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Hexagonal rod pyramid: deformations and natural oscillation frequency

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Abstract. A new scheme of a statically determinate dome truss is proposed. The purpose of the study is to obtain exact formulas for structural deflections under a uniform load and to find upper and lower analytical estimates of the first frequency of natural oscillations depending on the number of panels, sizes, and masses concentrated in the truss nodes. Calculation of forces in the truss rods is performed by cutting nodes. The system of equations in projections on the coordinate axes, compiled in the Maple software, includes the forces in the rods and the reactions of vertical supports located along two contours of the structure at the base. The amount of deflection and stiffness of the entire truss is calculated using the Mohr integral. To determine the lower estimate of the first frequency an approximate Dunkerley method is used. The formula for the upper limit of the first frequency is derived by the Rayleigh energy method. In the Rayleigh method, the shape of the deflection from the action of a uniformly distributed load is taken as the deflection of the truss. Displacements of loads are assumed to be only vertical. The overall dependence of the solution on the number of panels is obtained by induction on a series of solutions for trusses with a successively increasing number of panels. The operators of the Maple system of symbolic mathematics are used. Based on the calculation results, it was concluded that the distribution of forces over the structure rods does not depend on the number of panels. Asymptotes were found on the graphs of the obtained analytical dependences of the deflection on the number of panels for different truss heights. The estimates of the first natural frequency are compared with the numerical solution obtained from the analysis of the natural frequency spectrum. The coefficients of the frequency equation are found using the eigenvalue search operators in the Maple system. It is shown that the lower analytical estimate based on the calculation of partial frequencies differs from the numerical solution by no more than 54 %, and the upper estimate by the Rayleigh method has an error of about 2 %. The formula for the lower Dunkerley frequency estimate is simpler than the Rayleigh estimate.

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1. Introduction

The study of the stress state, deformations, and stability of spatial multifaceted dome structures are of both practical and theoretical interest [1, 2]. In the calculations of building structures of this type, as a rule, the finite element method is used [3, 4]. Regular (with periodic structure elements) statically determinate truss schemes are quite rare. The search for such schemes was even called "hunting" by R.G. Hutchinson and N.A. Fleck [3, 4]. One of the advantages of regular schemes [5] is that for them it is possible to analytically derive the dependences of the characteristics of the stress-strain state on their order (the number of periodicity elements, for example, the number of panels). To search for analytical solutions in the form of compact formulas, operators of symbolic mathematics, such as Maple, Mathematica [6], and others, are applicable. To obtain analytical solutions in the form of closed formulas, when modeling structures, it is necessary to make some simplifications. The construction must be statically determined. If at the same time it is regular, then for such a design it is possible to obtain calculation formulas for an arbitrary number of repeating elements. Regular trusses are, for example, planar or spatial trusses with identical panels or groups of panels. In this case, the analytical solution has a great advantage over the numerical one, not only due to the saving of computation time but also due to the fundamental possibility to calculate the truss with a very large number of panels without a loss of accuracy. A larger number of elements causes the inevitable effect of accumulation of rounding errors in the numerical calculation. Analytical solutions are especially effective for preliminary draft calculations, for assessing the accuracy of numerical solutions, and in truss optimization problems.

Analytical solutions for building structures do not always lead to the final compact design formula. In such works, an algorithm for calculations in the system of symbolic mathematics is given [7, 8]. The purpose of this work is to obtain formulas for calculating deflections and estimates of the first natural frequency of a three-dimensional truss. Most often, to solve such problems for planar [9–13], and three-dimensional [14] trusses, the induction method is used. Solutions of problems on the deformation of some planar arch trusses [15] with an arbitrary number of panels are obtained inductively and some problems on natural frequencies of regular structures are solved [16–19]. Calculation formulas for deflections and oscillation frequencies of spatial regular trusses were obtained in [20].

In the analytical form, it is impossible to obtain a solution directly from the analysis of the entire frequency spectrum in the general case.

Therefore, to estimate its lower limit of the first (lowest) frequency of natural oscillations, we will use the Dunkerley method [21–23], and for the upper one, the Rayleigh method [24, 25]. These approximate methods are based on the calculation of partial frequencies, for which it is not necessary to compose high-order characteristic polynomials (in terms of the number of degrees of freedom).

Pyramidal trusses are studied in connection with the design of structural panels (composite) in which the trusses act as a kind of reinforcement. Hexagonal trusses are also used in mesh coatings [26, 27]. Eccentrically braced frames and beam-type spatial trusses were studied numerically and experimentally in [28]. An overview of analytical solutions for planar statically determinate regular trusses is given in [29].

In this paper, we propose a new scheme of a statically determinate dome-type hexagonal spatial truss. The truss has architectural expressiveness and can be used in public buildings (circus, airport building, railway station, etc.).

The construction can be used as a basis for complicated statically indeterminate systems of this type.

2. Methods

2.1. The truss scheme

The truss in the form of a regular pyramid $2h_1 + h_2$ high with a hexagonal base side na contains $n_s = 36n - 15$ rods, including $6n$ vertical support posts h_1 high located along the outer contour of the structure and $6(n-2)$ posts $2h_1$ high supporting the upper contour (Fig. 1, 2). There are no posts in the corner nodes D of the upper contour. A similar hexagonal cover, but with a dome attached to the corner points of the outer (upper) contour, is considered in [30].

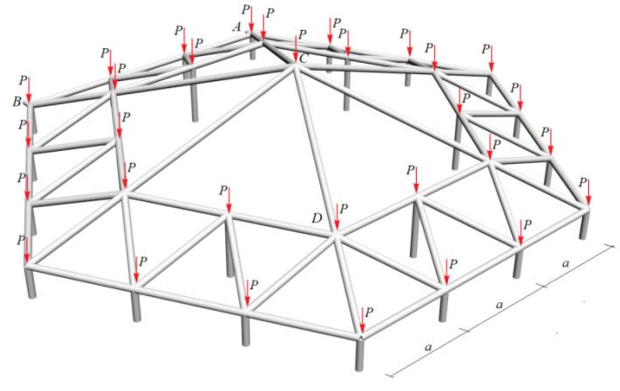

Figure 1. Truss scheme, *n* = 3, vertical load.

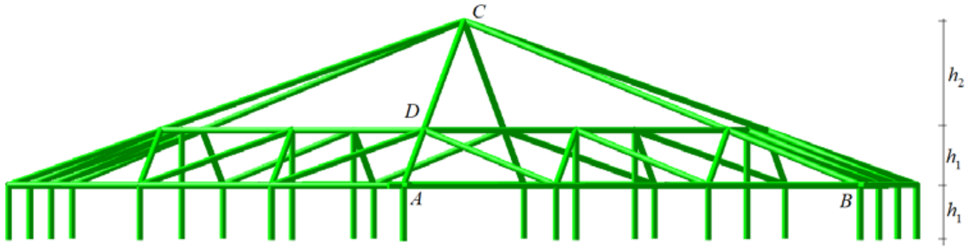

Figure 2. The truss dimensions, *n* = 3.

The lower horizontal rod contour consists of $6n$ rods of length a, the upper one consists of $6(n-1)$ similar rods. The braces connecting the contours have a length of $c=\sqrt{a^2+h^2}$. The lengths of six identical braces emanating from vertex C depend on the number of panels: $(n-1)c$. Corner node A rests on a spherical support hinge, modeled by three mutually perpendicular rods, one of which is a vertical post. Node B is a cylindrical hinge corresponding to two support bars. The following ratios of sizes are chosen: $h_1=h,\ \ h_2=(n-1)h$. All connections of the truss rods are hinged. An analytical calculation of the deformations of a spatial coating with a similar structure was performed in [20]. In [31] an optimization problem is solved for an irregular spatial truss of 25 rods.

Calculation of forces in rods in symbolic form is performed based on a program written in the language of computer mathematics Maple [32]. To do this, we introduce the coordinates of the nodes (Fig. 3), using the annular periodic structure of the truss:

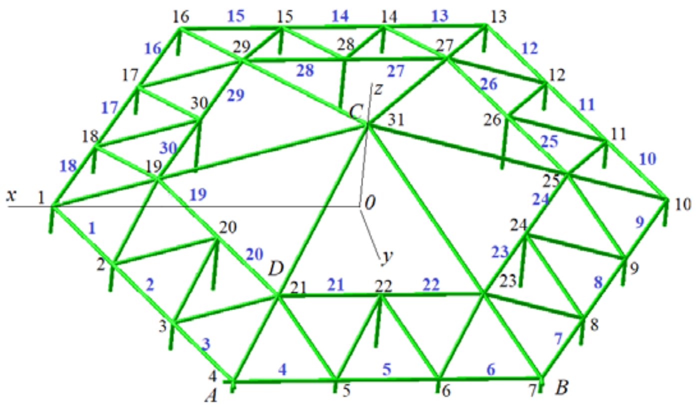

Figure 3. Numbers of nodes and rods of contours *n* = 3.

The coordinates of the nodes of the lower contour look like this:

$$x_{i+jn}=L\cos\phi-a(i-1)\cos\beta,$$
$$y_{i+jn}=L\sin\phi+a(i-1)\sin\beta,$$
$$z_{i+jn}=0,\ i=1,..,n,\ j=0,..,5,$$

where $L=na,\ \ \phi=j\pi/3,\ \ \beta=\pi/3-\phi.$

The coordinates of the nodes of the smaller (upper) contour:

$$x_{i+j(n-1)+6n}=(L-a)\cos\phi-a(i-1)\cos\beta,$$

$$y_{i+j(n-1)+6n}=(L-a)\sin\phi+a(i-1)\sin\beta,$$

$$z_{i+j(n-1)+6n}=h,\ \ i=1,...,n-1,\ \ j=0,...,5.$$

Vertex C coordinates: $x_{12n-5}=y_{12n-5}=0,\ z_{12n-5}=h_1+h_2.$

The order of connecting the bars of the lattice is entered into the program using ordered lists $\Phi_i,\ i=1,..,n_s$. of numbers of the ends of the corresponding bars, similar to how graphs are given in discrete mathematics. For example, the bars of the lower chord are encoded with the following vertex lists $\Phi_i=[i,\ i+1],\ i=1,..,6n-1, \Phi_{3n}=[6n,1]$.

Upper chord bar code:
$\Phi_{i+6n}=[i+6n,\ i+6n+1],\ i=1,\ ...,\ 6n-7,\ \ \Phi_{12n-6}=[12n-6,\ 6n+1]$.

The numbers of ends and other rods are set similarly.

2.2. Calculation of forces in bars

Let us represent the system of equilibrium equations of nodes in the projection on the coordinate axes in matrix form $\mathbf{GS}=\mathbf{\Psi}$, where $\mathbf{S}$ is the vector of unknown forces, including the reactions of the supports, $\mathbf{G}$ is the matrix of coefficients (projections of unit forces in the rods), $\mathbf{\Psi}$ is the vector of loads on the nodes. For each node in the matrix, three rows are assigned, corresponding to projections onto three coordinate axes. Similarly, in the elements of the load vector of the form Ψ_{3i-2}, where i is the number of the node, the loads on node i in the projection on the x-axis are written. Elements Ψ_{3i-1} contain projections of external forces in projection onto the y-axis. Vertical loads on nodes are recorded in elements Ψ_{3i}.

Matrix $\mathbf{G}$ elements are calculated according to the data on the structure of the connection of the bars and the coordinates of the nodes

$$g_{x,i}=\left(x_{\Phi_{i,1}}-x_{\Phi_{i,2}}\right)/l_i,\quad g_{y,i}=\left(y_{\Phi_{i,1}}-y_{\Phi_{i,2}}\right)/l_i,$$
$$g_{z,i}=\left(z_{\Phi_{i,1}}-z_{\Phi_{i,2}}\right)/l_i,\quad i=1,...,n_s+3,$$

where $l_i=\sqrt{l_{x,i}^2+l_{y,i}^2+l_{z,i}^2}$ is the length of the rod *i*. The number of rods also includes three horizontal support rods at angles A and B. The matrix of coefficients of equilibrium equations in projections is filled in rows. Every three lines correspond to the projection equations on the x, y, and z axes, respectively:

$$G_{3\Phi_{i,1}-2,i}=g_{x,i},\ G_{3\Phi_{i,1}-1,i}=g_{y,i},\ \ G_{3\Phi_{i,1},i}=g_{z,i},$$
$$G_{3\Phi_{i,2}-2,i}=-g_{x,i},\ G_{3\Phi_{i,2}-1,i}=-g_{y,i},\ G_{3\Phi_{i,2},i}=-g_{z,i}.$$

If a uniform vertical load is applied to the truss nodes (Fig. 1), then the non-zero elements of the load vector have the form: $\Psi_{3i}=P,\quad i=1,...,12n-5$. Numerical calculation of forces for a structure with $n=3,\ a=5.0\,\mathrm{m},\ h=1.0\,\mathrm{m}$ gives a picture of the distribution of forces shown in Fig. 4. Compressed rods are highlighted in blue, tension rods are highlighted in red. The thickness of the line is greater, the greater the modulus of force in the corresponding rod. The force value is related to the value of the nodal load P with an accuracy of two significant digits.

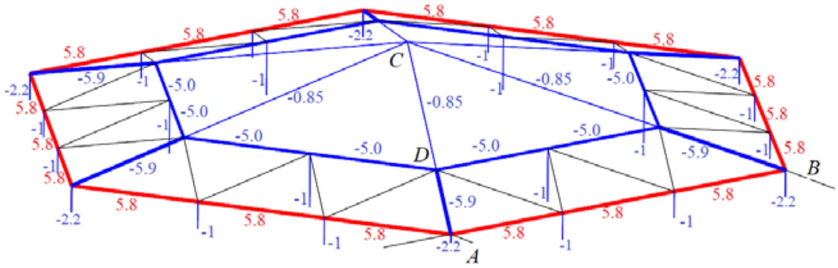

Figure 4. Distribution of forces in the rods, *n* = 3, *a* = 5.0 m, *h* = 1.0 m.

The upper contour of the truss under such a load is compressed, the lower one is stretched. The braces connecting the contours have zero forces. An interesting feature of the stressed state of the truss was noticed. The patterns of distribution of forces in the rods from a uniform load for trusses of different orders are similar. The order of the truss is equal to the number of rods in the side edge of the lower contour. The compressive forces in the rods of the upper contour for any n are equal to

$$S_i = -aP/h,\ i = 6n+1,...,12n-6. \tag{1}$$

The greatest tensile forces are observed in the lower belt

$$S_i = 7aP/(6h),\ i = 1,...,6n. \tag{2}$$

The six lower corner ribs are most compressed

$$S_i = -7Pc/(6h),\ \ i = 12n-5,...,17n-5. \tag{3}$$

The forces in the six upper rods of the dome, connected at the top C, are equal to $-Pc/(6h)$. The forces in the six corner support posts do not depend on the dimensions of the structure: $-13P/6$. The reactions of the supports of the intermediate posts along the outer (lower) and inner (upper) contours are equal to P.

2.3. Deflection

The formula for the dependence of the deflection of the top C on the dimensions of the structure, the load, and the number of panels will be obtained by induction. Under the deflection Δ_n we mean the vertical displacement of the node C of the truss of order n. To calculate the deflection value, we use the Mohr integral

$$\Delta_n = \sum_{j=1}^{n_S} \frac{S_j s_j l_j}{EF}, \tag{4}$$

where l_j is the length of the rod, S_j is the force in the *j*th rod from the action of the load, s_j is the force in the rod from the action of a single vertical force applied to the vertex C, E is the modulus of elasticity of the rods, F is the cross-sectional area. The summation is carried out over all the bars of the structure. The elastic moduli and cross-sectional areas are the same for all rods. Sequential calculation of the deflection of a series of trusses with an increasing number of panels gives the following results

$$\Delta_2 = P\left(14a^3 + 8c^3 + 13h^3\right)/\left(6h^2EF\right),$$
$$\Delta_3 = P\left(21a^3 + 9c^3 + 13h^3\right)/\left(6h^2EF\right),$$
$$\Delta_4 = P\left(28a^3 + 10c^3 + 13h^3\right)/\left(6h^2EF\right),$$
$$\Delta_5 = P\left(35a^3 + 11c^3 + 13h^3\right)/\left(6h^2EF\right),$$
$$\Delta_6 = P\left(42a^3 + 12c^3 + 13h^3\right)/\left(6h^2EF\right).$$

In the general case, we have the form of a formula for the deflection:

$$\Delta_n = P\left(C_1a^3 + C_2c^3 + C_3h^3\right)/\left(h^2EF\right). \tag{5}$$

The coefficients in this expression are functions of the number of panels n. The common members of the sequences they form can be found using the special operators *rsolve* and *rgf_findrecur* from the Maple system. Equally effective in finding common members of sequences are the operators of the Mathematica computer mathematics system. The common terms of the sequences of coefficients at $n = 2,3,...,7$ are linear concerning n

$$C_1 = 7n/6,\ \ C_2 = (n+6)/6,\ \ C_3 = 13/3. \tag{6}$$

Compared to similar well-known solutions for planar trusses, which have a shape that is non-linear in terms of the number of panels, the solution turned out to be much more compact. In part, this can be explained by the observed feature of the stress state of the structure, which does not depend on the number of panels. The same simple solution is obtained in the problem of the deflection of node D at the corner (not supported) point of the upper contour. The coefficients in (5) in this case have the form

$$C_1 = (13n-6)/27, \quad C_2 = 7/6, \quad C_3 = 13/6. \tag{7}$$

Let us plot the solution graphs (5), (6). Let us denote the total load on the truss $P_{sum} = P(12n-5)$ and the length of the outer side of the cover $L = na$. Let us introduce the designation for the dimensionless deflection: $\Delta' = \Delta_n EF/(P_{sum}L)$. The graphs of the curves of solutions (6) and (7) at $L = 50$ m show that in this setting the dimensionless deflection at points C and D decreases monotonically with an increase in the number of panels (Fig. 5).

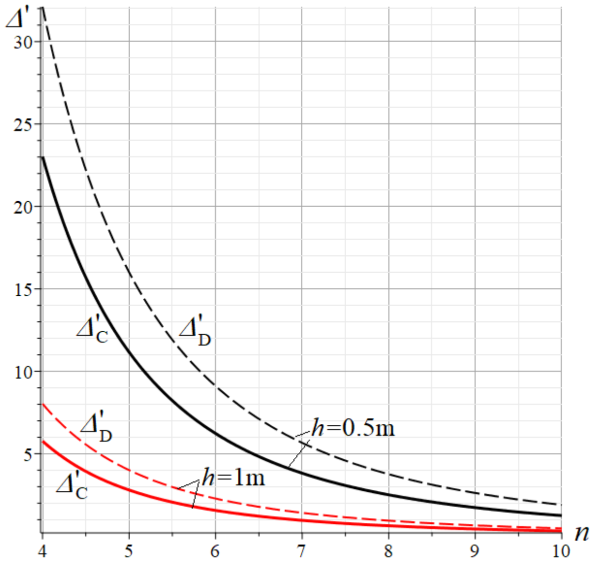

Figure 5. Dependence of dimensionless deflection of top C and node D on the number of panels.

For small n, the deflection of the non-supported node D is one and a half times greater than the deflection of the vertex C. Horizontal asymptotes of the solutions (ultimate deflection) are noted. Using the analytical form of the solution, using the operators of the Maple system, we obtain the lower limits of the relative deflections: $\lim\limits_{n\to\infty} \Delta'_C = h/(72L)$, $\lim\limits_{n\to\infty} \Delta'_D = 0$. It follows that curves Δ'_D and Δ'_C must intersect. Calculations show that the intersection of the curves for the height $h = 1.0$ m occurs at $n = 76$ and for $h = 0.5$m at $n = 151$.

In practice, there are problems about the deflection of a structure under the action of a load on only part of its surface. This corresponds, for example, to a snow load applied to one half of the roof (Fig. 6). The derivation of the formula in this case is no different from the previous tasks. The form of the solution obtained by induction on eight girders with a successively increasing number of panels on one edge coincides with solution (5). The coefficients look like:

$$C_1 = 5n/6, \quad C_2 = (4+n)/6, \quad C_3 = 3/2.$$

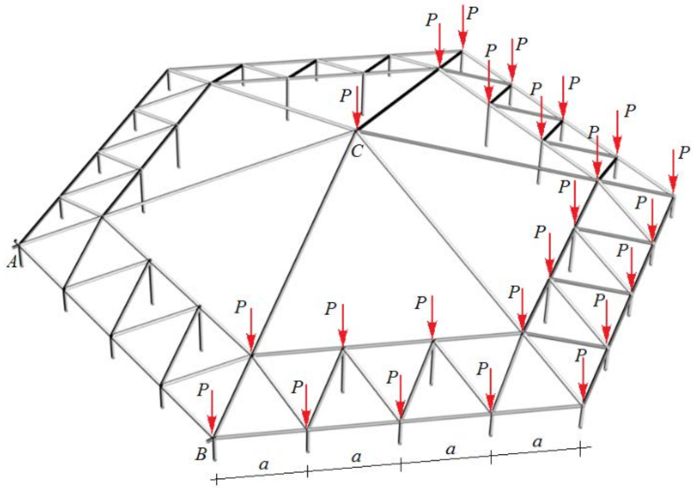

Figure 6. Vertical load on half of the truss surface, *n* = 4.

2.4. Natural oscillation frequency

The calculation of the first (lowest) frequency of natural oscillations is included in most dynamic calculations of the structure and is of independent interest. In addition to the direct calculation of the spectrum of natural frequencies, which is performed numerically [33–35], there are also known approximate methods for obtaining its upper and lower estimates of the first natural oscillation frequency [21–25]. These methods are based on the calculation of partial frequencies, the values of which can be found analytically. For regular constructions, analytical estimates can be generalized to an arbitrary number of panels using the [16] induction method.

The inertial properties of the truss are modeled by concentrated masses in the nodes. In the simplest setting, the masses of loads m are the same. Only vertical vibrations of nodes are considered. The number of degrees of freedom of the truss weight system of order n is equal to the number of nodes $K = 12n - 5.$

The dynamics of the system is described by a system of differential equations for the movement of goods in matrix form:

$$\mathbf{M}_K \ddot{\mathbf{Z}} + \mathbf{D}_K \mathbf{Z} = 0, \tag{8}$$

where $\mathbf{D}_K$ is the structural stiffness matrix, $\mathbf{Z}$ is the vector of vertical displacements of masses 1,..., K, $\mathbf{M}_K$ is the inertia matrix of size $K \times K,$ $\ddot{\mathbf{Z}}$ is the acceleration vector. The inertia matrix is proportional to the identity matrix $\mathbf{M}_K = m\mathbf{I}_K$ if the masses are the same. The elements of the compliance matrix $\mathbf{B}_K,$ which is the inverse of the stiffness matrix $\mathbf{D}_K,$ can be found using the Mohr integral

$$b_{i,j} = \sum_{\alpha=1}^{n_S} S_\alpha^{(i)} S_\alpha^{(j)} l_\alpha \Big/ (EF), \tag{9}$$

where $S_\alpha^{(i)}$ is the force in the rod α from the action of a unit vertical force at node $i.$ The problem can be reduced to the problem of matrix eigenvalues $\mathbf{B}_K.$ To do this, we multiply (8) from the left by $\mathbf{B}_K$ and, taking into account the replacement $\ddot{\mathbf{Z}} = -\omega^2 \mathbf{Z},$ which is valid for harmonic oscillations of the form

$$z_i = u_i \sin\left(\omega t + \varphi_0\right) \tag{10}$$

we obtain $\mathbf{B}_K \mathbf{Z} = \lambda \mathbf{Z},$ where $\lambda = 1\big/\left(m\omega^2\right)$ is the eigenvalue of the matrix $\mathbf{B}_K,$ ω is the natural frequency of oscillations.

From here we obtain the calculation formula for the frequency $\omega = \sqrt{1/(m\lambda)}.$

The forces $S_{\alpha}^{(i)}$ in the truss rods included in the elements of the matrix $\mathbf{B}_K$ are determined by solving the system of equations of the truss nodes, which also includes the reactions of the supports.

Consider approximate methods that give upper and lower estimates of the first frequency.

2.5. Energy method. Top rating

The Rayleigh formula, which follows from the equality of the maximum values of the kinetic and potential energies, has the form:

$$T_{\max} = \Pi_{\max}. \tag{11}$$

The kinetic energy of a system consisting of K identical masses m located at the nodes of the structure has the form: $T = \sum_{i=1}^{K} m v_i^2 \Big/ 2.$

The vertical velocity v_i of the mass *i* according to (7) has the form: $v_i = \dot{z}_i = \omega u_i \sin\left(\omega t + \varphi_0\right).$

The maximum kinetic energy corresponds to equality $\max\left(\sin\left(\omega t + \varphi_0\right)\right) = 1.$ From here we get:

$$T_{\max} = \omega^2 m \sum_{i=1}^{K} u_i^2 \Big/ 2, \tag{12}$$

where the amplitude of the vertical displacement is calculated using the Mohr integral:

$$u_i = \sum_{\alpha=1}^{n_s} S_{\alpha}^{(P)} \tilde{S}_{\alpha}^{(i)} l_{\alpha} / (EF) = P \sum_{\alpha=1}^{n_s} \tilde{S}_{\alpha}^{(P)} \tilde{S}_{\alpha}^{(i)} l_{\alpha} / (EF) = P\tilde{u}_i \ .$$

The previous designations are used: $S_{\alpha}^{(P)}$ is force in the rod $\alpha = 1,...,n_s$ from the action of the load $P,$ uniformly distributed over the nodes, $\tilde{S}_{\alpha}^{(i)}$ is force in the same rod from a single (dimensionless) load applied to the mass with number $i,$ $\tilde{S}_{\alpha}^{(P)} = S_{\alpha}^{(P)} \Big/ P.$ The choice of a uniformly distributed load is determined by the proximity of the shape corresponding deflection to the form of vibrations of the system of weights with the first frequency. Thus, (12) takes the form:

$$T_{\max} = P^2 \omega^2 \sum_{i=1}^{K} m \tilde{u}_i^2 / 2, \tag{13}$$

where $\tilde{u}_i = u_i / P = \sum_{\alpha=1}^{n_s} \tilde{S}_{\alpha}^{(P)} \tilde{S}_{\alpha}^{(i)} l_{\alpha} \Big/ \left(EF\right)$ is amplitude of displacements of the mass with number i under the action of a distributed load, referred to the value of $P.$ The potential energy of deformation of the truss rods has the form:

$$\Pi_{\max} = \sum_{\alpha=1}^{n_s} S_{\alpha}^{(P)} \Delta l_{\alpha} \Big/ 2 = \sum_{\alpha=1}^{n_s} \left(S_{\alpha}^{(P)}\right)^2 l_{\alpha} \Big/ \left(2EF\right).$$

Due to the linearity of the problem concerning loads, we have $S_{\alpha}^{(P)} = P \sum_{i=1}^{N} \tilde{S}_{\alpha}^{(i)}.$ From here we get:

$$\begin{aligned} \Pi_{\max} &= P^2 \sum_{\alpha=1}^{n_s} \tilde{S}_{\alpha}^{(P)} \sum_{i=1}^{K} \tilde{S}_{\alpha}^{(i)} l_{\alpha} \Big/ \left(2EF\right) = \\ &= P^2 \sum_{i=1}^{K} \sum_{\alpha}^{n_s} \tilde{S}_{\alpha}^{(P)} \tilde{S}_{\alpha}^{(i)} l_{\alpha} \Big/ \left(2EF\right) = P^2 \sum_{i=1}^{N} \tilde{u}_i \Big/ 2. \end{aligned} \tag{14}$$

From (11), (13), (14) we obtain a formula for the upper estimate of the first oscillation frequency of the truss (the Rayleigh formula):

$$\omega_R^2 = \sum_{i=1}^{K} \tilde{u}_i \Bigg/ \sum_{i=1}^{K} m\tilde{u}_i^2. \tag{15}$$

We find the displacements $\tilde{u}_i$ as functions of n. We generalize the solution obtained for a different number of panels concerning n. Consider the sums $\sum_{i=1}^{K} \tilde{u}_i$ and $\sum_{i=1}^{K} \tilde{u}_i^2$ separately.

The calculation of displacement for trusses with the different number of panels shows that the form of the solution $\sum_{i=1}^{K} \tilde{u}_i$ does not depend on n. Note that irregular trusses do not have this property. The numerator in (15) can be represented as:

$$\sum_{i=1}^{K} \tilde{u}_i = \left(C_a a^3 + C_c c^3 + C_h h^3\right) \Big/ \left(h^2 EF\right),$$

or in a more compact form

$$\sum_{i=1}^{K} \tilde{u}_i = \sum_{\alpha=[a,c,h]} mC_\alpha \alpha^3 \Bigg/ \left(h^2 EF\right), \tag{16}$$

where the coefficients C_a, C_c, C_h are obtained by induction, generalizing a series of solutions for different n:

$$n = 2,\quad \sum_{i=1}^{K} \tilde{u}_i = \left(134a^3 + 50c^3 + 205h^3\right) \Big/ \left(6h^2 EF\right),$$

$$n = 3,\quad \sum_{i=1}^{K} \tilde{u}_i = \left(219a^3 + 51c^3 + 313h^3\right) \Big/ \left(6h^2 EF\right),$$

$$n = 4,\quad \sum_{i=1}^{K} \tilde{u}_i = \left(304a^3 + 52c^3 + 421h^3\right) \Big/ \left(6h^2 EF\right),$$

$$n = 5,\quad \sum_{i=1}^{K} \tilde{u}_i = \left(389a^3 + 53c^3 + 529h^3\right) \Big/ \left(6h^2 EF\right),$$

$$n = 6,\quad \sum_{i=1}^{K} \tilde{u}_i = \left(474a^3 + 54c^3 + 637h^3\right) \Big/ \left(6h^2 EF\right),\ldots$$

As a result, we have the coefficients

$$C_a = \left(85n - 36\right),\ \ C_c = \left(n + 48\right)/6,\ \ C_h = \left(108n - 11\right)/6. \tag{17}$$

The denominator (15) has a more complex form:

$$\sum_{k=1}^{N} m\tilde{u}_k^2 = \sum_{\alpha,\beta=[a,c,h]} mC_{\alpha\beta} \alpha^3 \beta^3 \Bigg/ \left(h^4 E^2 F^2\right), \tag{18}$$

where

$$\begin{gathered} C_{aa} = (1063n^2 - 936n + 216)/36, \\ C_{cc} = \left(n^2 + 12n + 330\right)/36, \\ C_{hh} = \left(1080n + 253\right)/36, \\ C_{ac} = \left(7n^2 + 588n - 252\right)/36, \\ C_{ah} = \left(1105n - 468\right)/36, \\ C_{\mathrm{ch}} = \left(13n + 624\right)/36. \end{gathered} \tag{19}$$

Thus, the upper estimate of the first frequency of the truss, depending on the number of panels, can be obtained by the formula:

$$\omega_R = h\sqrt{\frac{EF \sum\limits_{\alpha=[a,c,h]} C_\alpha \alpha^3}{m \sum\limits_{\alpha,\beta=[a,c,h]} C_{\alpha\beta}\alpha^3\beta^3}} \tag{20}$$

with coefficients (17), (19) depending only on the construction order n.

2.6. Dunkerley score

We obtain a lower estimate of the first frequency of oscillations using the Dunkerley formula:

$$\omega_D^{-2} = \sum_{i=1}^{K} \omega_i^{-2}, \tag{21}$$

where ω_i is the oscillation frequency of one mass m located at node i. To calculate the partial frequencies ω_i, we compose equation (8) in the scalar form:

$$m\ddot{z}_i + D_i z_i = 0,$$

where D_i is the scalar stiffness coefficient (i is the mass number). The frequency of vibrations of the load is $\omega_i = \sqrt{D_i/m}$. The stiffness coefficient, the reciprocal of the compliance coefficient, is determined by the Mohr integral (4):

$$\delta_i = 1/D_i = \sum_{\alpha=1}^{n_S} \left(\tilde{S}_\alpha^{(i)}\right)^2 l_\alpha \Big/ \left(EF\right).$$

Arguing in the same way as when calculating the frequencies of a system with many degrees of freedom, we obtain

$$\begin{gathered} \omega_D^{-2} = \sum_{i=1}^{K} \omega_i^{-2} = m\sum_{i=1}^{K} \frac{1}{D_i} = m\sum_{i=1}^{K} \delta_i = \\ = m\sum_{i=1}^{K}\sum_{\alpha=1}^{n_S} \left(\tilde{S}_\alpha^{(i)}\right)^2 l_\alpha \Big/ \left(EF\right) = m\textstyle\sum_n \Big/ \left(h^2 EF\right). \end{gathered} \tag{22}$$

Let us successively calculate the sums $\sum_n = h^2 \sum\limits_{i=1}^{K}\sum\limits_{\alpha=1}^{n_S} \left(\tilde{S}_\alpha^{(i)}\right)^2 l_\alpha$ for n = 2, 3, 4, ...

$$\sum_2=\frac{94a^3+154c^3+173h^3}{12},$$
$$\sum_3=\frac{279a^3+231c^3+601h^3}{18},$$
$$\sum_4=\frac{1124a^3+584c^3+2495h^3}{48},$$
$$\sum_5=\frac{4715a^3+1739c^3+10555h^3}{150},...$$

Let us calculate the common terms of the sequences of coefficients in these expressions.

We get $\sum_n=\sum_{\alpha=[a,c,h]} r_\alpha \alpha^3$, where

$$\begin{aligned} r_a &= \left(49n^2-60n+18\right)/(6n), \\ r_c &= \left(n^3+36n^2+186n-216\right)/\left(6n^2\right), \\ r_h &= \left(108n^3-107n^2-60n+30\right)/\left(6n^2\right). \end{aligned} \tag{23}$$

When deriving expressions for the coefficients (23), the operators for compiling and solving the recursive equations of the Maple system were used. Some complication was the dependence on n not only of the numerators of the sequence members but also of their denominators. Maple system operators are adapted to define common members of such sequences. The result in the form of expressions (23) was obtained only because it was possible to guess the form of the denominators.

Thus, we obtain the lower estimate for the first frequency according to Dunkerley:

$$\omega_D=h\sqrt{\frac{EF}{m\sum_{\alpha=[a,c,h]} r_\alpha \alpha^3}}. \tag{24}$$

The form of the Dunkerley estimate (24) almost coincides with formula (20) obtained by the Rayleigh method, but formula (24) is much simpler. The desired coefficients are contained here only in the denominator.

3. Results and Discussion

To estimate the error of the estimates found, consider an example of a truss with n panels for $a=$ 6.0 m and $h=1.0$ m. The stiffness of the steel rods of the truss will be taken $EF=1.8\cdot 10^5$ kN. Fig. 7 plots the dependences of the upper estimate of the first frequency ω_R of natural oscillations of the truss (20), obtained by the Rayleigh energy method, ω_D the Dunkerley estimate (24), and the numerical solution ω_1, found as the minimum frequency of the entire frequency spectrum.

The numerical value of the lowest natural frequency ω_1 of a system with $K=12n-5$ degrees of freedom is so close to the Rayleigh estimate that for $n>3$ the curves merge. To refine the error estimates, we introduce the relative values $\varepsilon_D=|\omega_D-\omega_1|/\omega_1$, $\varepsilon_R=|\omega_R-\omega_1|/\omega_1$.

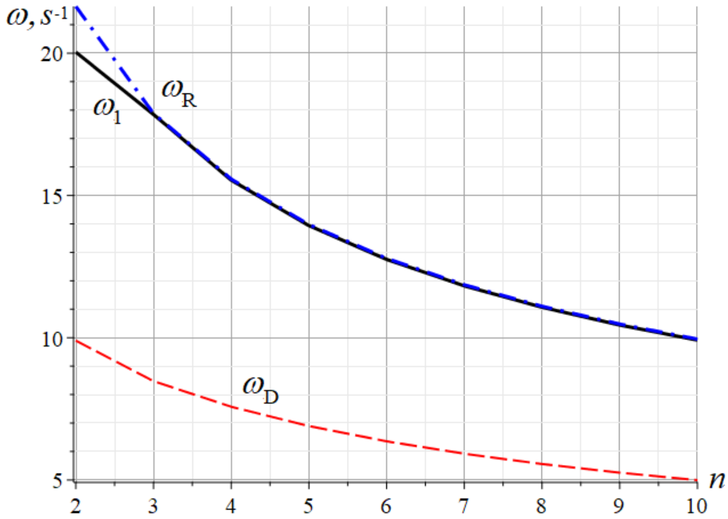

Figure. 7. The first oscillation frequency obtained in three ways.

Depending on the number of panels, the error of the Dunkerley solution is relatively large (Fig. 8), but it changes little, increasing sharply only at $n = 3$. The Rayleigh estimate error (Fig. 9) is very small and, in principle, it can be neglected, considering the analytical solution (20) to be exact.

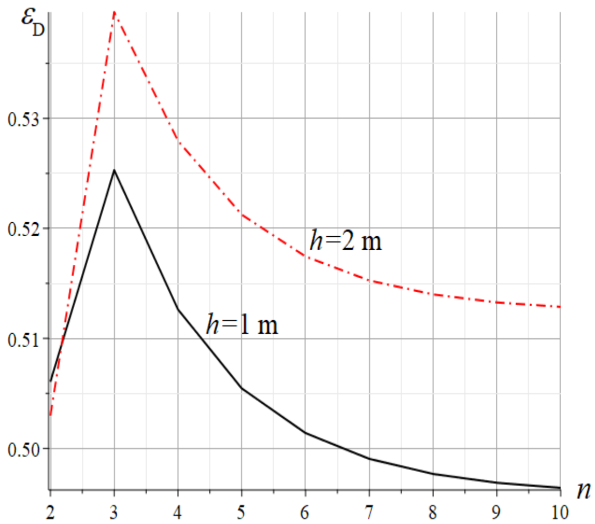

Figure 8. Dunkerley's estimation.

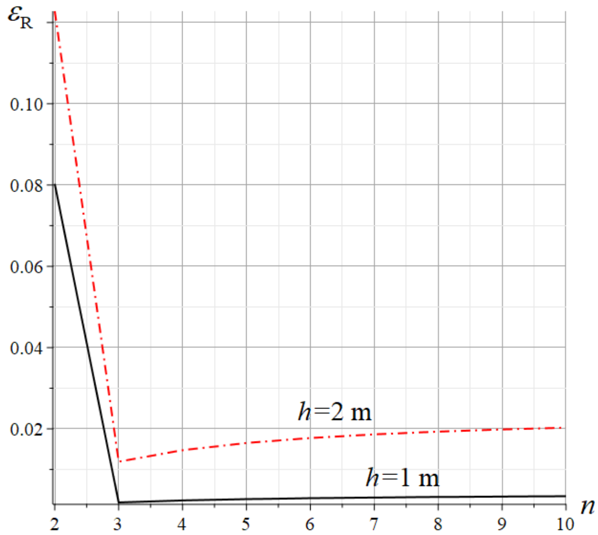

Figure 9. Rayleigh estimation.

With an increase in the number of panels up to a certain number *n*, the Dunkerley estimate error slightly decreases, and then, starting from $n = 12$ at $h = 2\,\text{m}$, it grows a little.

The error in the Rayleigh estimate increases slightly while remaining very small. The error of both methods increases with the height of the truss.

3.1. Discussion

A new scheme of a statically determined truss of a three-dimensional hexagonal cover is proposed. The truss can only partly be considered regular. It has only a regular base with two hexagonal horizontal contours, while the upper part has the shape of a regular hexagonal pyramid of rods connected at the vertex C. Despite this, the inductive method using a computer mathematics system made it possible to obtain analytical exact solutions for the deflections of its characteristic peaks and a two-sided estimate of the first frequency.

The design under consideration can be used in coverings of public buildings and structures of a large span, for example, arenas, buildings of stations and airports, circuses.

The external static indeterminacy, which means that the reactions of the supports can only be found from the solution of the joint system of equilibrium equations for all nodes simultaneously with the forces in the rods, did not complicate the solution. On the contrary, some obtained formulas turned out to be linear in the number of panels, which distinguishes this solution from similar formulas even for planar trusses. This is largely due to the property of the truss itself. An important property of similarity of the stressed states of trusses of various orders is noticed. The calculation showed that the pattern of force distribution over the structure rods does not depend on the number of panels. When n changes, the forces in the chords do not change, the formulas for the forces (1) – (3) and the reactions of the supports do not contain the number of panels n.

Comparison with the numerical solution of the complete problem of oscillations of a mass system with many degrees of freedom confirmed the well-known fact [16] that the Rayleigh formula for the upper estimate gives much greater accuracy than the Dunkerley method for the lower estimate of the first frequency.

4. Conclusion

Main results of the work:

1. A scheme of a spatial statically determinate rod structure of the dome type has been developed. Analytical solutions for deflection are obtained for both symmetrical and asymmetric loads. The truss design does not contain a central support post and can be used in large-span structures without central supports.

2. The calculation showed that the pattern of force distribution over the structure rods does not depend on the number of panels.

3. Formulas are derived both for the deflections of the truss and the boundaries of its first frequency of natural oscillations for an arbitrary number of panels. The obtained estimate of the first frequency from above has very high accuracy. Formulas can be used to evaluate numerical solutions for a very large number of panels, that is, precisely in those cases where the accumulation of calculation errors in numerical form is most likely.

4. The closed analytical form of the obtained formulas allows the use of mathematical analysis tools to identify their features and to search for combinations of design parameters that are optimal in terms of strength, rigidity, or stability.

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