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DERIVATION OF THE LORENTZ TRANSFORMATION FOR DETERMINATION OF SPACE CONTRACTION

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Abstract. The paper demonstrates the incompleteness of the relativistic space contraction formula for space-time coordinates transformation between inertial frames of reference in the existing normally acknowledged version of the theory of special relativity. The reason is the lack of a formula expressing the space contraction in terms of the cosines of the direction of an object's movement. Thus a modified version of the special relativity theory has been presented through a natural extension of the Lorentz transformation to three dimensions of space. The new transformations are based on the same mathematical operation as the Lorentz transformation, and provide mathematical information similar to that of a single-axis movement. An expression for the new transformation between inertial frames, when there was simultaneous relative motion in three directions, was introduced. The presence of relative motion in three directions between a parallelepiped and an observer (were introduced) made it possible to obtain the formulas of the directions of the ends of the parallelepiped. In fact, the Lorentz transformation was interpreted, revealing transformation of space-time coordinates in three dimensions. This offers ample scope for finding the space contraction within moving frame of reference.

Keywords: frame of reference, Lorentz transformation, special relativity, space contraction

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ВЫВОД ПРЕОБРАЗОВАНИЯ ЛОРЕНЦА ДЛЯ ОПРЕДЕЛЕНИЯ СЖАТИЯ ПРОСТРАНСТВА

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Аннотация. В статье показано, что формула релятивистского сжатия пространства для преобразования пространственно-временных координат между инерциальными системами отсчета в существующей общепризнанной версии специальной теории относительности является неполной. Причина состоит в отсутствии формулы, выражающей сжатие пространства через косинусы направления движения объекта. Поэтому представлена модифицированная версия специальной теории относительности путем естественного расширения преобразования Лоренца до трех измерений пространства.

Новые преобразования основаны на той же математической операции, что и преобразование Лоренца, и обеспечивают математическую информацию, аналогичную таковой для движения по одной оси. Вводится выражение для нового преобразования между инерциальными системами отсчета, когда одновременно происходит относительное движение по трем направлениям. Наличие относительного движения по трем направлениям между параллелепипедом и наблюдателем (введены) позволяет получить формулы для косинусов направлений концов параллелепипеда. Фактически интерпретировано преобразование Лоренца, позволяющее преобразовывать пространственно-временные координаты в трех измерениях, а это дает возможность определять сжатие пространства в движущейся системе отсчета.

Ключевые слова: система отсчета, преобразование Лоренца, специальная теория относительности, сжатие пространства

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Introduction

In present time, the special theory of relativity [1, 2] has produced the most profound effect on the entire field of physics to describe relativistic nature of mass, space and time. But its creators Hendrik Antoon Lorentz and Albert Einstein (both of them are Nobel Prize winners in physics in 1902 and 1921, respectively) [3] could not provide accurate information about space contraction, although there was all necessary information at that time. Before to enter into shortcomings of existing version of special theory of relativity, it would be useful to perceive the logic of its formulation and the process by which it was formulated. The Lorentz transformation [4, 5] plays a crucial role in the formulation of the special theory of relativity.

In order to get Lorentz transformation equations, we consider two coordinate systems: S and S'. The S' system must move in the positive X-direction at a uniform velocity $\mathbf{V}$ (Fig. 1). Then the Lorentz transformation equations will be generally formulated as a one-dimensional system whose motion is assumed to be aligned with X-direction [6]. There is a relative motion only in X-axis and not in Y- and Z-directions. The space and time coordinates of S and S' are interrelated by the Lorentz transformation equations, as follows:

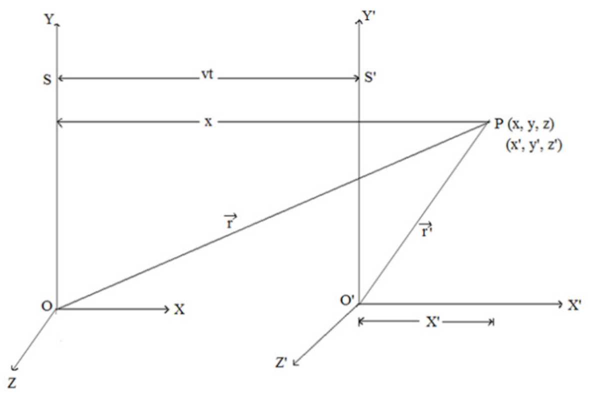

Fig. 1. An event at a point P observed from frames S and S'

$$X' = \frac{X - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad t' = \frac{t - \frac{VX}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad Y' = Y,\ Z' = Z.$$

Here, the complete mathematical calculations related with the relativistic phenomenon of an object [7, 8] are performed only for X-axis. Moreover, the length contraction [9, 10] or the object length will appear to be shortened only along X-axis in the S' system for the observer. Hence, the moving frame of reference is constrained to one-dimensional motion [11, 12]. That is why it is necessary to set the motion of travelling frame S' in such a way that there would be relative motion in X-, Y- and Z-axes simultaneously. For new transformation, instead of the above equation, we have simply the following one:

$$x' = \frac{X - \frac{VXt}{\sqrt{X^2 + Y^2 + Z^2}}}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad t' = \frac{t - V\frac{\sqrt{X^2 + Y^2 + Z^2}}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}},$$

$$y' = \frac{y - \frac{Vyt}{\sqrt{X^2 + Y^2 + Z^2}}}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad z' = \frac{z - \frac{Vzt}{\sqrt{X^2 + Y^2 + Z^2}}}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

The behavior of space contraction along each direction is well accounted by this formula. It agrees with the ordinary transformation as the relative motion between inertial frames constrained along single dimension. It manifests the inadequacy of ordinary transformation, namely, the need for expression of space contraction in three dimensions of space to relate with direction cosines.

In this paper, the velocity of the moving frame of reference has different components along each direction that are thoroughly put on mathematical calculation in an effort to reach a rational conclusion regarding space contraction [13, 14]. Pure mathematical methods are used as main tools in achieving the objective of this study. In section "Methods used", we show how the transformation equations between inertial frames can be formulated for extension of the Lorentz transformation to higher dimensions. Following this, we develop the new formulas for space-time coordinate transformation along X-, Y- and Z-directions in three dimensions of space. In subsequent section "Results and discussion", we give a detailed description of our results including association of space contraction with direction cosines of position vectors. In section "Conclusions" the concluding remarks are presented.

Methods used

The light wavefront equation. Let two frames S and S', such as S' frame of Refs. [15, 16], move with a uniform velocity (see Fig. 2). Let origin O and O' of two coordinate systems coincide at $t = t' = 0$ and a source of light flashes at the origin O at $t = 0$, when O and O' coincide. Then, in view of constancy of electromagnetic wave speed c with respect to the motion of two frames of reference, each observer at O and O' claims to be at the center of the spreading spherical wavefront of light pulses [17, 18]. When the light is at the point P, let the space-time coordinate be measured by the observers O and O' (X, Y, Z, t) and (X', Y', Z', t') respectively. Since both the observers are at the center of the same expanding wavefront, the wavefront equation of the frames S and S' should take the form [19, 20]:

$$R^2 - c^2t^2 = (R')^2 - c^2\,(t')^2; \tag{1}$$

$$X^2 + Y^2 + Z^2 - c^2t^2 = (X')^2 + (Y')^2 + (Z')^2 - c^2\,(t')^2.$$

Transformation equation in three dimensions. In three dimensions, there is a relative motion along X-, Y- and Z-directions simultaneously. Let R and R' be a position of point P measured from frames of references S and S' respectively. Let ψ and ψ' be angles made by the radius-vectors R and R' with Z- and Z'-axes in the S and S' frames of reference, respectively. Since **V** and c are the velocity of the moving frame and the light wavefront along radius-vectors R and R', just as $V\cdot\cos\psi$ and $V\cdot\cos\psi'$ are the Z- and Z'-components of velocity, $V\cdot\sin\psi$ and $V\cdot\sin\psi'$ are the velocity components along $r = \sqrt{X^2 + Y^2}$ and $r' = \sqrt{(X')^2 + (Y')^2}$ in XY- plane.

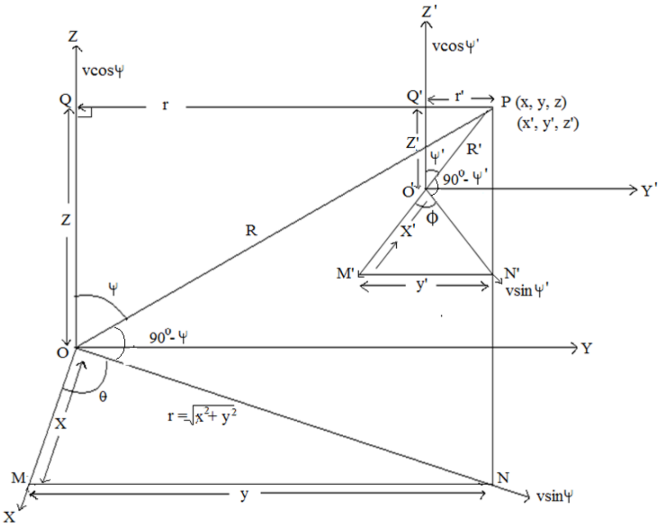

Fig. 2. The inertial systems S and S' in the three-dimensional space

Let θ and ϕ be the angles of radius-vectors **r** and **r**$'$ with X- and X'-axes respectively. Thus, $V\cdot\sin\psi\cdot\cos\theta$ and $V\cdot\sin\psi'\cdot\cos\phi$ are the X- and X'-components of velocity **V**. In a similar manner, $V\cdot\sin\psi\cdot\sin\theta$ and $V\cdot\sin\psi'\cdot\sin\phi$ are the Y- and Y'- components (respectively) of the velocity **V**.

Equations for the transformation of the frame S into frame S' are as follows:

$$Y' = Y - V_y t,\ X' = X - V_x t,\ Z' = Z - V_z t,$$

where $V_z = V\cdot\cos\psi$ and $V\cdot\sin\psi$ are the velocity components in Z-axis and on the XY-plane, respectively.

Therefore, the velocity components in X- and Y-axes must be $V\cdot\cos\theta\cdot\sin\psi$ and $V\cdot\sin\theta\cdot\sin\psi$, respectively.

In this case, the above transformation equations take the form (see Fig. 2)

$$Z' = Z - Vt\cdot\cos\psi,\ Y' = Y - Vt\cdot\sin\theta\cdot\sin\psi,\ X' = X - Vt\cdot\cos\theta\cdot\cos\psi;$$

now in the triangle OPQ,

$$\cos\psi = \frac{OQ}{OP} = \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}} = \frac{Z}{R},\ \sin\psi = \frac{PQ}{OP} = \frac{\sqrt{X^2 + Y^2}}{\sqrt{X^2 + Y^2 + Z^2}} = \frac{r}{R};$$

in the triangle OMN,

$$\cos\theta = \frac{OM}{ON} = \frac{X}{\sqrt{X^2 + Y^2}} = \frac{X}{r},\ \sin\theta = \frac{MN}{ON} = \frac{Y}{\sqrt{X^2 + Y^2}} = \frac{Y}{r}.$$

Substituting their values into the above transformation equations (of frame S into S'), we get

$$Z' = Z - \frac{VtZ}{R}, \tag{2}$$

$$Y' = Y - \frac{VtrY}{Rr} = Y - \frac{VtY}{R}, \tag{3}$$

$$X' = X - \frac{VtrX}{Rr} = X - \frac{VtX}{R}. \tag{4}$$

Again, transformation equations of S' into S are expressed as

$$Z = Z' + V_z' t',\ Y = Y' + V_y' t',\ X = X' + V_x' t',$$

where $V_z' = V \cdot \cos\psi'$ and $V \cdot \sin\psi'$ are the velocity components in Z'-axis and $X'Y'$-plane, respectively.

Therefore, the velocity components along X'- and Y'-axes should be $V \cdot \cos\phi \cdot \sin\psi'$ and $V \cdot \sin\phi \cdot \sin\psi'$ respectively, in which case the above transformation equations take the form

$$Z = Z' + Vt' \cdot \cos\psi',\ Y = Y' + Vt' \cdot \sin\phi \cdot \sin\psi',\ X = X' + Vt' \cdot \cos\phi \cdot \sin\psi';$$

now in the triangle $O'P'Q'$,

$$\cos\psi' = \frac{O'Q'}{O'P} = \frac{Z'}{R'} = \frac{Z'}{\sqrt{(X')^2 + (Y')^2 + (Z')^2}},\ \sin\psi' = \frac{PQ'}{O'P} = \frac{r'}{R'} = \frac{\sqrt{(X')^2 + (Y')^2}}{\sqrt{(X')^2 + (Y')^2 + (Z')^2}};$$

in the triangle $O'M'N'$,

$$\cos\phi = \frac{O'M'}{O'N'} = \frac{X'}{\sqrt{(X')^2 + (Y')^2}} = \frac{X'}{r'},\ \sin\phi = \frac{M'N'}{O'N'} = \frac{y'}{\sqrt{(X')^2 + (Y')^2}} = \frac{Y'}{r'}.$$

Substituting these expressions in the above transformation equations of frame S into S' we get

$$Z = Z' + \frac{Vt'Z'}{R'}, \tag{5}$$

$$Y = Y' + \frac{Vr'}{R'} \cdot \frac{V't'}{r'} = Y' + \frac{VY't'}{R'}, \tag{6}$$

$$X = X' + \frac{Vr'}{R'} \cdot \frac{X't'}{r'} = X' + \frac{VX't'}{R'}. \tag{7}$$

Now radius-vector R' in S' frame of reference when the relative motion along X-, Y- and Z-axes takes place, is

$$(R')^2 = (X')^2 + (Y')^2 + (Z')^2.$$

Substituting expressions from transformation Eqs. (2), (3) and (4) here, we get

$$(R')^2 = \left(X - \frac{VtX}{R}\right)^2 + \left(Y - \frac{VtY}{R}\right)^2 + \left(Z - \frac{VtZ}{R}\right)^2,$$

$$\text{or } (R')^2 = X^2 - \frac{2VtX^2}{R} + \frac{V^2X^2t^2}{R^2} + Y^2 - \frac{2VtY^2}{R} + \frac{V^2Y^2t^2}{R^2} + Z^2 - \frac{2VtZ^2}{R} + \frac{V^2Z^2t^2}{R^2},$$

$$\text{or } (R')^2 = X^2 + Y^2 + Z^2 - 2tV\left(\frac{X^2 + Y^2 + Z^2}{R}\right) + t^2V^2\left(\frac{X^2 + Y^2 + Z^2}{R^2}\right).$$

According to Fig. 2, radius-vector R in S frame of reference is given by

$$R^2 = X^2 + Y^2 + Z^2,$$

then $(R')^2 = R^2 - \frac{2VtR^2}{R} + \frac{t^2V^2R^2}{R^2}$, or $(R')^2 = R^2 - 2tVR + t^2V^2$, or $(R')^2 = (R - tV)^2$,

$$R' = R - tV. \tag{8}$$

This is a transformation equation of frame S' into frame S in term of the radius-vector when there is a relative motion along X-, Y- and Z-axes simultaneously.

Again, the radius-vector in S frame of reference is

$$R^2 = X^2 + Y^2 + Z^2.$$

Substituting the expressions from transformation Eqs. (2), (3) and (4), we get

$$R^2 = \left(X' + \frac{VX't'}{R'}\right)^2 + \left(Y' + \frac{VY't'}{R'}\right)^2 + \left(Z' + \frac{VZ't'}{R'}\right)^2,$$

$$\text{or } R^2 = X^2 + \frac{2Vt'(X')^2}{R'} + \frac{V^2(X')^2 t'^2}{R'^2} + (Y')^2 +$$

$$+ \frac{2Vt'(Y')^2}{R'} + \frac{V^2(Y')^2(t')^2}{R'} + (Z')^2 + \frac{2Vt'(Z')^2}{R'} + \frac{V^2(Z')^2(t')^2}{R},$$

$$\text{or } R^2 = (X')^2 + (Y')^2 + (Z')^2 + 2Vt'\left[\frac{(X')^2 + (Y')^2 + (Z')^2}{R'}\right] + V^2(t')^2\left[\frac{(X')^2 + (Y')^2 + (Z')^2}{(R')^2}\right].$$

According to Fig. 2, radius-vector R' in S' frame of reference is given by

$$(R')^2 = (X')^2 + (Y')^2 + (Z')^2,$$

then $R^2 = R' + \dfrac{2Vt'(R')^2}{R} + \dfrac{V^2(t')^2(R')^2}{(R')^2}$, or $R^2 = (R')^2 + 2Vt'R' + V^2t'^2$, or $R^2 = (R' + Vt')^2$,

$$R = R' - Vt'. \tag{9}$$

This is a transformation equation of frame S' into S in terms of the radius-vector when there is the relative motion along X- and Y-axes simultaneously.

The Lorentz transformation equation. The transformation equations related to R' and R can be written from Eqs. (8) and (9) as follows:

$$R' = \lambda(R - Vt),\ R = \lambda'(R' + Vt'),$$

where λ and λ' are independent of R, R' and t.

Furthermore,

$$R = \lambda'(R' + Vt'),\ R = \lambda'[\lambda(R - Vt) + Vt'],$$

$$\text{or } R = \lambda'\lambda R - \lambda'\lambda Vt + \lambda'Vt', \text{ or } t' = \frac{R}{\lambda'V} - \frac{\lambda'\lambda R}{\lambda'V} + \frac{\lambda'\lambda Vt}{\lambda'V}, \tag{10}$$

$$\text{or } t' = \lambda t + \frac{R}{\lambda'V} - \frac{\lambda R}{V}, \text{ or } t' = \lambda\left[t - \frac{R}{V}\left(1 - \frac{1}{\lambda\lambda'}\right)\right].$$

Eq. (1) gives the wavefront equation as

$$X^2 + Y^2 + Z^2 - c^2t^2 = (X')^2 + (Y')^2 + (Z')^2 - c^2(t')^2,$$

$$\text{or } R^2 - c^2t^2 = (R')^2 - (c')^2(t')^2.$$

Substituting R' and t' expressions from Eqs. (9) and (10), we get

$$R^2 - c^2t^2 = \lambda^2(R - Vt)^2 - c^2\lambda^2\left[t - \frac{R}{V}\left(1 - \frac{1}{\lambda\lambda'}\right)\right]^2,$$

or $R^2 - c^2t^2 = \lambda^2 R^2 - 2\lambda^2 RVt + \lambda^2 V^2 t^2 - c^2\lambda^2 t^2 + 2c^2\lambda^2 t \dfrac{R}{V}\left(1-\dfrac{1}{\lambda\lambda'}\right) - c^2\lambda^2 \dfrac{R^2}{V^2}\left(1-\dfrac{1}{\lambda\lambda'}\right)^2,$

or $R^2 - c^2t^2 = R^2\left[\lambda^2 - \dfrac{c^2\lambda^2}{V^2}\left(1-\dfrac{1}{\lambda\lambda'}\right)^2\right] + Rt\left[-2\lambda^2 V + \dfrac{2c^2\lambda^2}{V}\left(1-\dfrac{1}{\lambda\lambda'}\right)\right] + t^2\left(\lambda^2V^2 - c^2\lambda^2\right).$

Equating the coefficients of R^2, Rt and t^2 on both sides of the equations, we get

$$\lambda^2 - \frac{c^2\lambda^2}{V^2}\left(1-\frac{1}{\lambda\lambda'}\right)^2 = 1, \tag{11}$$

$$2\lambda^2 V + \frac{2c^2\lambda^2}{V}\left(1-\frac{1}{\lambda\lambda'}\right) = 0, \tag{12}$$

$$\lambda^2V^2 - \lambda^2c^2 = -c^2. \tag{13}$$

From Eq. (13) it follows that

$$-\lambda^2(c^2 - V^2) = -c^2,\ \lambda^2 = \frac{c^2}{c^2 - V^2} = \frac{1}{1-\dfrac{V^2}{c^2}},$$

$$\lambda = \frac{1}{\sqrt{1-\dfrac{V^2}{c^2}}}. \tag{14}$$

Furthermore,

$$\lambda^2 = \frac{1}{1-\dfrac{V^2}{c^2}},\ \text{or}\ 1-\frac{V^2}{c^2} = \frac{1}{\lambda^2},$$

$$\text{or}\ \frac{V^2}{c^2} = 1-\frac{1}{\lambda^2}. \tag{15}$$

From Eq. (12), we have an equation

$$-V + \frac{c^2}{V^2}\left(1-\frac{1}{\lambda\lambda'}\right) = 0,\ \text{or}\ \frac{-V^2 + c^2\left(1-\dfrac{1}{\lambda\lambda'}\right)}{V} = 0,\ \text{or}\ V^2 = c^2\left(1-\frac{1}{\lambda\lambda'}\right),$$

$$\text{or}\ \frac{V^2}{c^2} = 1-\frac{1}{\lambda\lambda'}. \tag{16}$$

Comparing Eqs. (15) and (16) we get

$$1-\frac{1}{\lambda^2} = 1-\frac{1}{\lambda\lambda'},\ \text{or}\ \frac{1}{\lambda} = \frac{1}{\lambda'},$$

$$\text{or}\ \lambda = \lambda' = \frac{1}{\sqrt{1-\dfrac{V^2}{c^2}}}.$$

Therefore, the required transformation equations are, as follows:

$$R'=\frac{R-Vt}{\sqrt{1-\frac{V^2}{c^2}}},\quad \text{or}\ \sqrt{\left(X'\right)^2+\left(Y'\right)^2+\left(Z'\right)^2}=\frac{\sqrt{\left(X'\right)^2+\left(Y'\right)^2+\left(Z'\right)^2}-Vt}{\sqrt{1-\frac{V^2}{c^2}}};$$

$$t'=\lambda\left[t-\frac{R}{V}\left(1-\frac{1}{\lambda\lambda'}\right)\right],\ t'=\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[t-\frac{R}{V}\cdot\frac{V^2}{c^2}\right],$$

$$t'=\frac{t-\frac{RV}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}},\ t'=\frac{t-V\frac{\sqrt{X^2+Y^2+Z^2}}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}.$$

The inverse Lorentz transformation equations when the relative motion takes place along X- and Y-directions simultaneously, are

$$R=\frac{R'+Vt'}{\sqrt{1-\frac{V^2}{c^2}}},$$

$$\text{or}\ \sqrt{X^2+Y^2+Z^2}=\frac{\sqrt{\left(X'\right)^2+\left(Y'\right)^2+\left(Z'\right)^2}+Vt'}{\sqrt{1-\frac{V^2}{c^2}}};\tag{17}$$

$$t=\frac{t'+V\frac{\sqrt{\left(X'\right)^2+\left(Y'\right)^2+\left(Z'\right)^2}}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}.$$

These equations convert a measurement made in frame S' into those in frame S. Thus, it is quite clear from these equations, that the measurements of space and time in the three-dimensional space are included X-, Y- and Z-axes in one equation.

Results and discussion

The transformation equations relating to radius-vectors R and R' when the relative motion along X-, Y- and Z-axes takes place, can be written using Eqs. (8) and (9), as follows:

$$R'=\lambda\left(R-tV\right)=\frac{R-tV}{\sqrt{1-\frac{V^2}{c^2}}},\ R=\lambda'\left(R'+t'V\right)=\frac{R'+t'V}{\sqrt{1-\frac{V^2}{c^2}}},\ R=\lambda'\left(R'+Vt'\right),$$

or $R^2 = (\lambda')^2(R' + Vt')^2$, or $R^2 = (\lambda')^2\, [(R')^2 + 2Vt'\, R' + V^2(t')^2]$,

$$\text{or}\ R^2=\left(\lambda'\right)^2\left[\left(R'\right)^2+\frac{2Vt'\left(R'\right)^2}{R'}+\frac{V^2t'^2\left(R'\right)^2}{\left(R'\right)^2}\right].$$

Now, substituting the expression

$$(R')^2 = (X')^2 + (Y')^2 + (Z')^2$$

we get, as follows:

$$R^2=(\lambda')^2\left\{\left[(X')^2+(Y')^2+(Z')^2+2Vt'\right]\left[\frac{(X')^2+(Y')^2+(Z')^2}{R'}\right]+\right.$$
$$\left.+V^2(t')^2\left[\frac{(X')^2+(Y')^2+(Z')^2}{(R')^2}\right]\right\},$$

or

$$R^2=(\lambda')^2\left[(X')^2+\frac{2Vt'(X')^2}{R'}+\frac{V^2(X')^2t'^2}{(R')^2}+(Y')^2+\frac{2Vt'(Y')^2}{R'}+\frac{V^2(Y')^2(t')^2}{(R')^2}+\right.$$
$$\left.+(Z')^2+\frac{2Vt'(Z')^2}{R'}+\frac{V^2(Z')^2(t')^2}{(R')^2}\right],$$

or $R^2=(\lambda')^2\left[\left(X'+\frac{VX't'}{R'}\right)^2+\left(Y'+\frac{VY't'}{R'}\right)^2+\left(Z'+\frac{VZ't'}{R'}\right)^2\right],$

or $R^2=(\lambda')^2\left(X'+\frac{VX't'}{R'}\right)^2+(\lambda')^2\left(Y'+\frac{VY't'}{R'}\right)^2+(\lambda')^2\left(Z'+\frac{VZ't'}{R'}\right)^2,$

$$R^2=X^2+Y^2+Z^2,$$

where $X^2=(\lambda')^2\left(X'+\frac{VX't'}{R'}\right)^2,\ X=\lambda'\left(X'+\frac{VX't'}{R'}\right),\ X=\frac{X'+\frac{VX't'}{R'}}{\sqrt{1-\frac{V^2}{c^2}}},$ (18)

and $Y^2=(\lambda')^2\left(Y'+\frac{VY't'}{R'}\right)^2,\ Y=\lambda'\left(Y'+\frac{VY't'}{R'}\right),\ Y=\frac{Y'+\frac{VY't'}{R'}}{\sqrt{1-\frac{V^2}{c^2}}},$ (19)

and $Z^2=(\lambda')^2\left(Z'+\frac{VZ't'}{R'}\right)^2,\ Z=\lambda'\left(Z'+\frac{VZ't'}{R'}\right),\ Z=\frac{Z'+\frac{VZ't'}{R'}}{\sqrt{1-\frac{V^2}{c^2}}}.$ (20)

Consider two frames of reference S and S' coinciding at $t=0$. Let the S' frame of reference move at a constant velocity $\mathbf{V}$ in space (see Fig. 3). Let (X_1, Y_1, Z_1) and (X_1', Y_1', Z_1') be positions of point P in a parallelepiped measured by S and S' frames, respectively, as well as (X_2, Y_2, Z_2) and (X_2', Y_2', Z_2') be positions of point Q in the given parallelepiped measured by S and S' frames.

The length of the parallelepiped is

$$L_0=X_2-X_1. \tag{21}$$

This length L_0 in the stationary frame of reference is known as the proper length. The inverse Lorentz transformation of X-coordinates of the parallelepiped is given by Eq. (18), in which case it is

$$X_1=\frac{X_1'+\frac{VX_1't'}{R_1'}}{\sqrt{1-\frac{V^2}{c^2}}},\quad X_2=\frac{X_2'+\frac{VX_2't'}{R_2'}}{\sqrt{1-\frac{V^2}{c^2}}},$$

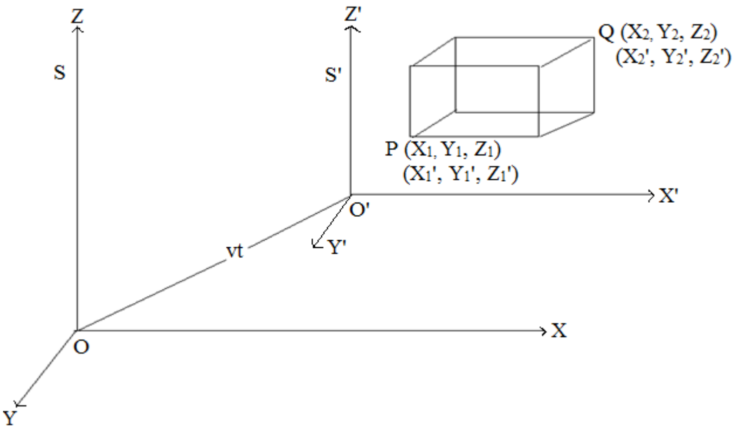

Fig. 3. The illustration explaining the formulation of the problem of space contraction

where X_1' and X_2' are the coordinates of the parallelepiped length in frame S' moving at constant velocity $\mathbf{V}$, as noted, simultaneously, at the same moment t'.

Substituting the expressions into Eq. (21) we get

$$L_0 = \frac{X_2' + \dfrac{VX_2't'}{R_2'}}{\sqrt{1-\dfrac{V^2}{c^2}}} - \frac{X_1' + \dfrac{VX_1't'}{R_1'}}{\sqrt{1-\dfrac{V^2}{c^2}}}, \quad L_0 = \frac{X_2' - X_1' + Vt'\left(\dfrac{X_2'}{R_2'} - \dfrac{X_1'}{R_1'}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}},$$

$$\text{or } L_0 = \sqrt{1-\frac{V^2}{c^2}} = X_2' - X_1' + Vt'\left(\frac{X_2'}{R_2'} - \frac{X_1'}{R_1'}\right),$$

$$\text{or } X_2' - X_1' = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(\frac{X_2'}{R_2'} - \frac{X_1'}{R_1'}\right).$$

According to definition of direction cosines,

$$\frac{X_2'}{R_2'} = \frac{X_2'}{\sqrt{(X_2')^2 + (Y_2')^2 + (Z_2')^2}}$$

is the expression for the cosine of the angle between the line joining the point O' to point Q, and X'-axis.

Hence,

$$\frac{X_2'}{\sqrt{(X_2')^2 + (Y_2')^2 + (Z_2')^2}} = l_2'$$

is the direction cosine of the line joining point O' to Q.

Similarly,

$$\frac{X_1'}{\sqrt{(X_1')^2 + (Y_1')^2 + (Z_1')^2}} = l_1'$$

is the direction cosine of the line joining point O' to P.

Now, the above equation takes the form

$$X_2' - X_1' = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'(l_2' - l_1'), \quad L = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'(l_2' - l_1'), \tag{22}$$

where L is the parallelepiped length (22) for an observer in frame S' (called improper length).

Similarly, the breadth of the parallelepiped is

$$B_0 = Y_2 - Y_1. \tag{23}$$

This breadth B_0 of the parallelepiped in the stationary frame of reference is known as the proper breadth. The inverse Lorentz transformation of Y-coordinates of the parallelepiped is given by Eq. (23) as follows:

$$Y_1 = \frac{Y_1' + \frac{VY_1't'}{R_1'}}{\sqrt{1-\frac{V^2}{c^2}}},\quad Y_2 = \frac{Y_2' + \frac{VY_2't'}{R_2'}}{\sqrt{1-\frac{V^2}{c^2}}},$$

where Y_1' and Y_2' are the coordinates of the parallelepiped breadth in frame S' moving at a constant velocity $\mathbf{V}$ (as noted) simultaneously at the same moment t'.

Substituting the expressions into Eq. (23) we get

$$B_0 = \frac{Y_2' + \frac{VY_2't'}{R_2'}}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{Y_1' + \frac{VY_1't'}{R_1'}}{\sqrt{1-\frac{V^2}{c^2}}},\quad B_0 = \frac{Y_2' - Y_1' + Vt'\left(\frac{Y_2'}{R_2'} - \frac{Y_1'}{R_1'}\right)}{\sqrt{1-\frac{V^2}{c^2}}},$$

$$\text{or } B_0\sqrt{1-\frac{V^2}{c^2}} = Y_2' - Y_1' + Vt'\left(\frac{Y_2'}{R_2'} - \frac{Y_1'}{R_1'}\right),\ \text{or } Y_2' - Y_1' = B_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(\frac{Y_2'}{R_2'} - \frac{Y_1'}{R_1'}\right).$$

According to definition of direction cosines,

$$\frac{Y_2'}{R_2'} = \frac{Y_2'}{\sqrt{(X_2')^2 + (Y_2')^2 + (Z_2')^2}}$$

is the expression for the cosine of the angle between the line joining point O' to point Q, and Y'-axis.

Hence,

$$\frac{Y_2'}{\sqrt{(X_2')^2 + (Y_2')^2 + (Z_2')^2}} = m_2'$$

is the direction cosine of the line joining point O' to Q.

Similarly,

$$\frac{Y_1'}{\sqrt{(X_1')^2 + (Y_1')^2 + (Z_1')^2}} = m_1'$$

is the direction cosine of the line joining point O' to point P.

Hence, the above equation takes the form, as follows:

$$Y_2' - Y_1' = B_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(m_2' - m_1'\right),$$

$$B = B_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(m_2' - m_1'\right), \tag{24}$$

where B is the parallelepiped breadth for an observer in frame S' (called the improper breadth).

Similarly, the height of the parallelepiped is

$$H_0 = Z_2 - Z_1. \tag{25}$$

This height H_0 of the parallelepiped in the stationary frame of reference is known as the proper height.

The inverse Lorentz transformation of Z-coordinates of the parallelepiped is given by Eq. (22):

$$Z_1 = \frac{Z_1' + \frac{VZ_1't'}{R_1'}}{\sqrt{1-\frac{V^2}{c^2}}},\ Z_2 = \frac{Z_2' + \frac{VZ_2't'}{R_2'}}{\sqrt{1-\frac{V^2}{c^2}}},$$

where Z_1' and Z_2' are the coordinates of the parallelepiped height in frame S' moving at constant velocity $\mathbf{V}$ as noted simultaneously, at the same moment t'.

Substituting the expressions into Eq. (25) we get

$$H_0 = \frac{Z_2' + \frac{VZ_2't'}{R_2'}}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{Z_1' + \frac{VZ_1't'}{R_1'}}{\sqrt{1-\frac{V^2}{c^2}}},\ H_0 = \frac{Z_2' - Z_1' + Vt'\left(\frac{Z_2'}{R_2'} - \frac{Z_1'}{R_1'}\right)}{\sqrt{1-\frac{V^2}{c^2}}},$$

$$\text{or } H_0\sqrt{1-\frac{V^2}{c^2}} = Z_2' - Z_1' + Vt'\left(\frac{Z_2'}{R_2'} - \frac{Z_1'}{R_1'}\right),\ \text{or } Z_2' - Z_1' = H_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(\frac{Z_2'}{R_2'} - \frac{Z_1'}{R_1'}\right).$$

According to definition of direction cosines,

$$\frac{Z_2'}{R_2'} = \frac{Z_2'}{\sqrt{(X_2')^2 + (Y_2')^2 + (Z_2')^2}}$$

is the expression for the cosine of the angle between the line joining point O' to point Q, and Z'-axis.

Hence,

$$\frac{Z_2'}{\sqrt{(X_2')^2 + (Y_2')^2 + (Z_2')^2}} = n_2'$$

is the direction cosine of the line joining point O' to Q.

Similarly,

$$\frac{Z_1'}{\sqrt{(X_1')^2 + (Y_1')^2 + (Z_1')^2}} = n_1'$$

is the direction cosine of the line joining point O' to P.

Now, the above equation takes the form, as follows:

$$Z_2' - Z_1' = H_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(n_2' - n_1'\right),$$

$$H = H_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(n_2' - n_1'\right), \tag{26}$$

where H is the parallelepiped height for an observer in frame S' (called the improper height).

From Eqs. (22), (24) and (26), one can see that the length, breadth and height contractions of the parallelepiped depend upon the direction cosines of the angles formed between the straight line and the coordinate axes.

Eq. (26) vanishes when there is no relative motion along Z-axis, as follows:

$$n_2' - n_1' = 0, \text{ or } \frac{Z_2'}{\sqrt{(X_2')^2+(Y_2')^2+(Z_2')^2}} = \frac{Z_1'}{\sqrt{(X_1')^2+(Y_1')^2+(Z_1')^2}} = 0, \text{ or } z_2' = z_1' = 0.$$

Thus,

$$H_0 = Z_2' - Z_1' = 0$$

in the absence of relative motion along Z-axis.

Similarly,

$$B_0 = Y_2' - Y_1' = 0$$

in the absence of relative motion along Y-axis.

Table

Main findings of space contraction

Motion between inertial frames along axes	Equation of space contraction		
	Along X-axis (length contraction)	Along Y-axis (breadth contraction)	Along Z-axis (height contraction)
X, Y, Z	$L = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left[\frac{X_2'}{\sqrt{(X_2')^2+(Y_2')^2+(Z_2')^2}} - \frac{X_1'}{\sqrt{(X_1')^2+(Y_1')^2+(Z_1')^2}}\right]$	$B = B_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left[\frac{Y_2'}{\sqrt{(X_2')^2+(Y_2')^2+(Z_2')^2}} - \frac{Y_1'}{\sqrt{(X_1')^2+(Y_1')^2+(Z_1')^2}}\right]$	$H = H_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left[\frac{Z_2'}{\sqrt{(X_2')^2+(Y_2')^2+(Z_2')^2}} - \frac{Z_1'}{\sqrt{(X_1')^2+(Y_1')^2+(Z_1')^2}} = n_1'\right]$
X, Y only $Z = Z'$	$L = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left[\frac{X_2'}{\sqrt{(X_2')^2+(Y_2')^2}} - \frac{X_1'}{\sqrt{(X_1')^2+(Y_1')^2}}\right]$ (contraction along X-axis)	$B = B_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left[\frac{Y_2'}{\sqrt{(X_2')^2+(Y_2')^2}} - \frac{Y_1'}{\sqrt{(X_1')^2+(Y_1')^2}}\right]$ (contraction along Y-axis)	$H = 0$ (no contraction along Z-axis)
X only $Y = Y'$ $Z = Z'$	$L = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left[\frac{X_2'}{\sqrt{(X_2')^2}} - \frac{X_1'}{\sqrt{(X_1')^2}}\right]$, or $L = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'(1-1)$, or $L = L_0\sqrt{1-\frac{V^2}{c^2}}$ (contraction along X-axis only)	$B = 0$ (no contraction along Y-axis)	

The table represents the space contraction along X-, Y- and Z-axes with different special cases of relative motion between the inertial frames.

It should be noted from the third Table row that the length contraction takes place along X-axis only when the motion between inertial frames is constrained by a one-dimensional system. Thus, the modified transformation is a generalization that includes the ordinary transformation as a special case (in the one-dimensional system).

Conclusions

In this paper, a new transformation between inertial frames has been thoroughly devised as a continuation of the well-known Lorentz transformation. It is applied to three-dimensional space and encompasses each coordinate axis. The most fundamental success of this paper is the understanding of the details of space contraction along each coordinate axis in a three-dimensional form. For one-dimensional case in ordinary transformation (say X-axis), the length contraction takes place only along X-axis, and it is given by formula

$$L = L_0\sqrt{1-\frac{V^2}{c^2}}.$$

If the relative motion between inertial frames is confined to three-dimensional space under modified transformation, then the corresponding proposed modified space contraction along X-, Y- and Z-axes will be given by Eqs. (22), (24) and (26) respectively, namely:

$$L = L_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(l_2'-l_1'\right),\ B = B_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(m_2'-m_1'\right),$$

$$H = H_0\sqrt{1-\frac{V^2}{c^2}} - Vt'\left(n_2'-n_1'\right).$$

It is worth noting that the relativistic formula in modified version of the Lorentz transformation comes through the dependence of the space contraction on direction cosines. Therefore, the extension of the Lorentz transformation to three dimensions of space provides a comprehensive understanding of space contraction. The formulas illuminated here will have many applications to perceive the structure of space-time and it will be of interest in many other areas of theoretical physics.

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