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FORMULATION OF THE LORENTZ TRANSFORMATION EQUATIONS IN THE THREE DIMENSIONS OF SPACE

C. B. Khadka ✉

Tribhuvan University, Kathmandu, Nepal

✉ chandrabahadur9988@gmail.com

Abstract. The Lorentz transformation of space and time between two inertial frames of reference is one of the pillars of the special theory of relativity. Until now, the Lorentz transformation equations have been considered on the base of one-dimensional motion between the inertial frames. The goal of this article is to extend the ordinary one-dimensional Lorentz transformation to motion along *X*-, *Y*-, and *Z*-directions, i.e., to achieve a space-time coordinate transformation in the three-dimensional (3D) space. We particularly discovered the modified Lorentz transformation equations along 3 directions, and this helped us to analyze the space contraction phenomena of a cuboid due to the relative motion between the inertial frames in the 3D space. As a final point, this study concluded that all length, breadth and height of a cuboid appeared to be shortened to the observer if there is the relative motion between the cuboid and an observer in the 3D space.

Keywords: frame of reference, Lorentz transformation, space contraction, special relativity theory

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ФОРМУЛИРОВКА УРАВНЕНИЙ ПРЕОБРАЗОВАНИЯ ЛОРЕНЦА В ТРЕХ ПРОСТРАНСТВЕННЫХ ИЗМЕРЕНИЯХ

Ч. Б. Хадка ✉

Многопрофильный колледж Три-Чандра Университета Трибхувана,

г. Катманду, Непал

✉ chandrabahadur9988@gmail.com

Аннотация. Уравнения преобразования пространства и времени по Лоренцу между двумя инерциальными системами отсчета служат краеугольным камнем специальной теории относительности. До настоящего времени эти преобразования рассматривались на основе одномерного движения между указанными системами отсчета. Цель данной статьи – распространить эти уравнения на движения вдоль трех направлений: *X*, *Y* и *Z*, т. е. так модифицировать одномерное преобразование Лоренца, чтобы получить преобразования пространственно-временных координат в трехмерном пространстве. Для этого использовано совместное применение полярной и декартовой систем координат для нахождения расположения точки, что должно обеспечивать полное преобразование пространственно-временных координат вдоль каждой оси. Модификация уравнений позволила автору проанализировать явления сжатия пространства (через введение кубоида и наблюдателя), вызванные относительным движением в трехмерном пространстве между инерциальными системами отсчета. Проведенное исследование привело к выводу, что все габариты кубоида кажутся наблюдателю уменьшенными, если между ним и кубоидом осуществляется относительное движение в трехмерном пространстве.

Ключевые слова: система отсчета, преобразование Лоренца, сжатие пространства, специальная теория относительности

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Introduction

The Lorentz transformation equations [1], which are regarded as a fundamental mathematical tool to develop the special theory of relativity, were represented in their form by H. Poincaré [2], and subsequently by A. Einstein [3] as follows:

$$X' = \gamma\left(X - VT\right),\ T' = \gamma\left(T - \frac{XV}{c^2}\right),\ Y' = Y,\ Z' = Z, \tag{1}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$ denotes the Lorentz coefficient.

There were significant contributions to the development of Lorentz transformation. H. Poincarй [2] recognized the relativity of simultaneity and studied the group theoretic properties. In the course of the last several decades, Lorentz transformation equations have been extended and generalized in many directions with several applications in many branches of relativistic mechanics. In Ref. [4], the authors presented the Lorentz transformation equations by changing the way of synchronizing clocks in inertial frame of reference. These transformation equations were further analyzed for one-way speed of light in vacuum by F. Selleri in Ref. [5]. In Ref. [6], the authors explained the Lorentz transformation by changes occurring in the wave characteristics of matter as it changes the inertial frames. There have been numerous publications [7, 8] looking for an agreement between the equivalence principle and nonrelativistic quantum theory with no electromagnetic field involved (see also Ref. [9] for a discussion involving relativistic effect). Ref. [10] presents a mechanism of accelerating relativistic charged particles using multifrequency modulated circularly polarized laser pulses directed along the propagation direction of a constant uniform magnetic field. Ref. [11] also presents alternative Hamiltonian and Lagrangian formalisms for relativistic mechanics using proper time and proper Lagrangian coordinates in 1 + 1 dimensions as parameters of evolution. The dynamics of a relativistic particle not having an electric charge and being under the action of an external force was analyzed on the basis of the special theory of relativity in Ref. [12]. The author of Ref. [13] determined a change in mass of an object inside the gravitational field using relativistic consideration. Ref. [14] also modified Newton's second law of motion by developing the novel formula of linear momentum, force and kinetic energy. Ref. [15] presented an original derivation of Lorentz transformation in the three-dimensional space. There have been a lot of publications on special relativity with important theoretical results. Since the Lorentz transformation equations in the most references have been based on one dimensional motion between inertial frames governed by Eq. (1), it is desirable to have an alternative method easily performed for deriving the Lorentz transformation equations when the motion between inertial frames takes place in the three-dimensional space.

Therefore, the goal of this paper is to propose an efficient way to formulate and study the transformation equation along *X*-, *Y*- and *Z*-directions by introducing relative motion between inertial frames in the three-dimensional space.

For the new transformation, instead of the above Eq. (1), we simply put forward the following one:

$$X' = \gamma X\left(1 - \frac{VT}{\sqrt{X^2+Y^2+Z^2}}\right),\ T' = \gamma\left(T - \frac{V\sqrt{X^2+Y^2+Z^2}}{c^2}\right),$$
$$Y' = \gamma Y\left(1 - \frac{VT}{\sqrt{X^2+Y^2+Z^2}}\right),\ Z' = \gamma Z\left(1 - \frac{VT}{\sqrt{X^2+Y^2+Z^2}}\right). \quad (2)$$

It is clearly seen that the spatial coordinates along Y- and Z-directions in the former Eq. (1) are absolute as well as the transformation equation of time depends only on X-coordinate, while the spatial coordinates along X-, Y- and Z-directions in modified Eq. (2) are relative as well as the transformation equation of time depends on all X-, Y- and Z-coordinates.

With above background, we present the organization of this paper as follows. In the next section "Method used", we will introduce the motion between inertial frames in the three dimensional-space and present the space-time coordinates transformation along X-, Y- and Z-directions. Following this, in section "Results and discussion", we will mathematically demonstrate the space contraction of cuboid along X-, Y- and Z-directions and will also reveal that the former transformation equation, namely, Eq. (1) arises only as a special case. In section "Conclusions" the concluding remarks are presented.

Methods used

Geometry analysis. Let us consider the relative velocity v of moving frame F' with respect to stationary frame F in the three-dimensional space (Fig. 1).

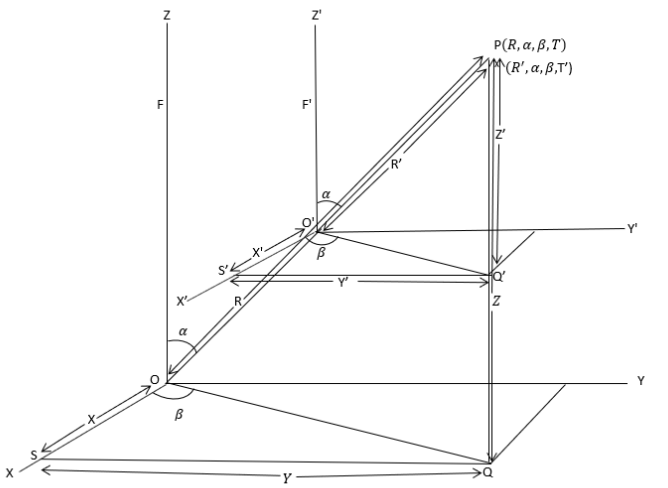

Fig. 1. Graphic presentation of movement between inertial frames in the three dimensions of space. Basic geometric notations are given

Let T and T' be the time recorded in two frames. Let the origin O and O' of two reference system coincide at $T = T' = 0$. Now let us suppose that a source of light is located at the origin O in the frame F, from which a light wavefront is emitting at a time $T = 0$. When the light reaches point P, the polar space-time coordinates measured from frame F and F' are (R, α, β, T) and (R', α, β, T') respectively (see Fig. 1). The time required for the light signal to travel the distance OP in frame F is given by equations

$$T = OP/c,\ T = R/c, \text{ or } R = cT. \quad (3)$$

This Eq. (3) represents the equation of wavefront in frame F. According to the special theory of relativity, the light velocity will be c in the second frame F'. Hence, in F' frame the time required for the light signal to travel the distance $O'P$ will be given by the following equations:

$$T' = O'P/c,\ T' = R'/c, \text{ or } R' = cT'. \tag{4}$$

This Eq. (4) represents the equation of wavefront in frame F'. Now, we can easily calculate the mathematical relationship between the Cartesian coordinates (x, y, z) of the point P and its spherical coordinates (R, α, β) in F frame of reference using trigonometry, and they are given by the following equations:

$$Z = R\cdot\cos\alpha, \tag{5}$$

$$Y = R\cdot\sin\alpha\cdot\sin\beta, \tag{6}$$

$$X = R\cdot\sin\alpha\cdot\cos\beta. \tag{7}$$

Putting the value of $R = cT$ from Eq. (3) into above equations we get:

$$Z = cT\cdot\cos\alpha, \tag{8}$$

$$Y = cT\cdot\sin\alpha\cdot\sin\beta, \tag{9}$$

$$X = cT\cdot\sin\alpha\cdot\cos\beta. \tag{10}$$

Also, the value of radius vector OP that connects points O and P in Fig. 1 is given by equations:

$(OP)^2 = (OQ)^2 + (QP)^2$, or $(OP)^2 = (OS)^2 + (SQ)^2 + (QP)^2$,

or $R^2 = X^2 + Y^2 + Z^2$, or $R = \sqrt{X^2 + Y^2 + Z^2}$. (11)

Similarly, we can easily calculate the mathematical relationship between the Cartesian coordinates (X', Y', Z') of the point P and its spherical coordinates (R', α, β) in F' frame of reference using trigonometry, and they are given by the following equations:

$$Z' = R'\cdot\cos\alpha, \tag{12}$$

$$Y' = R'\cdot\sin\alpha\cdot\sin\beta, \tag{13}$$

$$X' = R'\cdot\sin\alpha\cdot\cos\beta. \tag{14}$$

Putting the value of $R' = cT'$ from Eq. (4) into above equations we get

$$Z' = cT'\cdot\cos\alpha, \tag{15}$$

$$Y' = cT'\cdot\sin\alpha\cdot\sin\beta, \tag{16}$$

$$X' = cT'\cdot\sin\alpha\cdot\cos\beta. \tag{17}$$

The value of radius vector $O'P$ that connects points O' and P in Fig. 1 is also given by equations:

$(O'P)^2 = (O'Q)^2 + (Q'P)^2$, or $(O'P)^2 = (O'S')^2 + (S'Q')^2 + (Q'P)^2$,

or $(R')^2 = (X')^2 + (Y')^2 + (Z')^2$, or $R' = \sqrt{(X')^2 + (Y')^2 + (Z')^2}$. (18)

Transformation equations in relativistic mechanics. The following equation can be clearly written if we refer to Fig. 1:

$$O'P = OP - OO', \text{ or } R' = R - vT.$$

This equation in relativistic mechanics with Lorentz factor γ can be written as follows:

$$R' = \gamma(R - vT). \tag{19}$$

Putting the T value from Eq. (3) we get:

$$R' = \gamma(R - v\,R/c), \text{ or } R' = \gamma R(1 - v/c). \tag{20}$$

This is the relativistic transformation equation from frame F to F' in terms of radius vectors R and R'.

Similarly, the following equation can be clearly written if we refer to Fig. 1:

$$OP = O'P + OO', \text{ or } R = R' + vT'.$$

This equation in relativistic mechanics with Lorentz factor γ can be written as follows:

$$R = \gamma(R' + vT'). \tag{21}$$

Putting the value of T' from Eq. (4) we get:

$$R = \gamma\left(R' + V\frac{R'}{c}\right), \text{ or } R = \gamma R'\left(1 + \frac{V}{c}\right). \tag{22}$$

This is the relativistic transformation equation from frame F' to F in terms of radius vectors R' and R. Now, putting the R' value from Eq. (20) we get:

$$R = \gamma^2 R\left(1 - \frac{V}{c}\right)\left(1 + \frac{V}{c}\right), \text{ or } 1 = \gamma^2\left(1 - \frac{V^2}{c^2}\right), \text{ or } \gamma^2 = \frac{1}{1 - \frac{V^2}{c^2}}, \text{ or } \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \tag{23}$$

which is the required value of the Lorentz factor.

Putting this value of γ in Eqs. (19) and (21), we get:

$$R' = \frac{R - VT}{\sqrt{1 - \frac{V^2}{c^2}}}, \tag{24}$$

$$R = \frac{R' + VT'}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{25}$$

Multiplying both sides of Eq. (24) by $\cos\alpha$ we get:

$$R' \cdot \cos\alpha = \frac{R \cdot \cos\alpha - VT\cos\alpha}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

Putting $R' \cdot \cos\alpha = Z'$ from Eq. (12) we get:

$$Z' = \frac{R \cdot \cos\alpha - VT \cdot \cos\alpha}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{26}$$

Putting also $R \cdot \cos\alpha = Z$ from Eq. (5) we get:

$$Z' = \frac{Z - VT \cdot \cos\alpha}{\sqrt{1 - \frac{V^2}{c^2}}}, \tag{27}$$

$$\text{or } Z' = \frac{Z - VT\left(Z/R\right)}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Putting also the value of R from Eq. (11),

$$Z' = \frac{Z - \frac{VTZ}{\sqrt{X^2+Y^2+Z^2}}}{\sqrt{1-\frac{V^2}{c^2}}}, \text{ or } Z' = \frac{Z\left(1-\frac{VT}{\sqrt{X^2+Y^2+Z^2}}\right)}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{28}$$

and again, multiplying both sides of Eq. (24) by sinα·sinβ, we get:

$$R'\cdot\sin\alpha\cdot\sin\beta = \frac{R\cdot\sin\alpha\cdot\sin\beta - VT\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Putting R'·sinα·sinβ = Y' from Eq. (13) we get:

$$Y' = \frac{R\cdot\sin\alpha\cdot\sin\beta - VT\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{29}$$

Putting also R·sinα·sinβ = Y from Eq. (6) we get:

$$Y' = \frac{Y - VT\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{30}$$

$$\text{or } Y' = \frac{Y - VT\left(Y/R\right)}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Putting also the value of R from Eq. (11)

$$Y' = \frac{Y - \frac{VTY}{\sqrt{X^2+Y^2+Z^2}}}{\sqrt{1-\frac{V^2}{c^2}}}, \text{ or } Y' = \frac{Y\left(1-\frac{VT}{\sqrt{X^2+Y^2+Z^2}}\right)}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{31}$$

again, multiplying both sides of Eq. (24) by sinα·cosβ, we get:

$$R'\cdot\sin\alpha\cdot\cos\beta = \frac{R\cdot\sin\alpha\cdot\cos\beta - VT\cdot\sin\alpha\cdot\cos\beta}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Putting R'·sinα·cosβ = X' from Eq. (14) we get:

$$X' = \frac{R\cdot\sin\alpha\cdot\cos\beta - VT\cdot\sin\alpha\cdot\cos\beta}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{32}$$

Putting also $R\cdot\sin\alpha\cdot\cos\beta = X$ from Eq. (7) we get:

$$X' = \frac{X - VT\cdot\sin\alpha\cdot\cos\beta}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{33}$$

$$\text{or } X' = \frac{X - VT\left(X/R\right)}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Putting also the value of R from Eq. (11) we get:

$$X' = \frac{X - \frac{VTX}{\sqrt{X^2+Y^2+Z^2}}}{\sqrt{1-\frac{V^2}{c^2}}}, \text{ or } X' = \frac{X\left(1 - \frac{VT}{\sqrt{X^2+Y^2+Z^2}}\right)}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{34}$$

Eqs. (28), (31) and (34) are transformation equations along X-, Y- and Z-directions from frame F to F' respectively.

Similarly, multiplying both sides of Eq. (25) by $\cos\alpha$ we get:

$$R\cdot\cos\alpha = \frac{R'\cdot\cos\alpha + VT'\cdot\cos\alpha}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Putting $R\cdot\cos\alpha = Z$ from Eq. (5) we get:

$$Z = \frac{R'\cdot\cos\alpha + VT'\cdot\cos\alpha}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{35}$$

Putting also $R'\cdot\cos\alpha = Z'$ from Eq. (12) we get:

$$Z = \frac{Z' + VT'\cdot\cos\alpha}{\sqrt{1-\frac{V^2}{c^2}}}, \text{ or } Z = \frac{Z' + VT'\left(Z'/R'\right)}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{36}$$

Putting also the value of R' from Eq. (18), we get:

$$Z = \frac{Z' + \frac{VT'Z'}{\sqrt{\left(X'\right)^2+\left(Y'\right)^2+\left(Z'\right)^2}}}{\sqrt{1-\frac{V^2}{c^2}}}, \text{ or } Z = \frac{Z'\left(1 + \frac{VT'}{\sqrt{\left(X'\right)^2+\left(Y'\right)^2+\left(Z'\right)^2}}\right)}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{37}$$

Again, multiplying both sides of Eq. (25) by $\sin\alpha\cdot\sin\beta$, we get:

$$R\cdot\sin\alpha\cdot\sin\beta = \frac{R'\cdot\sin\alpha\cdot\sin\beta + VT'\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Putting $R\cdot\sin\alpha\cdot\sin\beta = Y$ from Eq. (6) we get:

$$Y = \frac{R' \cdot \sin\alpha \cdot \sin\beta + VT' \cdot \sin\alpha \cdot \sin\beta}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{38}$$

Putting also $R' \cdot \sin\alpha \cdot \sin\beta = Y'$ from Eq. (13) we get:

$$Y = \frac{Y' + VT' \cdot \sin\alpha \cdot \sin\beta}{\sqrt{1 - \frac{V^2}{c^2}}}, \tag{39}$$

$$\text{or } Y = \frac{Y' + VT'\left(Y'/R'\right)}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

Putting also the value of R' from Eq. (18) we get:

$$Y = \frac{Y' + \frac{VT'Y'}{\sqrt{\left(X'\right)^2 + \left(Y'\right)^2 + \left(Z'\right)^2}}}{\sqrt{1 - \frac{V^2}{c^2}}}, \text{ or } Y = \frac{Y'\left(1 + \frac{VT'}{\sqrt{\left(X'\right)^2 + \left(Y'\right)^2 + \left(Z'\right)^2}}\right)}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{40}$$

Again, multiplying both sides of Eq. (25) by $\sin\alpha \cdot \cos\beta$ we get:

$$R \cdot \sin\alpha \cdot \cos\beta = \frac{R' \cdot \sin\alpha \cdot \cos\beta + VT' \cdot \sin\alpha \cdot \cos\beta}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

Putting $R \cdot \sin\alpha \cdot \cos\beta = X$ from Eq. (7) we get:

$$X = \frac{R' \cdot \sin\alpha \cdot \cos\beta + VT' \cdot \sin\alpha \cdot \cos\beta}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{41}$$

Putting also $R' \cdot \sin\alpha \cdot \cos\beta = X'$ from equation Eq. (14) we get:

$$X = \frac{X' + VT' \cdot \sin\alpha \cdot \cos\beta}{\sqrt{1 - \frac{V^2}{c^2}}}, \text{ or } X = \frac{X' + VT'\left(X'/R'\right)}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{42}$$

Putting also the value of R' from Eq. (18) we get:

$$X = \frac{X' + \frac{VT'X'}{\sqrt{\left(X'\right)^2 + \left(Y'\right)^2 + \left(Z'\right)^2}}}{\sqrt{1 - \frac{V^2}{c^2}}}, \text{ or } X = \frac{X'\left(1 + \frac{VT'}{\sqrt{\left(X'\right)^2 + \left(Y'\right)^2 + \left(Z'\right)^2}}\right)}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{43}$$

Eqs. (37), (40) and (43) are the transformation equations along X-, Y- and Z-directions from frame F' to F respectively.

Results and discussion

We have already derived the transformation equations along X-, Y- and Z-directions when the relative motion between inertial frames is in the three-dimensional space. In this section, our derived transformation equations are further analyzed to demonstrate the space contraction phenomena of a cuboid in the three-dimensional space.

For that, consider two frames F and F' that are superimposed at a time $T = T' = 0$. In the course of time, frame F' moves in the three-dimensional space at velocity v. Let a cuboid keep in the frame F' (Fig. 2).

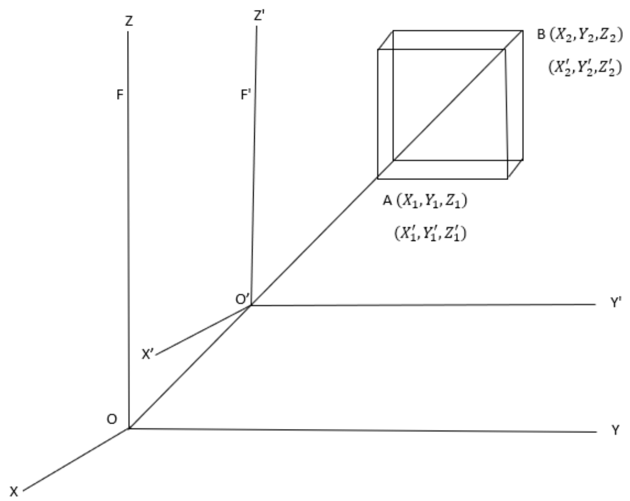

Fig. 2. Graphic presentation of space contraction of a cuboid in the three dimensions of space

Let us suppose that α is the angle between Z-axis and a line connecting the origins of the inertial frames of reference and A- and B- corners of the cuboid. The space coordinates of the opposite corners A and B as measured by observer O' of frame F' are (X_1', Y_1', Z_1') and (X_2', Y_2', Z_2') respectively, while the space coordinates of the opposite corners A and B as measured by observer O of frame F are (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) respectively. Therefore, the height of the cuboid along Z-axis as seen by observer O' in frame F' is

$$H_0 = Z_2' - Z_1'. \tag{44}$$

The height H_0 is called the proper height of cuboid. The apparent height of the cuboid from frame F at any instant of time T is

$$H = Z_2 - Z_1. \tag{45}$$

Now, using transformation Eq. (27) along Z-axis we get:

$$Z_2' = \frac{Z_2 - VT \cdot \cos\alpha}{\sqrt{1 - \frac{V^2}{c^2}}}, \tag{46}$$

$$Z_1' = \frac{Z_1 - VT \cdot \cos\alpha}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{47}$$

By putting Eqs. (46) and (47) in Eq. (44) we get:

$$H_0 = \frac{Z_2 - VT\cdot\cos\alpha}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{Z_1 - VT\cdot\cos\alpha}{\sqrt{1-\frac{V^2}{c^2}}}, \quad \text{or } H_0 = \frac{Z_2 - Z_1}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Substituting Eq. (45) in above equation we get:

$$H_0 = \frac{H}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Thus, the apparent height of the cuboid that is the height from frame F is

$$H = H_0\sqrt{1-\frac{V^2}{c^2}}. \tag{48}$$

Similarly, the breadth of the cuboid along Y-axis as seen by observer O' in frame F' is

$$B_0 = Y_2' - Y_1'. \tag{49}$$

The breadth B_0 is called the proper breadth of the cuboid. The apparent breadth of the cuboid from frame F at any instant of time T is

$$B = Y_2 - Y_1. \tag{50}$$

Now, using transformation Eq. (30) along Y-axis,

$$Y_2' = \frac{Y_2 - VT\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{51}$$

$$Y_1' = \frac{Y_1 - VT\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}}, \tag{52}$$

by putting Eqs. (51) and (52) in Eq. (49) we get:

$$B_0 = \frac{Y_2 - VT\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{Y_1 - VT\cdot\sin\alpha\cdot\sin\beta}{\sqrt{1-\frac{V^2}{c^2}}}, \quad \text{or } B_0 = \frac{Y_2 - Y_1}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Substituting Eq. (50) in the above equation we get:

$$B_0 = \frac{B}{\sqrt{1-\frac{V^2}{c^2}}}.$$

Thus, the apparent breadth of the cuboid that is the breadth from frame F is

$$B = B_0\sqrt{1-\frac{V^2}{c^2}}. \tag{53}$$

Similarly, the length of the cuboid along X-axis as seen by observer O' in frame F' is

$$L_0 = X_2' - X_1'. \tag{54}$$

The length L_0 is called the proper length of the cuboid. The apparent length of the cuboid from frame F at any instant of time T is

$$L = X_2 - X_1. \tag{55}$$

Now, using transformation Eq. (33) along X-axis,

$$X_2' = \frac{X_2 - VT \cdot \sin\alpha \cdot \cos\beta}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \tag{56}$$

$$X_1' = \frac{X_1 - VT \cdot \sin\alpha \cdot \cos\beta}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \tag{57}$$

by putting Eqs. (56) and (57) in Eq. (54) we get:

$$L_0 = \frac{X_2 - VT \cdot \sin\alpha \cdot \cos\beta}{\sqrt{1 - \dfrac{V^2}{c^2}}} - \frac{X_1 - VT \cdot \sin\alpha \cdot \cos\beta}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \text{ or } L_0 = \frac{X_2 - X_1}{\sqrt{1 - \dfrac{V^2}{c^2}}}.$$

Substituting Eq. (55) in the above equation, we get:

$$L_0 = \frac{L}{\sqrt{1 - \dfrac{V^2}{c^2}}}.$$

Thus, the apparent length of the cuboid that is the length from frame F is

$$L = L_0 \sqrt{1 - \frac{V^2}{c^2}}. \tag{58}$$

Eqs. (48), (53) and (58) give corresponding contraction of the cuboid along X-, Y- and Z-axes when relative motion between the inertial frames is in the three-dimensional space. In order to derive the relativistic form of the time coordinate, let rewrite Eq. (24) as follows:

$$R' = \frac{R - VT}{\sqrt{1 - \dfrac{V^2}{c^2}}}.$$

Putting the T value from Eq. (3) we get:

$$R' = \frac{R - V\dfrac{R}{c}}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \text{ or } R' = \frac{R - \dfrac{V\sqrt{X^2 + Y^2 + Z^2}}{c}}{\sqrt{1 - \dfrac{V^2}{c^2}}}.$$

Now putting the values of R and R' from Eqs. (3) and (4) we get:

$$CT' = \frac{CT - \dfrac{V\sqrt{X^2+Y^2+Z^2}}{c}}{\sqrt{1-\dfrac{V^2}{c^2}}}, \text{ or } T' = \frac{T - \dfrac{V\sqrt{X^2+Y^2+Z^2}}{c^2}}{\sqrt{1-\dfrac{V^2}{c^2}}}. \tag{59}$$

In order to get the inverse transformation equation of time, let interchange the coordinates and replace v by $-v$ in the above Eq. (59):

$$T = \frac{T' + \dfrac{V\sqrt{(X')^2+(Y')^2+(Z')^2}}{c^2}}{\sqrt{1-\dfrac{V^2}{c^2}}}. \tag{60}$$

Eqs. (59) and (60) are the required transformation equations of time in the three-dimensional space.

Table

Space coordinate transformation in the three-dimensional space

<table>
<tr><td rowspan="2">Motion between frames along axes</td><td colspan="3">Space coordinate transform equation</td></tr>
<tr><td>Along X-direction</td><td>Along Y- direction</td><td>Along Z- direction</td></tr>
<tr><td>X, Y, Z</td><td>$X' = \dfrac{\sin\alpha\cos\beta(R-VT)}{\sqrt{1-V^2/c^2}}$</td><td>$Y' = \dfrac{\sin\alpha\sin\beta(R-VT)}{\sqrt{1-V^2/c^2}}$</td><td>$Z' = \dfrac{\cos\alpha(R-VT)}{\sqrt{1-V^2/c^2}}$</td></tr>
<tr><td rowspan="3">X, Y only,
$\alpha = \dfrac{\pi}{2}$</td><td>$X' = \dfrac{\cos\beta(R-VT)}{\sqrt{1-V^2/c^2}}$</td><td>$Y' = \dfrac{\sin\beta(R-VT)}{\sqrt{1-V^2/c^2}}$</td><td>$Z' = \dfrac{\cos\dfrac{\pi}{2}(R-VT)}{\sqrt{1-V^2/c^2}} = 0$</td></tr>
<tr><td>$X' = \dfrac{\cos\beta\left(\sqrt{X^2+Y^2+Z^2}-VT\right)}{\sqrt{1-V^2/c^2}}$</td><td>$Y' = \dfrac{\sin\beta\left(\sqrt{X^2+Y^2+Z^2}-VT\right)}{\sqrt{1-V^2/c^2}}$</td><td rowspan="3">see Eq. (5):
$Z = \cos\alpha$
$Z = R\cos\dfrac{\pi}{2} = 0$</td></tr>
<tr><td>$X' = \dfrac{\cos\beta\left(\sqrt{X^2+Y^2}-VT\right)}{\sqrt{1-V^2/c^2}}$</td><td>$Y' = \dfrac{\sin\beta\left(\sqrt{X^2+Y^2}-VT\right)}{\sqrt{1-V^2/c^2}}$</td></tr>
<tr><td>X only,
$\alpha = \dfrac{\pi}{2}$
$\beta = 0$</td><td>$X' = \dfrac{R-VT}{\sqrt{1-V^2/c^2}}$
$X' = \dfrac{\sqrt{X^2+Y^2+Z^2}-VT}{\sqrt{1-V^2/c^2}}$
$X' = \dfrac{X-VT}{\sqrt{1-V^2/c^2}}$</td><td>$Y' = 0$, $Y = 0$ (see also Eq. (6))</td></tr>
</table>

The Table represents the Lorentz transformation equations along X-, Y- and Z-axes with different special cases of relative motion between inertial frames. The second row of the Table gives the Lorentz transformation equations along X- and Y-axes when the motion between inertial frames is in the two-dimensional XY-plane. Similarly, the third row of the Table represents the transformation equation in one dimensional motion along X-axis which is exactly the same transformation equation along X-axis as that obtained in the ordinary one-dimensional Lorentz transformation.

Conclusions

In summary, a method has been obtained for the formulation of the Lorentz transformation equations in the three-dimensional space by using the concept of both polar and Cartesian coordinates. The space-time coordinates transformation equations along X-, Y- and Z-directions were separately presented and proved as displayed in Eqs. (37), (40) and (43) respectively. Also, it was explicitly verified in Eqs. (48), (53) and (58) that the length, breadth and height of a cuboid got contracted if there was simultaneous relative motion between inertial frames along X-, Y- and Z-directions, being of the following form:

$$L = L_0\sqrt{1-\frac{V^2}{c^2}},\ B = B_0\sqrt{1-\frac{V^2}{c^2}},\ H = H_0\sqrt{1-\frac{V^2}{c^2}}.$$

From these equations, we can conclude that it is definitely possible to observe the simultaneous space contraction of the cuboid along X-, Y- and Z-directions. We hope that the Lorentz transformation equations discovered by such a different mathematical method in the three-dimensional space will enrich the scientific literature connected by relativistic mechanics, particularly those which are related to the space-time coordinates transformation between inertial frames in the three-dimensional space.

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THE AUTHOR

KHADKA Chandra Bahadur
Department of Physics, Tri-Chandra Multiple College, Tribhuvan University, Kathmandu, Nepal.
Kathmandu, 44600, Nepal
chandrabahadur9988@gmail.com

СВЕДЕНИЯ ОБ АВТОРЕ

ХАДКА Чандра Бахадур – *сотрудник кафедры физики Многопрофильного колледжа Три-Чандра Университета Трибхувана, г. Катманду, Непал.*
Катманду, 44600, Непал
chandrabahadur9988@gmail.com

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