a result we formulate a maximin problem of optimization with the normalized criteria.

$$\lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}} \boldsymbol{min}_{\boldsymbol{k} \in \boldsymbol{K}} \lambda_k\left(X\right). \tag{5.23}$$

At the second stage, we transform problem (5.23) into a standard problem of mathematical programming:

$$\lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}} \lambda, \quad \rightarrow \quad \lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}} \lambda, \tag{5.24}$$

$$\lambda - \lambda_k\left(X\right) \le 0,\, k = \overline{1, K}, \quad \rightarrow \quad \lambda - \frac{f_k\left(X\right) - f_k^0}{f_k^* - f_k^0} \le 0,\, k = \overline{1, K}, \tag{5.25}$$

$$G\left(X\right) \le B,\, X \ge 0, \quad \rightarrow \quad G\left(X\right) \le B,\, X \ge 0. \tag{5.26}$$

In λ-problem (5.24)–(5.26), the vector of unknown X has the dimension of $N+1$: $X = \{X, x_1, ..., x_N\}$.

Step 5. Solution of λ-problem.

λ-problem (4.24)–(4.26) is a standard problem of convex programming and for its solution we use the standard methods.

As a result of solving λ-problem, we obtain: $\boldsymbol{X^o} = \{\boldsymbol{X^o}, \boldsymbol{\lambda^o}\}$ is an optimum point;

$f_k\left(X^o\right),\, k = \overline{1, K}$ are values of the criteria in this point;

$\lambda_k\left(X^o\right) = \dfrac{f_k\left(X\right) - f_k^0}{f_k^* - f_k^0},\, k = \overline{1, K}$ are values of relative estimates;

λ^o is the maximum relative estimate which is the maximum lower level for all relative estimates of $\lambda_k(X^o)$, or the guaranteed result in relative units. λ^o guarantees that all relative estimates of $\lambda_k(X^o)$ are greater than or equal to λ^o:

$$\lambda_k\left(X^o\right) \ge \lambda^o,\, k = \overline{1, K} \quad \text{or} \quad \boldsymbol{\lambda^o} \le \lambda_k\left(X^o\right),\, k = \overline{1, K},\, X^o \in \boldsymbol{S},$$

and according to Theorem 1 the $X^o = \{\boldsymbol{\lambda^o}, x_1, ..., x_N\}$ point is a Pareto optimum.

5.4. Mathematical method of the solution of a vector problem with criterion priority

Step 1. We solve a vector problem with equivalent criteria. The algorithm of the solution is presented in Section 5.3.

As a result, we obtain:

optimum points by each criterion separately $X_k^*,\, k = \overline{1, K}$ and values of criterion functions in these points of $f_k^* = f_k\left(X_k^*\right),\, k = \overline{1, K}$, which represent the boundary of a set of Pareto optimal points;

anti-optimum points by each criterion of $X_k^0 = \left\{x_j,\, j = \overline{1, N}\right\}$ and the worst unchangeable part of each criterion of $f_k^0 = f_k\left(X_k^0\right),\, k = \overline{1, K}$;

$\boldsymbol{X^o} = \{X^o, \lambda^o\}$, an optimum point, as a result of the solution of VPMP at equivalent criteria, i.e., the result of the solution of a maximin problem and the λ-problem constructed on its basis;

λ^o, the maximum relative estimate that is the maximum lower level for all relative estimates of $\lambda_k(X^o)$, or the guaranteed result in relative units, λ^o guarantees that all relative estimates of $\lambda_k(X^o)$ are equal to or greater than λ^o:

$$\lambda^o \le \lambda_k\left(X^o\right), k=\overline{1,K}, X^o \in S. \tag{5.27}$$

The decision-maker carries out the analysis of the results of the solution of the vector problem with equivalent criteria.

If the obtained results satisfy the decision-maker, then the process concludes, otherwise subsequent calculations are performed.

In addition, we calculate:

in each point $X_k^*, k=\overline{1,K}$ we determine values of all criteria of:

$q=\overline{1,K}\left\{f_q\left(X_k^*\right), q=\overline{1,K}\right\}, k=\overline{1,K},$ and relative estimates $\lambda\left(X^*\right)=\left\{\lambda_q\left(X_k^*\right), q=\overline{1,K},\right.$ $\left.k=\overline{1,K}\right\}, \lambda_k\left(X\right)=\dfrac{f_k\left(X\right)-f_k^0}{f_k^*-f_k^0}, \forall k \in \boldsymbol{K}:$

$$F\left(X^*\right)=\begin{vmatrix} f_1\left(X_1^*\right)\ldots f_K\left(X_1^*\right) \\ \ldots \\ f_1\left(X_K^*\right)\ldots f_K\left(X_K^*\right) \end{vmatrix}, \lambda\left(X^*\right)=\begin{vmatrix} \lambda_1\left(X_1^*\right)\ldots \lambda_K\left(X_1^*\right) \\ \ldots \\ \lambda_1\left(X_K^*\right)\ldots \lambda_K\left(X_K^*\right) \end{vmatrix}. \tag{5.28}$$

Matrices of criteria of $F(X^*)$ and relative estimates of $\lambda(X^*)$ show the values of each criterion of $k=\overline{1,K}$ upon transition from one optimum point $X_k^*, k \in \boldsymbol{K}$ to another $X_q^*, q \in \boldsymbol{K}$, i.e., on the border of a great number of Pareto.

at an optimum point at equivalent criteria X^o we calculate values of criteria and relative estimates:

$$f_k\left(X^o\right), k=\overline{1,K}; \lambda_k\left(X^o\right), \overline{1,K}, \tag{5.29}$$

which satisfy the inequality of Eq. (4.27). In other points $X \in \boldsymbol{S}^o$, in relative units the criteria of $\lambda = \boldsymbol{min}_{\boldsymbol{k} \in \boldsymbol{K}} \lambda_k(X)$ are always less than λ^o, given the λ-problem of Eqs. (5.24)–(5.26).

This information is also a basis for further study of the structure of a great number of Pareto.

Step 2. Choice of priority criterion of $q \in \boldsymbol{K}$.

We know from the theory (see Theorem 1) that at an optimum point X^o there are always two most inconsistent criteria, $q \in \boldsymbol{K}$ and $v \in \boldsymbol{K}$, for which an exact equality holds in relative units:

$\lambda^o = \lambda_q(X^o) = \lambda_v(X^o)$, $q, v \in \boldsymbol{K}, X \in \boldsymbol{S}$. Others are subject to inequalities:

$$\lambda^o \le \lambda_k\left(X^o\right), \forall k \in \boldsymbol{K}, q \ne v \ne k.$$

As a rule, the criterion which the decision-maker would like to improve is part of this couple, and such a criterion is called a priority criterion, which we designate $q \in \boldsymbol{K}$.

Step 3. We determine numerical limits of the change in the value of the criterion priority $q \in \boldsymbol{K}$.

For priority criterion $q \in \boldsymbol{K}$, we use the matrix of Eq. (5.22) to determine the numerical limits of the change of the criterion value:

$$f_q\left(X^o\right) \le f_q\left(X\right) \le f_q\left(X_q^*\right), q \in \boldsymbol{K}, \tag{5.30}$$

where $f_q\left(X_q^*\right)$ derives from the matrix of Eq. (4.28) $F(X^*)$, all criteria showing values measured in physical units, $f_k\left(X^o\right), k=\overline{1,K}$ from Eq. (4.29), and, in relative units of

$$\lambda_q\left(X^o\right) \le \lambda_q\left(X\right) \le \lambda_q\left(X_q^*\right), q \in \boldsymbol{K}, \tag{5.31}$$

where $\lambda_q\left(X_q^*\right)$ derives from the matrix $\lambda(X^*)$, all criteria showing values measured in relative units (we note that $\lambda_q\left(X_q^*\right)=1$), $\lambda_q(X^o)$ from Eq. (5.22).

As a rule, Eqs. (5.30) and (5.31) are displayed for analysis.

Step 4. Choice of the value of priority criterion (decision-making).

The decision-maker carries out the analysis of the results of calculations of Eq. (5.28) and from the inequality of Eq. (5.30) chooses the numerical value f_q of the criterion of $q \in \boldsymbol{K}$:

$$f_q\left(X^o\right) \le f_q \le f_q\left(X_q^*\right), q \in \boldsymbol{K}. \tag{5.32}$$

For the chosen value of the criterion of f_q, it is necessary to determine a vector of unknown X^o. For this purpose, we carry out the subsequent calculations.

Step 5. Calculation of a relative estimate.

For the chosen value of the priority criterion of f_q, the relative estimate is calculated as:

$$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0}, q \in \boldsymbol{K}, \tag{5.33}$$

which upon transition from point X^o to X_q^*, according to Eq. (5.27), lies in the limits:

$$\lambda_q\left(X^o\right) \le \lambda_q \le \lambda_q\left(X_q^*\right) = 1.$$

Step 6. Calculation of the coefficient of linear approximation.

Assuming a linear nature of the change of criterion of $f_q(X)$ in Eq. (5.30) and according to the relative estimate of $\lambda_q(X)$ in Eq. (5.31), using standard methods of linear approximation we calculate the proportionality coefficient between $\lambda_q(X^o)$, λ_q, which we will refer to as ρ:

$$\rho = \frac{\lambda_q - \lambda_q\left(X^o\right)}{\lambda_q^* - \lambda_q^0}, q \in \boldsymbol{K}.$$

Step 7. Calculation of coordinates of priority criterion with the value f_q.

In accordance with Eq. (5.32), the coordinates of the X^q priority criterion point lie within the following limits: $X^o \le X_q \le X_q^*, q \in \boldsymbol{K}$. Assuming a linear nature of change of the vector $X_q = \left\{x_1^q, \ldots, x_N^q\right\}$ we determine coordinates of a point of priority criterion with the value f_q with the relative estimate of Eq. (5.32):

$$X_q = \left\{x_1^q = x_1^o + \rho\left(x_q^*(1) - x_1^o\right), \ldots, x_N^q = x_N^o + \rho\left(x_q^*(N) - x_N^o\right)\right\},$$

where $X^o = \left\{x_1^o, \ldots, x_N^o\right\}$, $X_q^* = \left\{x_q^*(1), \ldots, x_q^*(N)\right\}$.

Step 8. Calculation of the main indicators of a point X_q.

For the obtained point x_q, we calculate:

all criteria in physical units $F^q = \left\{f_k\left(X_q\right), k = \overline{1,K}\right\}$,

all relative estimates of criteria

$$\lambda^q = \left\{\lambda_k^q, k = \overline{1,K}\right\}, \lambda_k\left(X_q\right) = \frac{f_k\left(X_q\right) - f_k^0}{f_k^* - f_k^0}, k = \overline{1,K},$$

the vector of priorities $P^q = \left\{ p_k^q = \dfrac{\lambda_q\left(X_q\right)}{\lambda_k\left(X_q\right)}, k = \overline{1,K} \right\}$,

the maximum relative estimate $\lambda^{oq} = \min\left(p_k^q \lambda_k\left(X_q\right), k = \overline{1,K}\right)$.

Any point from Pareto's set $\boldsymbol{X}_t^o = \left\{\lambda_t^o, X_t^o\right\} \in \boldsymbol{S}^o$ can be calculated in a similar way.

Analysis of results. The calculated value of criterion $f_q\left(X_t^o\right), q \in \boldsymbol{K}$ is usually not equal to the set f_q. The error of the choice of $\Delta f_q = \left|f_q\left(X_t^o\right) - f_q\right|$ is defined by the error of linear approximation.

Conclusion

Thus, the following results are obtained in the article.

1. The study and analysis of the management of the multi-level system of the state within the digital economy has shown that the activities of state authorities aimed at managing the economic and social development of the state include the relationship between the state, regions (subjects of the Russian Federation) and municipalities should be an integral part of the national policy of the state. We defined "Digital economy" and showed its use in the management of a multi-level system of the state.

2. We carried out an analysis of economic and mathematical models of the company's development and on its basis we developed the theoretical foundations of mathematical modeling of the development and management of an industrial enterprise: a cluster that represents the lower level of state management. We proposed a mathematical model of the functioning of an industrial enterprise (cluster), represented by a vector problem of mathematical programming. This model allows making an optimal decision considering a set of criteria (economic indicators) in the aggregate.

3. To solve the vector problem of mathematical programming, a mathematical apparatus based on the normalization of criteria and the principle of guaranteed results is presented. The presented mathematical modeling apparatus provides, first, an opportunity to solve one of the most important problems of the theory of the firm - making an optimal decision based on a certain set of economic indicators (criteria) in the aggregate, secondly, the numerical model of the enterprise allows you to assess the dynamics of production development, economic indicators, relative growth rates, and collectively assess the investment required for such production growth, and, thirdly, allows you to form a strategic plan for innovative development of the enterprise, taking into account extensive and intensive factors (technologies) of production development.

The direction of research related to the strategic development of the multi-level socio-economic system of the state in the digital economy creates a methodological basis for the development and implementation of management decisions focused on the priority areas of economic development of the state. In this direction, the structure of the multi-level hierarchical system of the state economy is developed, aimed at solving the problems of forecasting, strategic planning and management at individual levels of the state within the digital economy. To solve the problems of forecasting and planning, a mathematical apparatus based on the theory and methods of vector optimization has been developed. At the lower level of the hierarchical system of the state economy, a mathematical model of a corporation (cluster) is studied, which takes into account both extensive and intensive factors of production development. In the cluster model, the input data is, first, statistical information and, second, technological information of the production preparation. It is shown that in the aggregate, statistical, technological information, as well as the relationship with consumers of products and with the financial (banking) sphere, is characterized as "digital economy".

The direction of further research is related to the practical implementation of mathematical models, vector optimization methods and their use in the practice of forecasting, planning and development of all departments of the state and industrial corporations (clusters) in particular.

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