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Beam on a two-parameter elastic foundation: simplified finite element model

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Abstract. When calculating beams resting on a solid elastic foundation, the simplest foundation models proposed by Winkler-Zimmerman and Vlasov-Leontyev are often used. These hypotheses have been repeatedly subjected to well-founded criticism, because they do not take into account the inclusion in the work of some areas of the base and do not allow determining reactive pressures at the ends of the foundation beam and beyond it. In order to clarify these hypotheses, many authors have proposed some other models that allow smoothing out the shortcomings of these models to varying degrees. This article proposes a new numerical approach to solving the problem of a beam on a two-parameter elastic foundation. To calculate the beam, the finite element method has been used. A separate rod has been proposed as a finite element for solving the bending state of the beam on a two-parameter model of an elastic foundation. There has been presented the construction of the stiffness matrix of this finite element. The elastic foundation is assumed to be linear, homogeneous and isotropic and is taken into account using the parameters *r, s*. The reactions of the elastic base, deflections and angles of rotation, the formulas for calculating bending moments and transverse forces have been determined. There have been given examples of static calculation of a beam on an elastic two-parameter foundation for the action of various loads. These examples demonstrate the effectiveness of the developed method. Reliability of the method proposed by the authors has been verified on test examples, and good agreement has been obtained with the well-known models of Winkler and Vlasov.

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1. *Introduction*

Structures in the form of beams and slabs on the yielding bases are widely used in various branches of technology, in particular in construction. The main problem in such tasks is to take into account the interaction in the "structure-foundation" system. In this case, the difficulty consists in selecting an appropriate mathematical model for the assigned engineering problem.

The study of the stress-strain state of foundation beams is mainly based on the Winkler model [1–3] in the form of two main options: a physical model of a two-parameter support structure [4–6], as well as a foundation beam model in the form of an elastic half-space [7, 8]. The well-known Winkler model does not take into account the bending deformation of the foundation itself, the presence of shear deformation, as well as the structural continuity of the soil; at the same time, the disadvantage is that the subsidence of the base is taken into account only within the area of the foundation structure [9, 10].

When using a foundation model with an elastic half-space, the force interaction between adjacent loads is taken into account, which corresponds to the realities of engineering problems. Based on a similar model, Kontomaris and Malamu [11] determine the contact forces between a rigid stamp in the form of a ball and an elastic body (base). At the same time, Baraldi and Tullini [12] propose a different variant in the form of a numerical model of a two-dimensional contact without taking into account friction between the bodies in the volume of a three-dimensional elastic half-space; meanwhile, bases with isotropic properties were studied to determine deformations (subsidence) and physical relationships between the components of the force vector. The other authors [13–15] introduced original parameters derived from the elastic half-space model to describe various models of elastic anisotropy, taking into account effects, especially of transverse (shear) anisotropy. Still other scientists [16–18] studied the bending of foundation beams by the direct Galerkin method based on power series, the finite element method, etc.

The authors of works [19–22] studied the analysis of beams on elastic foundation of Timoshenko in addition to the classical theory of beams. Theodore [23] developed an original computer code based on the finite element method for beams on an elastic foundation using the Matlab software package. Scientist Dinev [24] proposed an analytical method of calculating a beam of finite length on an elastic foundation based on a variational interpretation of the expression for the total potential energy functional. S. Limkatanyu et al. presented a nonlinear Winkler-based beam element with improved displacement shape functions that was capable of representing the nonlinear interaction mechanics between the beam and the foundation [25]. Sánchez and Roesset proposed a more accurate model of a beam on the elastic foundation for laterally loaded piles and used a consistent boundary matrix to evaluate the computing accuracy [26]. Results of other approaches, such as iterative methods [27], discrete singular convolution method [28], and boundary element method [29], can be found in the literature

The presented models include an elastic foundation of the Winkler, Vlasov, and Pasternak types. An ideal model for calculating the foundation was obtained. Still, with the development of the scale of construction, the interaction of soil and foundation beams, slabs, and other elements requires deeper theoretical studies. The development of more realistic foundation models and simplified methods is very important for the safe and economical design of such type of structure. Therefore, it is important to develop mathematical models and methods for assessing the stress-strain state of beams lying on an elastic base, considering their geometric and physical characteristics.

The present study takes an attractive approach for beams resting on an elastic foundation. The study aims to develop a simplified finite element model for calculating a beam on an elastic foundation is proposed. According to the proposed model, an elastic foundation is considered a single-layer model, the properties of which are described by two elastic characteristics.

The novelty and importance of this article lie in the following:

- a simplified finite element model is proposed, taking into account the elastic foundation, which is fundamentally different from the other models;
- obtaining a simplified model of an elastic foundation adopted for modeling the mechanical behavior of the soil;
- using the finite element method, in which the basic equations are derived, the stiffness matrix is determined, and the boundary conditions for beams of finite length resting on an elastic foundation are taken into account;
- simplifying methods of calculating beams on an elastic foundation for their wider implementation in engineering practice;
- carrying out a comparative analysis based on a simplified model of an elastic foundation and on traditional models of Winkler and Vlasov.

2. Methods

The initial differential equation for beam bending on a two-parameter elastic foundation is written in the form

$$\frac{d^4W_0(x)}{dx^4}-2r^2\frac{d^2W_0(x)}{dx^2}+S^4\cdot W_0(x)=\frac{q(x)}{EJ}. \qquad (1)$$

The $2r^2$ and s^4 dimensionless elastic characteristics are determined in the work [30] as:

$$2r^2=\frac{6(1-\nu^2)P_1l^2}{h^2}\cdot\frac{\overline{E}h^2}{Eh_0^2},\quad s^4=\frac{6(1-\nu^2)kP_0l^4}{h^4}\cdot\frac{\overline{E}h^3}{Eh_0^3}.$$

The effect of the elastic foundation on the beam is taken into account by two parameters P_0, P_1 and is determined in [30] by the formula

$$P_0=\frac{2\cdot m+\frac{h_0}{h}k(1-\nu)}{2m-(1-\nu)^2},\quad P_1=\frac{k\frac{h_0}{h}+(1-\nu)}{2m-(1-\nu)^2},\quad m=2+\frac{(1-\nu^2)}{k}\frac{\overline{E}h}{Eh_0},$$

where ν is the Poison's ratio, k is the deformed state parameter that depends on the boundary conditions.

The parameter k depends on the boundary conditions and is determined as follows [31]:

– for hinge-supported beam $k=\pi^2\frac{h^2}{\ell^2}$;

– for fixed-end beam $k=4\pi^2\frac{h^2}{\ell^2}$;

– for cantilever beam $k=\frac{\pi^2}{4}\frac{h^2}{\ell^2}$;

– for a statically indeterminate beam $k=\pi^2\frac{h^2}{\ell^2}$.

Original differential equation (1) differs from the classical equation of beam bending on the elastic foundation of the Winkler model in its structure there is an additional term containing the second derivative, which allows taking into account the effect of shear stresses.

The beam on a two-parameter elastic foundation is calculated by the finite element method. From the mathematical point of view, the finite element method (FEM) is a numerical procedure for finding approximate solutions to boundary value problems for partial differential equations.

When using the FEM, there are the following assumptions underpinning this development:

– the finite element has a unit length and has two nodes at its ends;
– the finite element is connected with other elements only in nodes;
– loading the elements occurs only in nodes.

3. Results and Discussion

3.1. Finite element formulation

A finite element selected from beam structures on an elastic foundation is considered. This element has the length ℓ, the width h_0 and the bending stiffness EJ. Here E is the modulus of elasticity of the material, and J is the axial moment of inertia. The thickness of the elastic base is denoted as h, the modulus of elasticity of the material is $\overline{E}$ (Fig. 1).

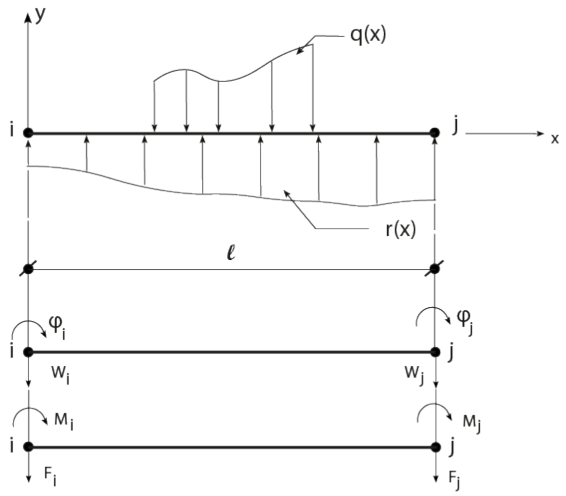

Figure 1. A finite element on an elastic foundation.

The deformed state of this element is determined by the nodal displacements $(W_i, \varphi_i, W_j, \varphi_j)$, and the stressed state is determined by the nodal forces (F_i, M_i, F_j, M_j).

The deflection function has the following form

$$W(x) = q_1(x)W_i + q_2(x)\varphi_i + q_3(x)W_j + q_4(x)\varphi_j$$

$$q_1(x) = \left(1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}\right),\; q_2(x) = \ell\left(\frac{x}{\ell} - 2\frac{x^2}{\ell^2} + \frac{x^3}{\ell^3}\right),$$
$$q_3(x) = \left(3\frac{x^2}{\ell^2} - 2\frac{x^3}{\ell^3}\right),\; q_4(x) = \ell\left(-\frac{x^2}{\ell^2} + \frac{x^3}{\ell^3}\right), \tag{2}$$

where $q_1(x), q_2(x), q_3(x), q_4(x)$ are the coordinate functions.

The function of deflections (2) can be presented in the vector form

$$W(x) = \vec{q}^T \cdot \vec{V},\; \vec{q}^T = \left|q_1\; q_2\; q_3\; q_4\right|, \vec{V}^T = \left|W_i\; \varphi_i\; W_j\; \varphi_j\right|, \tag{3}$$

where $\vec{q}^T$ is the transposed vector of the coordinate functions, $\vec{V}$ is the vector of nodal displacements of the finite element.

The potential energy of the finite element on a two-parameter elastic foundation is found as follows

$$U_0 = \frac{1}{2}\int_0^l \frac{M^T M dx}{EJ} + \frac{1}{2}\int_0^l -\left(\frac{dW}{dx}\right)^T 2r^2\left(\frac{dW}{dx}\right)dx + \frac{1}{2}\int_0^l W^T s^4 W dx, \tag{4}$$

where *M(x)* is the moment of flexion.

Based on equation (3), the potential energy (4) had written in the following form

$$U_0 = \frac{1}{2}\vec{V}^T \cdot K \cdot \vec{V} - \frac{1}{2}\cdot 2r^2 \cdot \vec{V}^T \cdot \hat{K} \cdot \vec{V} + \frac{1}{2}\cdot s^4 \cdot \vec{V}^T \cdot \check{K} \cdot \vec{V},$$

where

$$K = \int_0^l \left(\frac{d^2\vec{q}}{dx^2}\right)\cdot(EJ)\cdot\left(\frac{d^2\vec{q}}{dx^2}\right)^T dx,\; \hat{K} = \int_0^l \left(\frac{d\vec{q}}{dx}\right)\cdot\left(\frac{d\vec{q}}{dx}\right)^T dx,\; \check{K} = \int_0^l \vec{q}\cdot\vec{q}^T dx.$$

The finite element stiffness matrix elements $\left(K, \hat{K}, \check{K}\right)$ are determined by the following formulas

$$K_{ij} = \int_0^l \left(\frac{d^2\vec{q}_i}{dx^2} \right) \cdot (EJ) \cdot \left(\frac{d^2\vec{q}_j}{dx^2} \right) dx,$$
$$\hat{K}_{ij} = \int_0^l \left(\frac{d\vec{q}_i}{dx} \right) \cdot \left(\frac{d\vec{q}_j}{dx} \right) dx, \tag{5}$$
$$\overset{\vee}{K}_{ij} = \int_0^l \vec{q}_i \cdot \vec{q}_j dx.$$

The work of the nodal forces is determined as follows

$$A_0 = \frac{1}{2} \vec{V}^T \cdot \vec{F}_0, \quad \vec{F}_0^T = \left| F_i \quad M_i \quad F_j \quad M_j \right|. \tag{6}$$

From the condition of the finite element equilibrium $(A_0 = U_0)$ there is the basic dependence

$$A_0 = U_0: \; \vec{F}_0 = K_0 \cdot \vec{V}, \tag{7}$$

where K_0 is the finite element stiffness matrix taking into account the elastic foundation.

Based on equation (2) and formula (5), the stiffness matrix of the finite element on the elastic foundation is determined as follows

$$K_0 = K - 2r^2 \cdot \hat{K} + s^4 \cdot \overset{\vee}{K},$$
$$K = \frac{EJ}{l^3} \begin{vmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{vmatrix},$$
$$\hat{K} = \frac{1}{30l} \begin{vmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{vmatrix}, \tag{8}$$
$$\overset{\vee}{K} = \frac{l}{420} \begin{vmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{vmatrix}.$$

The beam internal forces are presented as follows

$$M(x) = -EJ \cdot \frac{d^2W(x)}{dx^2},$$
$$Q(x) = -EJ \cdot g_0 \cdot \frac{d^3W(x)}{dx^3},, \tag{9}$$
$$g_0 = 1 + \frac{6\left(1-v^2\right)P_1}{k^2} \frac{\bar{E}h^2}{Eh_0^2},$$

where M is a bending moment, Q is a shear force, g_0 is the parameter of the beam shear force.

One of the following boundary conditions must be satisfied at the beam edges.

1) If the beam edge has a full contact (touches fully) with the elastic foundation, then the boundary conditions as follows

$$W_* = \frac{Q \cdot L^3}{3\bar{E}J_0}, \;\; f_* = \frac{M \cdot L}{\bar{E}J_0}, \;\; J_0 = \frac{h^3}{12},$$

where W_*, φ_* are the beam edge displacements, L is the foundation length beyond the beam, $\bar{E}J_0$ is rigidity with the foundation deflection.

2) If the ends of the beam are hinge-supported, then the boundary conditions as follows

$$W = 0, M = 0.$$

3) If the ends of the beam are fixed, then the boundary conditions as follows

$$W = 0, \theta = 0.$$

4) If the ends of the beam are free, then the boundary conditions as follows

$$M = 0, Q = 0.$$

The stiffness matrix of the internal forces is defined as follows

$$\begin{bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{bmatrix} = \begin{bmatrix} -Q(0) \\ M(0) \\ Q(l) \\ -M(l) \end{bmatrix} = \frac{EJ}{l^3} \begin{bmatrix} 12g_0 & 6lg_0 & -12g_0 & 6lg_0 \\ 6l & 4l^2 & -6l & 2l^2 \\ -12g_0 & -6lg_0 & 12g_0 & -6lg_0 \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \cdot \begin{bmatrix} W_i \\ \phi_i \\ W_j \\ \phi_j \end{bmatrix},$$

$$\bar{K} = \frac{EJ}{l^3} \begin{vmatrix} 12g_0 & 6lg_0 & -12g_0 & 6lg_0 \\ 6l & 4l^2 & -6l & 2l^2 \\ -12g_0 & -6lg_0 & 12g_0 & -6lg_0 \\ 6l & 2l^2 & -6l & 4l^2 \end{vmatrix}. \qquad (10)$$

Reactive pressures of the elastic foundation are determined as follows

$$R(x) = EJ \cdot \left(2r^2 \frac{d^2W(x)}{dx^2} - s^4 W(x) \right). \qquad (11)$$

The reactive pressure stiffness matrix of the elastic foundation is determined as follows

$$\begin{bmatrix} R(0) \\ R(l) \end{bmatrix} = \frac{EJ}{l^3} \cdot \begin{bmatrix} -6l \cdot 2r^2 - s^4 \cdot l^3 & -4l^2 \cdot 2r^2 & 6l \cdot 2r^2 & -2l^2 \cdot 2r^2 \\ 6l \cdot 2r^2 & 2l^2 \cdot 2r^2 & -6l \cdot 2r^2 - s^4 l^3 & 4l^2 \cdot 2r^2 \end{bmatrix} \cdot \begin{bmatrix} W_i \\ \varphi_i \\ W_j \\ \varphi_j \end{bmatrix},$$

$$\overset{0}{K} = \frac{EJ}{l^3} \begin{vmatrix} -6l \cdot 2r^2 - s^4 l^3 & -4l^2 \cdot 2r^2 & 6l \cdot 2r^2 & -2l^2 \cdot 2r^2 \\ 6l \cdot 2r^2 & 2l^2 \cdot 2r^2 & -6l \cdot 2r^2 - s^4 l^3 & 4l^2 \cdot 2r^2 \end{vmatrix}. \qquad (12)$$

The calculation of a beam on an elastic foundation by the finite element method is performed according to the following algorithm:

1 Dividing the beam into finite elements and numbering the nodes and elements from left to right.

2 Setting the directions of nodal displacements.

3 Forming vectors of nodal displacements and external forces.

4 Making the stiffness matrices of the beam elements.

5 Forming the general matrix of beam stiffness.

6 Determining the vector of nodal displacements of the beam.

7 Based on the main dependence, determining the vectors of nodal forces.

8 The internal forces of the beam are determined through the vector of nodal displacements and the matrix of internal forces.

9 Determining the reactive pressure of the elastic foundation using the reactive pressure matrix.

3.2. Numerical Results and Analysis

To verify the calculation accuracy, in order to compare the solution of the proposed simplified finite element model with other solutions of the Winkler and Vlasov model from the classical theory of beams on an elastic foundation, below there are several examples of solving engineering problems.

In these problems, beams on an elastic foundation with different boundary conditions, external loads, and reaction coefficients of the soil foundation are considered. In this case, the stiffness matrices, internal nodal forces, and reactions of the elastic foundation are obtained using an improved method by applying the corresponding equations (8), (9), and (11).

3.2.1. Example 1

A simply supported beam on an elastic foundation is considered. A beam of length $\ell = 10\ m$ $\ell = 10\ m$, width $b_0 = 1\ m$, and height $h_0 = 2\ m$, with the modulus of elasticity $E = 20 \cdot 10^5\ kPa$, is considered to be supported by a foundation having depth $h = 5\,m$, deformation modulus $\overline{E} = 40\,kPa$ and Poisson's ratio $\nu = 0.25$. The beam carries the external uniform vertical load $q = 200\dfrac{kN}{m}$.

The calculated values of vertical deformations, internal forces are given in Table 1–3; the largest values are given in Table 4. Analytical and numerical solutions are given for the case of a uniformly distributed load. The results obtained are compared with similar values obtained earlier using the Winkler and Vlasov models (Fig. 2–4).

Table 1. Vertical displacement values (simply supported beam on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	2	4	5	6	8	10
Wn	0	11.5997	18.5995	19.5308	18.5995	11.5997	0
Wa	0	11.5918	18.5869	19.5175	18.5869	11.5918	0
Ww	0	11.5623	18.5389	19.4671	18.5389	11.5623	0
Wv	0	11.5901	18.5839	19.5144	18.5839	11.5901	0

Table 2. Bending moment values (simply supported beam on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	2	4	5	6	8	10
Mn, 10^6	0	1.59989	2.40124	2.49912	2.40124	1.59989	0
Ma, 10^6	0	1.59887	2.39831	2.49824	2.39831	1.59887	0
Mw, 10^6	0	1.59503	2.39197	2.49156	2.39197	1.59503	0
Mv, 10^6	0	1.59869	2.39789	2.49779	2.39789	1.59869	0

Table 3. Shear force values (simply supported beam on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	2	4	5	6	8	10
Qn, 10^5	9.9955	5.9961	1.9993	0	-1.9991	-5.9958	-9.9952
Qa, 10^5	9.9941	5.9965	1.9988	0	-1.9988	-5.9965	-9.9941
Qw, 10^5	9.9734	5.9785	1.9918	0	-1.9918	-5.9785	-9.9734
Qv, 10^5	9.9930	5.9944	1.9979	0	-1.9979	-5.9944	-9.9930

Table 4. Maximum values of vertical displacements, bending moments and transverse forces (simply supported beam on an elastic foundation).

Modular Ratio	Property	Case

$\left(\frac{\overline{E}}{E}\right)$		Winkler Model	Vlasov Model	Analytical Model	Present finite element model
0.2	Max Vertical displacement	2.412	2.465	2.543	2.536
	Max Bending moment, (10^5)	3.047	3.177	3.255	3.202
	Max Shear force, (10^5)	2.346	2.493	2.571	2.506
0.1	Max Vertical displacement	4.247	4.335	4.413	4.402
	Max Bending moment, (10^5)	5.467	5.569	5.649	5.632
	Max Shear force, (10^5)	3.242	3.366	3.448	3.412
0.05	Max Vertical displacement	6.950	7.057	7.137	7.110
	Max Bending moment, (10^5)	8.923	9.053	9.135	9.121
	Max Shear force, (10^5)	4.426	4.573	4.655	4.624

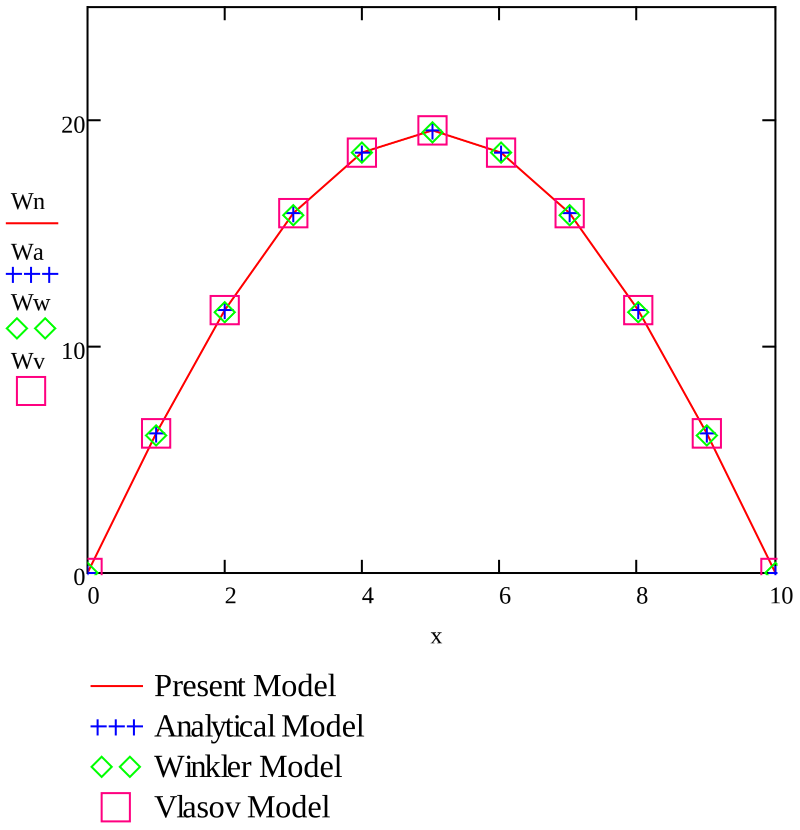

Figure 2. Vertical displacement of the beam on elastic foundation.

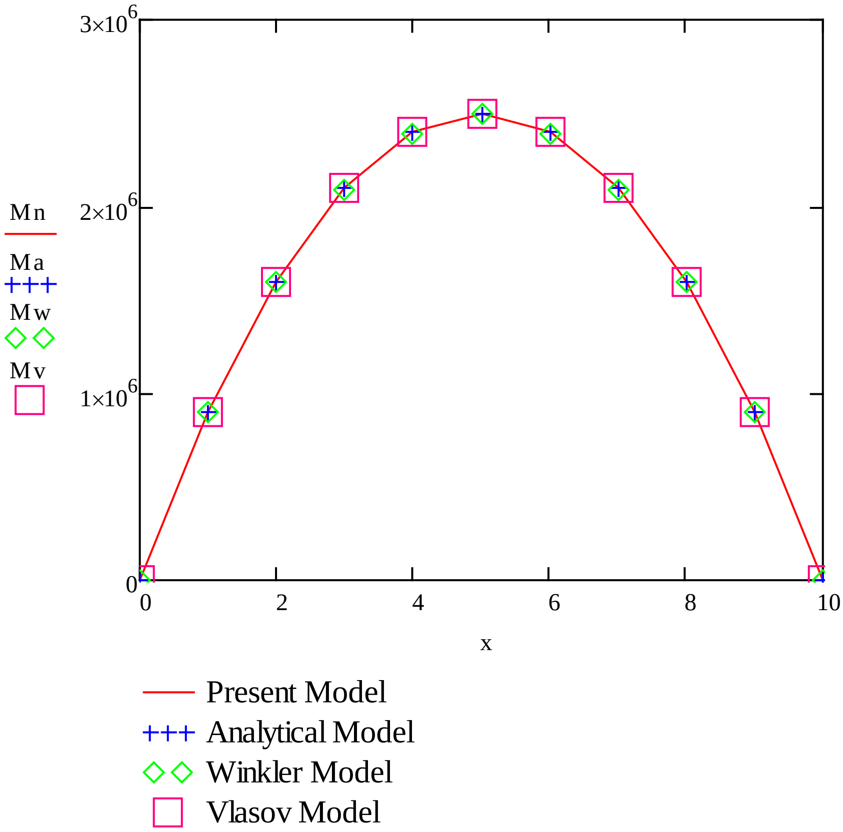

Figure 3. Bending moment of the beam on elastic foundation.

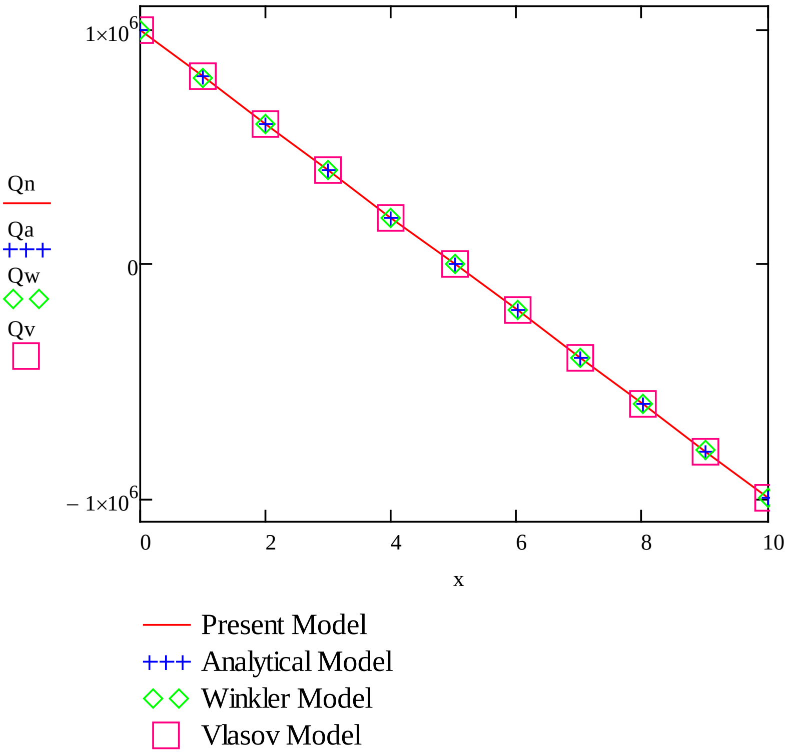

Figure 4. Shear force of the beam on elastic foundation.

The results of the reactive pressure of the elastic foundation are shown in Figure 5.

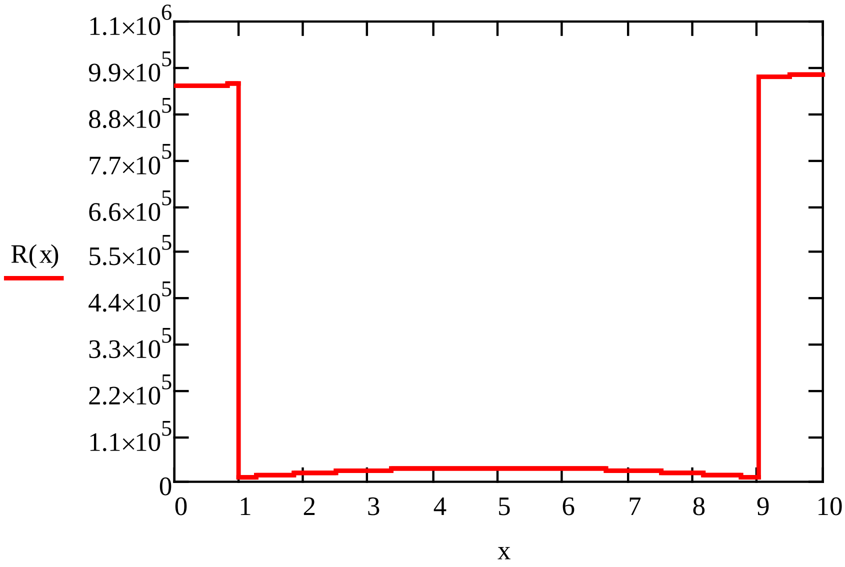

Figure 5. Reactive pressure of the elastic foundation.

3.2.2. Example 2

A fixed-end beam on an elastic foundation is considered. The uniformly distributed load was chosen as $q = 120\dfrac{kN}{m}$. A beam of length $\ell = 8\ m$, width $b_0 = 1\ m$, and height $h_0 = 2\ m$. The beam's material has a modulus of elasticity $E = 30 \cdot 10^5\ kPa$. The physical and geometry parameters of the elastic foundations were $\overline{E} = 30\,kPa$, $\nu = 0.25$ and $h = 4\,m$.

The comparison results of the beam displacement, bending moment, and shear force are shown in Tables 5–7 and the maximum value are given in Table 8, which indicate that the values obtained by the present method are in good agreement with the analytical, Winkler, and Vlasov models. Fig. 6–8 show the displacement, bending moment, and shear force diagrams.

Table 5. Vertical displacement values (fixed-end beam on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	1	2	4	6	7	8
Wn	0	0.12266	0.35999	0.63998	0.35999	0.12266	0
Wa	0	0.12248	0.35993	0.63987	0.35993	0.12248	0
Ww	0	0.12249	0.35995	0.63992	0.35995	0.12249	0
Wv	0	0.12249	0.35998	0.63997	0.35998	0.12249	0

Table 6. Bending moment values (fixed-end beam on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	1	2	4	6	7	8
Mn, 10^5	-6.39905	-2.19986	0.79995	3.19983	0.79995	-2.19986	-6.39905
Ma, 10^5	-6.39875	-2.19957	0.79984	3.19937	0.79984	-2.19957	-6.39875
Mw, 10^5	-6.39925	-2.19968	0.79992	3.19954	0.79992	-2.19968	-6.39925
Mv, 10^5	-6.39976	-2.19989	0.79998	3.19984	0.79998	-2.19989	-6.39976

Table 7. Shear force values (fixed-end beam on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	1	2	4	6	7	8
Qn, 10^5	4.79987	3.59982	2.39986	0	-2.39986	-3.59982	-4.79987
Qa, 10^5	4.79926	3.59944	2.39963	0	-2.39963	-3.59944	-4.79926
Qw, 10^5	4.79956	3.59958	2.39965	0	-2.39965	-3.59958	-4.79956
Qv, 10^5	4.79989	3.59986	2.39987	0	-2.39987	-3.59986	-4.79989

Table 8. Maximum values of vertical displacements, bending moments and transverse forces (fixed-end beam on an elastic foundation).

Modular Ratio	Property	Case: Winkler Model	Vlasov Model	Analytical Model	Present finite element model
0.2	Max Vertical displacement	0.359	0.361	0.373	0.368
	Max Bending moment, (10^5)	1.809	1.825	1.864	1.847
	Max Shear force, (10^5)	3.446	3.431	3.602	3.586
0.1	Max Vertical displacement	0.449	0.451	0.469	0.458
	Max Bending moment, (10^5)	2.255	2.287	2.344	2.298
	Max Shear force, (10^5)	3.985	3.959	4.039	3.916
0.05	Max Vertical displacement	0.526	0.529	0.530	0.532
	Max Bending moment, (10^5)	2.686	2.689	2.701	2.692
	Max Shear force, (10^5)	4.341	4.352	4.359	4.348

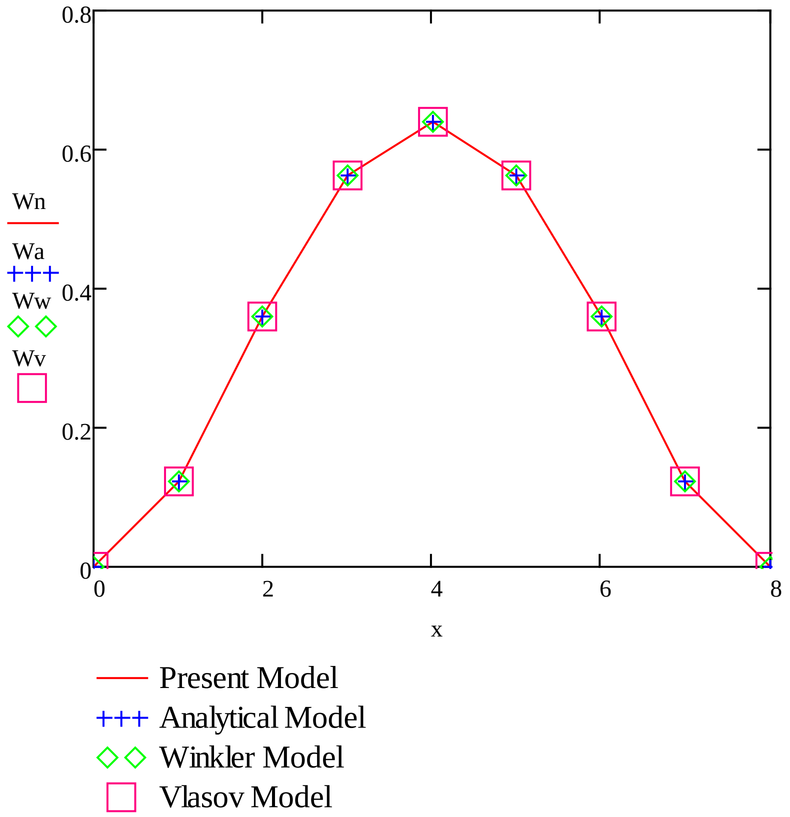

Figure 6. Vertical displacement of the beam on elastic foundation.

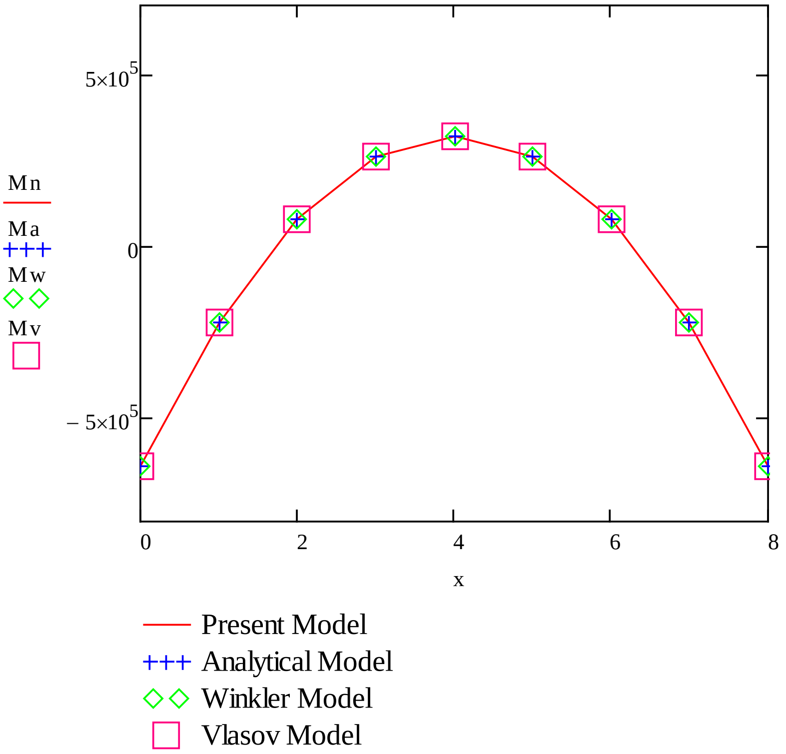

Figure 7. Bending moment of the beam on elastic foundation.

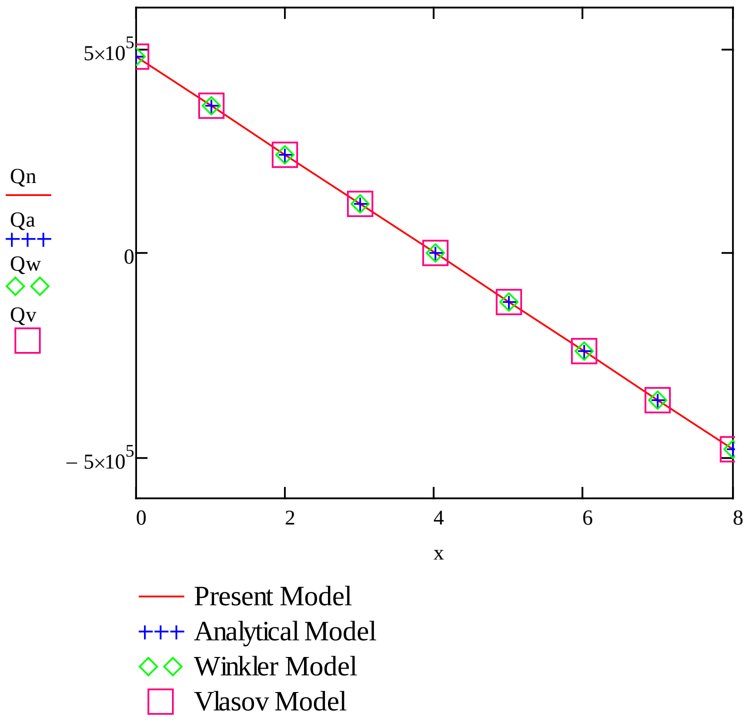

Figure 8. Shear force of the beam on elastic foundation.

3.2.3. Example 3.

A beam fixed at one end and supported at the other on an elastic foundation is considered. The beam on an elastic foundation was assumed to be subjected only to uniform vertical loads. The vertically uniform load was chosen as $q(x) = q_0 \frac{x}{\ell}$, $q_0 = 100 \frac{kN}{m}$. The physical and geometry parameters of the elastic foundations were deformation modulus $\overline{E} = 70\,kPa$ and Poissons ratio $\nu = 0.25$, and depth

$h = 6m$. A beam of length $\ell = 12\ m$, width $b_0 = 1\ m$, height $h_0 = 2\ m$, and modulus of elasticity $E = 60 \cdot 10^5\ kPa$.

Reliability of the results of the method proposed by the authors was assessed according to the results of Winkler and Vlasov (Tables 9–12). At the same time (Figs. 9–11) it was found that displacements along the normal to the base, the internal forces obtained on the basis of the 4 models described above, are in good agreement.

Table 9. Vertical displacement values (beam fixed at one end and supported at the other on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	2	4	6	8	10	12
Wn	0	0.33185	0.97638	1.48647	1.52865	0.98769	0
Wa	0	0.33033	0.97712	1.48400	1.52786	0.98545	0
Ww	0	0.32999	0.97612	1.48254	1.52644	0.98459	0
Wv	0	0.33043	0.97741	1.48444	1.52833	0.98576	0

Table 10. Bending moment values (beam fixed at one end and supported at the other on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	2	4	6	8	10	12
Mn, 10^5	-8.39987	-3.10986	1.51123	4.79962	6.08686	4.70752	0
Ma, 10^5	-8.39434	-3.10902	1.51009	4.79677	6.08479	4.70794	0
Mw, 10^5	-8.38555	-3.10597	1.50800	4.79152	6.07935	4.70493	0
Mv, 10^5	-8.39693	-3.10989	1.51048	4.79808	6.08666	4.70964	0

Table 11. Shear force values (beam fixed at one end and supported at the other on an elastic foundation).

Case	The length of the beam (ℓ, m)						
	0	2	4	6	8	10	12
Qn, 10^5	2.69975	2.53647	2.03312	1.19987	0.03341	-1.46785	-3.29982
Qa, 10^5	2.69842	2.53185	2.03214	1.19930	0.03331	-1.46581	-3.29807
Qw, 10^5	2.69530	2.52881	2.02978	1.19832	0.03397	-1.46408	-3.29665
Qv, 10^5	2.69916	2.53236	2.03250	1.19958	0.03345	-1.46606	-3.29919

Table 12. Maximum values of vertical displacements, bending moments and transverse forces (beam fixed at one end and supported at the other on an elastic foundation).

Modular Ratio	Property	Case			
		Winkler Model	Vlasov Model	Analytical Model	Present finite element model
0.2	Max Vertical displacement	0.540	0.545	0.561	0.551
	Max Bending moment, (10^5)	2.082	2.089	2.163	2.095
	Max Shear force, (10^5)	1.384	1.391	1.425	1.396
0.1	Max Vertical displacement	0.792	0.796	0.818	0.806
	Max Bending moment, (10^5)	3.131	3.139	3.157	3.148
	Max Shear force, (10^5)	1.730	1.735	1.757	1.744
0.05	Max Vertical displacement	1.050	1.059	1.074	1.069
	Max Bending moment, (10^5)	4.052	4.059	4.143	4.135
	Max Shear force, (10^5)	1.989	2.001	2.077	2.062

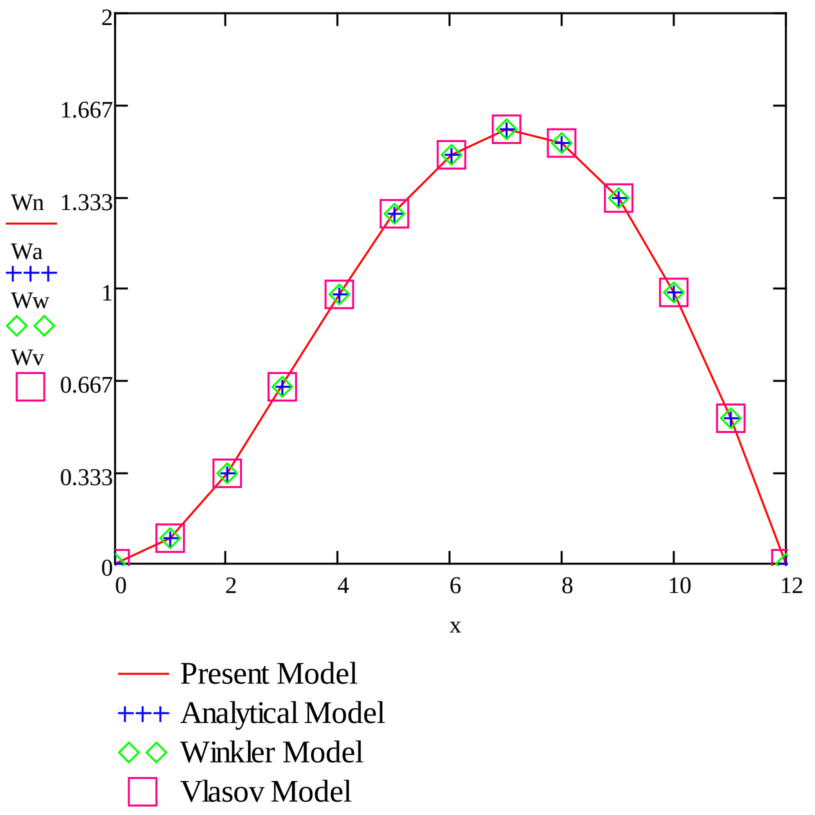

Figure 9. Vertical displacement of the beam on elastic foundation.

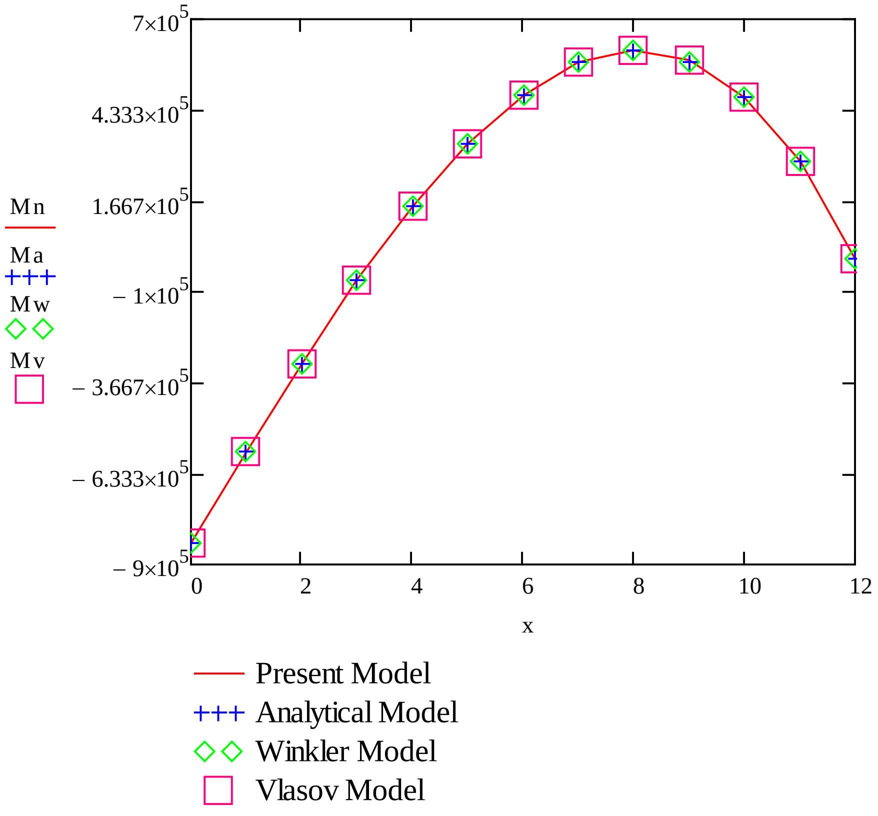

Figure 10. Bending moment of the beam on elastic foundation.

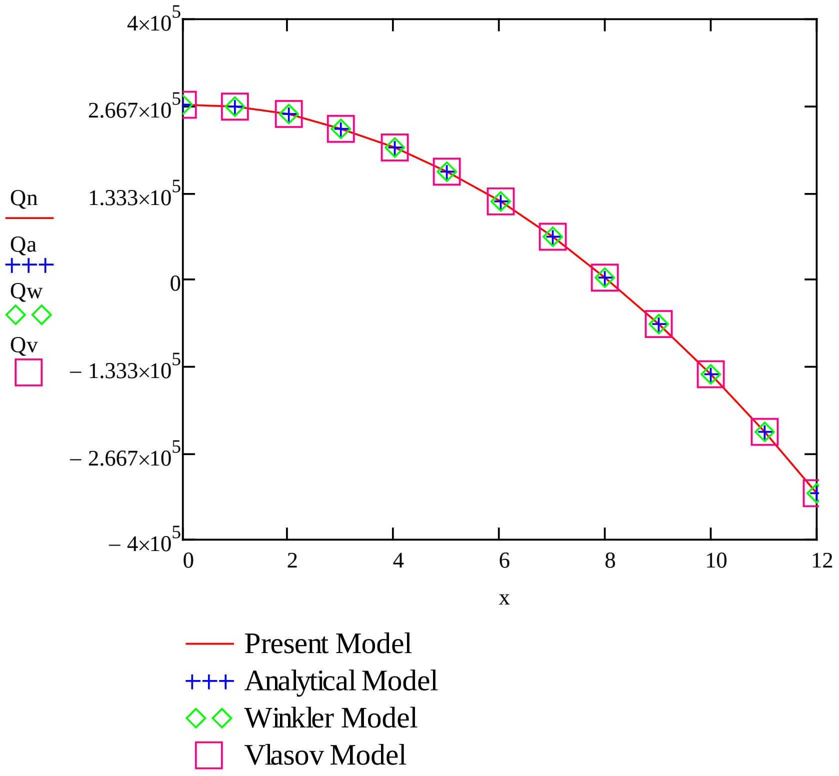

Figure 11. Shear force of the beam on elastic foundation.

In tables and figures Ww, Mw, Qw means vertical displacement, bending moment, and shear force by Winkler, respectively Wv, Mv, Qv vertical displacement, bending moment, and shear force by Vlasov. Vertical displacement (Wa), bending moment (Ma), and shear force (Qa) had determined by the analytical model [32]. The present finite element model found vertical displacement (Wn), bending moment (Mn) and shear force (Qn).

The presented examples show the advantages of the suggested approach for a numerical beam solution on an elastic foundation. The tables and figures show the excellent agreement of the proposed method with the results obtained by the Winkler and Vlasov models. These results are in good agreement with the results of the author's work, which were obtained using a different approach [32].

4. Conclusions

This article proposes a simplified model of the finite element method of solving the problem of the bending state of finite length beams interacting with a two-parameter base.

Some conclusions can be drawn from the results:

1. The elastic foundation is considered without increasing the degrees of freedom of the finite element.

2. On the basis of the proposed finite element model, which allows determining the deformation (displacement) of the force, an original (the author's) model has been developed.

3. The corresponding formulas for the finite element stiffness matrix, reactive pressure, internal forces, and vertical and nodal displacements are derived.

4. A simplified version of the elastic foundation model is defined. The elastic foundation was taken into account using two parameters.

5. Reliability of the method proposed by the author was evaluated on three test examples; the results obtained are in good agreement with the results obtained on the basis of the Winkler and Vlasov models.

The simplicity of mathematical techniques and the clarity of the scheme make the simplified finite element method under consideration very flexible and allow solving not only the main problems of calculating beams on an elastic foundation but several more complex issues.

Civil engineers use off-the-shelf software for calculations of foundation structures. The proposed finite element model may be of interest to software developers.

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