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INTEGRALS OF MOTION OF A RELATIVISTIC PARTICLE IN 1 + 1 DIMENSIONS WITH COUPLED PARAMETERS

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Abstract. The eigenfunctions and eigenvalues of the integrals of motion γ and θ have been studied. An invariant form of motion was obtained for the derivatives of γ and θ, with respect to the proper time and velocity of a relativistic particle (RP). The integrals γ and θ were shown to be mutually expressible. Inverse values $1/E$ and $1/P$ were introduced for the energy and momentum of a free RP. A one-to-one correspondence of the RP energy and momentum was obtained. The properties of the γ integral expressed in terms of $1/E$ and $1/P$ were determined as a functional dependence $\gamma = \gamma(1/E, 1/P)$. Forms of the motion equations depending on the γ and θ integrals were obtained using Lagrangian and Hamiltonian formalism. Based on the latter, a generalized integral of motion describing all types of motions in 1+1 dimensions was derived. Mutually expressive differential forms of RP motion were introduced.

Keywords: integral of motion, special relativity, Lagrangian and Hamiltonian formalisms

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ИНТЕГРАЛЫ ДВИЖЕНИЯ РЕЛЯТИВИСТСКОЙ ЧАСТИЦЫ В ИЗМЕРЕНИЯХ 1 + 1 СО СВЯЗАННЫМИ ПАРАМЕТРАМИ

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Аннотация. В работе исследованы собственные функции и собственные значения от интегралов движения γ и быстроты θ. Для производных от γ и θ получена форма движения относительно собственного времени и скорости частицы. Показано, что интегралы движения являются взаимно-выражаемыми. Введены обратные значения для энергии $E_g = 1/E$ и импульса $P_g = 1/P$ свободной релятивистской частицы. Найдены свойства интеграла движения γ, который выражается через обратный импульс P_g и обратную энергию E_g как функциональная зависимость $\gamma = \gamma(E_g, P_g)$. С использованием формализма Лагранжа и Гамильтона получены уравнения движения $L' = L'(q, Q)$ и $H' = H'(\theta)$. На основе Гамильтонова формализма выведен обобщенный интеграл движения $\gamma = \gamma(\theta)$. Введены взаимно-выражающиеся дифференциальные формы движения релятивистской частицы.

Ключевые слова: интеграл движения, специальная теория относительности, Лагранжев и Гамильтонов формализмы

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Introduction

The classical integral of motion of a relativistic particle γ_0 has the invariant form

$$\gamma_0 = \frac{mc\left(1 - f\mathbf{n\beta}\right)}{\sqrt{1-\beta^2}} = mc\exp\left(-f\theta\right),\ \mathbf{n\beta} = \frac{\mathbf{nV}}{c} = \tanh\theta,\ \gamma = \frac{\gamma_0}{mc} = \exp\left(-f\theta\right), \tag{1}$$

where $\mathbf{V}$ is the particle velocity, c is the light speed, m is the particle mass, $f = \pm 1$.

For analytical and numerical calculations in Refs. [1 – 8], the integral of motion is understood as a constant (γ = const), however, this case is satisfied only for a constant particle velocity ($\boldsymbol{\beta}$ = const). Ignoring the nonlinear functional dependence $\gamma = \gamma(\mathbf{r}, t)$, for finding the trajectory of a charged particle $\mathbf{R}(\mathbf{r}, t)$ in the field of an electromagnetic wave

$$\mathbf{R}\left(\mathbf{r},t\right) = \int \frac{\mathbf{p_E}}{\gamma}dt + \boldsymbol{\chi} = \int \frac{q\mathbf{E}\left(\mathbf{r},t\right)}{\omega\gamma}dt + \boldsymbol{\chi}, \tag{2}$$

where $\mathbf{p}_E$ is the momentum of particle oscillation in the field of an electromagnetic wave, $\mathbf{p}_E = q\mathbf{E}(\mathbf{r}, t)/\omega$; $\mathbf{E}(\mathbf{r}, t)$ is the electric field strength of the wave; q is the particle charge; ω is the particle oscillation frequency; $\boldsymbol{\chi}$ is a certain constant that determines the initial phase of the wave; $\mathbf{r}$ is the particle position at time t.

In many papers (see, for example, Refs. [1 – 8]), as a rule, integration is performed over the space-time coordinate $\xi = t - \mathbf{nr}/c$ and supposing that γ = const. In this case, there are no relativistic effects of particle acceleration.

The angular integral of motion θ is defined from Hamiltonian mechanics

$$\frac{\partial H'}{\partial\theta} = \sqrt{Q_\xi^+ Q_\xi^- - 1} = P = \mathbf{nP}, \tag{3}$$

where $H' = H/(mc^2) = \sqrt{Q_\xi^+ Q_\xi^-}$ is the Hamiltonian of a freely moving particle, $Q_\xi^+ = \frac{1}{1-\mathbf{n\beta}}$, $Q_\xi^- = \frac{1}{1+\mathbf{n\beta}}$, $\mathbf{n}$ is the normal vector, and $H'(\beta = 0) = 1$ [9, 10].

As is known, the integral angle or the so-called angular integral of motion is expressed from Eq. (3) and has the following form:

$$\theta = \cosh^{-1}\left(\sqrt{Q_\xi^+ Q_\xi^-}\right), \tag{4}$$

where $E = \sqrt{Q_\xi^+ Q_\xi^-}$ is the dimensionless particle energy, $E \geq 0$.

From Eqs. (3) and (4) it can be seen that the energy and momentum of the system take the following values:

$$E = \sqrt{Q_\xi^+ Q_\xi^-} = \cosh\theta,\ \mathbf{nP} = \sqrt{Q_\xi^+ Q_\xi^- - 1} = \sinh\theta, \tag{5}$$

which express the laws of conservation of energy and momentum for the particle

$$E^2 = P^2 + 1, \tag{6}$$

where

$$E = \varepsilon / mc^2 = \sqrt{Q_\xi^+ Q_\xi^-} = \frac{1}{\sqrt{1-\beta^2}}, \mathbf{P} = \mathbf{p}/mc = \mathbf{n}\sqrt{Q_\xi^+ Q_\xi^- - 1}. \tag{7}$$

From Eqs. (4) – (7) and Ref. [10], for θ, $\mathbf{r}$, t, and γ, we have the following functional relationship:

$$\theta = \theta(\mathbf{r}, t, \gamma). \tag{8}$$

Using the functional dependence (8), it is easy to show that the trajectory of the relativistic particle (see Eq. (3)) can be represented as a function of the integral angle θ:

$$\mathbf{R}\left(\theta\right) = \int \frac{\mathbf{p_E}\left(\theta\right)}{\gamma\left(\theta\right)}d\theta + \boldsymbol{\chi} = \int \frac{q\mathbf{E}\left(\theta\right)}{\omega\gamma\left(\theta\right)}d\theta + \boldsymbol{\chi}. \tag{9}$$

From Eq. (8) we also have a mutually functional dependence with the space-time coordinate $\xi = \xi(\theta)$.

It is of interest to study in this paper the mutually functional dependence of the laws of conservation of energy and momentum of a particle in a mutually invariant form depending on $\theta = \theta(\gamma)$ and $\gamma = \gamma(\theta)$, that is

$$E^2 = P^2 + 2\gamma E - \gamma^2, \tag{10}$$

where $\gamma^{-1} = 2E - \gamma$.

From the rapid development of such areas as the double special theory of relativity and gravitomagnetism [11 – 16], it follows that it is necessary to introduce differential forms, which in the future will help to give a more extensive answer to such a question as: "Do we observe the violation of the Lorentz invariant form in gamma-ray bursts?"

The purpose of this work is to search for new forms of the of motion and to study the mutual functional relationship between integrals. Based on the Hamiltonian and Lagrange formalisms, successively derive generalized formulas that are applicable to the motion of a particle in any configuration of electromagnetic fields. Thus, in Refs. [17 – 19], the effectiveness of using Lorentz coordinates in terms of rapidity has been demonstrated $\xi = \xi(\theta)$. In Ref. [20], Lorentz coordinates with associated parameters were applied to describe the dynamics of a charged particle in the field of a plane wave and in the field of a one-dimensional Gaussian laser beam.

The method of coupled parameters was first introduced in relativistic hydrodynamics [21], where these parameters were defined as physical quantities of a hydrodynamic system, such as pressure, volume, and entropy, depending on a single parameter, such as temperature. In the special theory of relativity, the method of coupled parameters has been applied in Refs. [10, 17 – 20] based on eigenfunctions and eigenvalues, where it has been demonstrated that the chosen parameters being related to the invariance of the obtained results across different reference frames.

In this work, we will also show that the use of integrals of motion with coupled parameters and their eigenvalues can describe the dynamics of a particle more efficiently than its presentation in the orthogonal form [22]. Further, based on the task, on the eigenfunctions and eigenvalues, we will find the eigenvalues of the integrals of motion γ and θ from the operators of differentiation with respect to time $\hat{t} = \frac{d}{dt}$, velocity $\hat{\beta} = \frac{d}{d\beta}$ and angular displacement $\hat{\theta} = \frac{d}{d\theta}$.

This work presents an approach that allows transitioning from certain eigenfunctions in the special theory of relativity with associated parameters to eigenvalues in various representations of the integrals of motion, which facilitate working with these eigenvalues.

A problem for eigenfunctions and eigenvalues from γ

We introduce the direct integral of motion from $\xi = t - \mathbf{nr}/c$ as a derivative with respect to time t:

$$\frac{d\xi}{dt} = Q_t = 1 - f\mathbf{n\beta}, \tag{11}$$

where

$$Q_t^+ = 1 - \mathbf{n\beta},\ Q_t^- = 1 + \mathbf{n\beta}, \tag{12}$$

conjugate expressions.

The relationship between Q_t^+, Q_t^- and Q_t is expressed by the following expressions:

$$Q_t^+ = 1 - f\left(1 - Q_t\right), Q_t^- = 1 + f\left(1 - Q_t\right), Q_t^+ + Q_t^- = 2. \tag{13}$$

Using Eqs. (7), (11), and (12), we represent the integral of motion (1) in the following form:

$$\gamma = \frac{Q_t}{\sqrt{Q_t^+ Q_t^-}} = Q_t \sqrt{Q_\xi^+ Q_\xi^-} = Q_t E. \tag{14}$$

Eigenfunctions and eigenvalues from the integral of motion γ (14) are obtained by differentiating with respect to time t, that is,

$$\frac{d\gamma}{dt} = \frac{dQ_t}{dt} E + Q_t \frac{dE}{dt}, \quad (15)$$

where

$$\frac{dQ_t}{dt} = -f\left(1 - Q_t\right)^2 = -f\left(1 - Q_t^+ Q_t^-\right) \quad (16)$$

is the function from Ref. [10].

The expression $\frac{dQ_t}{dt}E$ has the form in the closed Lorentz group, viz.

$$\frac{dQ_t}{dt} E = -f\left(1 - Q_t^+ Q_t^-\right)\sqrt{Q_\xi^+ Q_\xi^-} = -f\left(\sqrt{Q_\xi^+ Q_\xi^-} - \sqrt{Q_t^+ Q_t^-}\right). \quad (17)$$

Under the closed Lorentz group, we will understand when all relations in the equations are closed, that is, they have a connection $Q_t^+ Q_t^-$ or $Q_\xi^+ Q_\xi^-$.

Under the open Lorentz group, we will understand when there is one "free" integral of motion Q_t or Q_ξ in the equations. It is possible to choose an invariant form. For example, for the integral of motion (see Eq. (14)) we have

$$\gamma^+ \equiv Q_t^+ E \equiv E - \mathbf{nP} \equiv \text{inv}, \ \gamma^- \equiv Q_t^- E \equiv E + \mathbf{nP} \equiv \text{inv}. \quad (18)$$

The time derivative of the energy has the form [10]

$$\frac{dE}{dt} = P^3 \equiv \mathbf{nP}P^2. \quad (19)$$

The particle velocity can also be represented in the following form:

$$\boldsymbol{\beta} = f\mathbf{n}(1 - Q_t). \quad (20)$$

Expanding $\frac{dQ_t}{dt}E$ from Eq. (15) in an open group, we get

$$\frac{dQ_t}{dt} E = -f\boldsymbol{\beta}\mathbf{P}, \quad (21)$$

$$\frac{d\gamma}{dt} = -f\boldsymbol{\beta}\mathbf{P} + Q_t P^3, \quad (22)$$

$$Q_t P = \beta\gamma = \mathbf{n}\boldsymbol{\beta}\gamma, \quad (23)$$

$$\frac{d\gamma}{dt} = -f\boldsymbol{\beta}\mathbf{P} + \beta\gamma P^2 = -f\boldsymbol{\beta}\mathbf{P}\left(1 - f\gamma P\right). \quad (24)$$

In Eq. (24) on the right side, multiplying and dividing by γ we get the function $f_t(\gamma)$,

$$\frac{d\gamma}{dt} = f_t\left(\gamma\right)\gamma, \quad (25)$$

$$f_t\left(\gamma\right) = \frac{-f\boldsymbol{\beta}\mathbf{P}\left(1 - f\gamma P\right)}{\gamma} = -\frac{f\boldsymbol{\beta}\mathbf{P}}{\gamma} + \mathbf{n}\boldsymbol{\beta}P^2. \quad (26)$$

Representations $\boldsymbol{\beta}\mathbf{P}$ in the open Lorentz group have the form

$$\boldsymbol{\beta}\mathbf{P} = f\mathbf{n}(1 - Q_t)f\mathbf{n}(Q_\xi - 1)\gamma = \gamma(Q_\xi + Q_t - 2). \quad (27)$$

Due to the invariance in Eq. (26), for $\mathbf{n}\boldsymbol{\beta}P^2$ can be represented compactly $\mathbf{n}\boldsymbol{\beta}$ in an open group, and P^2 in a closed group,

$$\boldsymbol{\beta} = f\mathbf{n}(1 - Q_t), \ \ P^2 = Q_{\text{i}}^+ Q_{\text{i}}^- - 1. \quad (28)$$

Representation f_t (γ) in f_t (γ⁺). The representation of $\mathbf{n}\boldsymbol{\beta}P^2$ from Eq. (26) for the case $f = +1$ has the form

$$\boldsymbol{\beta}^{+} = \mathbf{n}\left(1 - Q_t^{+}\right), \tag{29}$$

$$\mathbf{n}\boldsymbol{\beta}^{+} P^2 = \left(1 - Q_t^{+}\right)\left(Q_\xi^{+} Q_\xi^{-} - 1\right) = P^2 - Q_\xi^{-} + Q_t^{+}. \tag{30}$$

Substituting Eqs. (27) and (28) into Eq. (26), and passing from representations $f_t(\gamma)$ to $f_t(\gamma^+)$, we obtain

$$f_t\left(\gamma^{+}\right) = -\left(Q_\xi^{+} + Q_t^{+} - 2\right) + P^2 - Q_\xi^{-} + Q_t^{+}, \tag{31}$$

$$f_t(\gamma^+) = -P^2. \tag{32}$$

From Eqs. (31) and (32) we also have the form

$$f_t\left(\gamma^{+}\right) = -\frac{Q_\xi^{+} + Q_\xi^{-}}{2} + 1. \tag{33}$$

Representation $f_t(\gamma)$ in $f_t(\gamma^-)$. The representation of $\mathbf{n}\boldsymbol{\beta}P^2$ from Eq. (26) for the case $f = -1$ has the form

$$\boldsymbol{\beta}^{-} = -\mathbf{n}\left(1 - Q_t^{-}\right), \tag{34}$$

$$\mathbf{n}\boldsymbol{\beta}^{-} P^2 = -\left(1 - Q_t^{-}\right)\left(Q_\xi^{+} Q_\xi^{-} - 1\right) = -P^2 + Q_t^{-} P^2, \tag{35}$$

$$Q_t^{-} P^2 = Q_t^{-}\left(Q_\xi^{+} Q_\xi^{-} - 1\right) = Q_\xi^{+} - Q_t^{-}, \tag{36}$$

$$\mathbf{n}\boldsymbol{\beta}^{-} P^2 = -\left(1 - Q_t^{-}\right)\left(Q_\xi^{+} Q_\xi^{-} - 1\right) = -P^2 + Q_\xi^{+} - Q_t^{-}, \tag{37}$$

$$f_t\left(\gamma^{-}\right) = \frac{-f\boldsymbol{\beta}\mathbf{P}\left(1 - f\gamma P\right)}{\gamma} = Q_\xi^{-} + Q_t^{-} - 2 - P^2 + Q_\xi^{+} - Q_t^{-}, \tag{38}$$

$$f_t\left(\gamma^{-}\right) = Q_\xi^{+} + Q_\xi^{-} - 2 - P^2, \tag{39}$$

$$f_t(\gamma^-) = P^2. \tag{40}$$

From Eqs. (39) and (40) there is also the following form:

$$f_t\left(\gamma^{-}\right) = \frac{Q_\xi^{+} + Q_\xi^{-}}{2} - 1. \tag{41}$$

Summarizing the obtained results – Eqs. (32) and (40), we have that

$$f_t(\gamma) = -fP^2. \tag{42}$$

Representation γ in γ^+ and γ^-. The representation of γ^+ and γ^- in the closed Lorentz group has the form

$$\gamma^+ \gamma^- = 1. \tag{43}$$

Differentiating Eq. (43) with respect to time, we obtain

$$\frac{d\left(\gamma^{+}\gamma^{-}\right)}{dt} = 0, \text{ or } \frac{d\gamma^{+}}{dt}\gamma^{-} + \frac{d\gamma^{-}}{dt}\gamma^{+} = 0. \tag{44}$$

The eigenfunction and eigenvalue problems for γ^+ and γ^- have the forms

$$\frac{d\gamma^{+}}{dt} = f_t\left(\gamma^{+}\right)\gamma^{+}, \; \frac{d\gamma^{-}}{dt} = f_t\left(\gamma^{-}\right)\gamma^{-}, \tag{45}$$

where $f_t(\gamma^+)$ and $f_t(\gamma^-)$ are eigenfunctions γ^+ and γ^- in time t.

Substituting Eq. (45) in Eq. (44) we obtain

$$f_t(\gamma^+) + f_t(\gamma^-) = 0. \tag{46}$$

Representation of γ^+ and γ^- in differential form with respect to Q_ξ^+ and Q_ξ^-. Using the relations $Q_t^+ + Q_t^- = 2$ and $Q_t Q_\xi = 1$, we represent γ^+ and γ^- in the form

$$\gamma^+ = \sqrt{2Q_\xi^- - 1},\ \gamma^- = \sqrt{2Q_\xi^+ - 1},$$
$$\gamma^- = \frac{1}{\sqrt{2Q_\xi^- - 1}},\ \gamma^+ = \frac{1}{\sqrt{2Q_\xi^+ - 1}}. \tag{47}$$

In differential form, γ^+ and γ^- with respect to Q_ξ^- and Q_ξ^+, have the following form:

$$\frac{d\gamma^+}{dQ_\xi^-} = \gamma^-,\ \frac{d\gamma^-}{dQ_\xi^+} = \gamma^+,\ \frac{d\gamma^-}{dQ_\xi^-} = -\left(\gamma^-\right)^3,\ \frac{d\gamma^+}{dQ_\xi^+} = -\left(\gamma^+\right)^3. \tag{48}$$

The fulfillment of properties of Eq. (48) can be easily verified by differentiating $\gamma^+\gamma^- = 1$, with respect to Q_ξ^- or Q_ξ^+.

The representation of $\frac{dQ_\xi^+}{dQ_\xi^-}$ and $\frac{dQ_\xi^-}{dQ_\xi^+}$ in differential form is as follows:

$$\frac{dQ_\xi^+}{dQ_\xi^-} = -\left(\gamma^-\right)^4,\ \frac{dQ_\xi^-}{dQ_\xi^+} = -\left(\gamma^+\right)^4. \tag{49}$$

Energy and momentum in γ^+ and γ^- have the following relationships:

$$\gamma^+ + \gamma^- = 2E = 2Q_{\xi\gamma}, \tag{50}$$

$$\gamma^- - \gamma^+ = 2\mathbf{nP} = 2f(Q_\xi - 1)\gamma. \tag{51}$$

Expanding Eq. (51), we obtain a following equation for $f = +1$:

$$\gamma^- - \gamma^+ = 2Q_\xi^+\gamma^+ - 2\gamma^+; \tag{52}$$

for $f = -1$ we have

$$\gamma^- - \gamma^+ = -2Q_\xi^-\gamma^- + 2\gamma^-. \tag{53}$$

Representation of functions $f_t(\gamma^+)$ and $f_t(\gamma^-)$ in γ^+ and γ^-. Let us represent the functions $f_t(\gamma^-)$ from Eq. (41) and $f_t(\gamma^+)$ from Eq. (33) using the invariant forms

$$Q_\xi^+ = \frac{\left(\gamma^-\right)^2}{2} + \frac{1}{2} \text{ and } Q_\xi^- = \frac{\left(\gamma^+\right)^2}{2} + \frac{1}{2},$$

in the representation γ^+ and γ^-, which have the form

$$f_t\left(\gamma^+\right) = -\frac{\left(\gamma^+\right)^2 + \left(\gamma^-\right)^2}{4} - \frac{1}{2}, \tag{54}$$

$$f_t\left(\gamma^-\right) = \frac{\left(\gamma^+\right)^2 + \left(\gamma^-\right)^2}{4} + \frac{1}{2}. \tag{55}$$

Adding and subtracting Eqs. (54) and (55), we have

$$f_t(\gamma^+) + f_t(\gamma^-) = 0, \tag{56}$$

$$f_t\left(\gamma^-\right) - f_t\left(\gamma^+\right) = \frac{\left(\gamma^+\right)^2 + \left(\gamma^-\right)^2}{2} + 1, \tag{57}$$

Alternately multiplying Eqs. (56) and (57) by $(\gamma^+)^2$ and $(\gamma^-)^2$, we obtain a mutually invariant form of eigenvalues:

$$f_t(\gamma^+)(\gamma^+)^2 + f_t(\gamma^-)(\gamma^+)^2 = 0,\ f_t(\gamma^+)(\gamma^-)^2 + f_t(\gamma^-)(\gamma^-)^2 = 0, \tag{58}$$

$$\left[f_t\left(\gamma^-\right)-f_t\left(\gamma^+\right)\right]\left(\gamma^+\right)^2=\frac{\left(\gamma^+\right)^4+1}{2}+\left(\gamma^+\right)^2, \tag{59}$$

$$\left[f_t\left(\gamma^-\right)-f_t\left(\gamma^+\right)\right]\left(\gamma^-\right)^2=\frac{1+\left(\gamma^-\right)^4}{2}+\left(\gamma^-\right)^2. \tag{60}$$

The invariance property of the energy and momentum of a particle expressed in terms of the integral of motion γ and Q_ξ. To write the invariance property as

$$f_t(\gamma^-)+f_t(\gamma^+)=P^2-P^2=0, \tag{61}$$

and

$$P^2=P^2, \tag{62}$$

let us represent P as an open Lorentz group in the form

$$(P^+)^2=(P^-)^2, \tag{63}$$

where the invariance property holds $2P=2P^+=2P^-=\gamma^--\gamma^+\equiv \text{inv}$.

The momentum P of the particle is represented in the following form:

$$P=f(Q_\xi-1)\gamma=\beta Q_\xi\gamma=\beta E=-f(Q_t-1)Q_\xi\gamma. \tag{64}$$

Substituting the values of the pulse in the open group $P^+=\left(Q_\xi^+-1\right)\gamma^+$ and $P^-=-\left(Q_\xi^--1\right)\gamma^-$ in Eq. (63), we obtain

$$\left[\left(Q_\xi^+-1\right)\gamma^+\right]^2=\left[\left(Q_\xi^--1\right)\gamma^-\right]^2, \tag{65}$$

$$\left(Q_\xi^+-1\right)^2\left(\gamma^+\right)^2=\left(Q_\xi^--1\right)^2\left(\gamma^-\right)^2, \tag{66}$$

$$\left[\left(Q_\xi^+\right)^2-2Q_\xi^++1\right]\left(\gamma^+\right)^2=\left[\left(Q_\xi^-\right)^2-2Q_\xi^-+1\right]\left(\gamma^-\right)^2. \tag{67}$$

Using the invariance property of the energy $E=Q_\xi\gamma=Q_\xi^+\gamma^+=Q_\xi^-\gamma^-$, in Eq. (67) we have

$$\left(-2Q_\xi^++1\right)\left(\gamma^+\right)^2=\left(-2Q_\xi^-+1\right)\left(\gamma^-\right)^2. \tag{68}$$

Substituting properties $Q_\xi^-=Q_\xi^+\left(\gamma^+\right)^2$ and $Q_\xi^+=Q_\xi^-\left(\gamma^-\right)^2$ in Eq. (68), we derive an invariant form

$$2\left(Q_\xi^+-Q_\xi^-\right)=\left(\gamma^-\right)^2-\left(\gamma^+\right)^2=\left(\gamma^++\gamma^-\right)\left(\gamma^--\gamma^+\right). \tag{69}$$

From the difference Q_ξ^+ and Q_ξ^-, we obtain

$$Q_\xi^+-Q_\xi^-=2\mathbf{n}\boldsymbol{\beta}Q_\xi^+Q_\xi^+. \tag{70}$$

From the invariance property of the velocities $\boldsymbol{\beta}=\boldsymbol{\beta}^-=\boldsymbol{\beta}^+$, we have the conservation law

$$Q_\xi^++Q_\xi^+=2Q_\xi^+Q_\xi^+, \tag{71}$$

also for $\boldsymbol{\beta}^-$

$$Q_\xi^+-Q_\xi^-=-2Q_\xi^+Q_\xi^++2Q_\xi^+, \tag{72}$$

as well for $\boldsymbol{\beta}^+$

$$Q_\xi^+-Q_\xi^-=2Q_\xi^+Q_\xi^+-2Q_\xi^-. \tag{73}$$

Substituting the values of the energy

$$E^2 = Q_\xi^+ Q_\xi^- = \frac{\left(\gamma^-\right)^2 + \left(\gamma^+\right)^2 + 2}{4}$$

in Eqs. (72) and (73), we obtain

$$Q_\xi^+ - Q_\xi^- = -\frac{\left(\gamma^-\right)^2 + \left(\gamma^+\right)^2 + 2}{2} + 2Q_\xi^+, \tag{74}$$

$$Q_\xi^+ - Q_\xi^- = \frac{\left(\gamma^-\right)^2 + \left(\gamma^+\right)^2 + 2}{2} - 2Q_\xi^-. \tag{75}$$

Substituting Eq. (74) into Eq. (69), we have the form

$$-\left(\gamma^-\right)^2 - \left(\gamma^+\right)^2 - 2 + 4Q_\xi^+ = \left(\gamma^-\right)^2 - \left(\gamma^+\right)^2. \tag{76}$$

Similarly, substituting Eq. (75) into Eq. (69), we obtain

$$\left(\gamma^-\right)^2 + \left(\gamma^+\right)^2 + 2 - 4Q_\xi^- = \left(\gamma^-\right)^2 - \left(\gamma^+\right)^2. \tag{77}$$

Adding Eqs. (76) and (77), we have the value of the invariant form

$$2Q_\xi^+ - 2Q_\xi^- = \left(\gamma^-\right)^2 - \left(\gamma^+\right)^2, \tag{78}$$

Subtracting Eq. (76) from Eq. (77), we have

$$\frac{1}{2}\left[\left(\gamma^-\right)^2 + \left(\gamma^+\right)^2\right] + 1 = Q_\xi^+ + Q_\xi^-. \tag{79}$$

Invariant forms of the integrals γ^+ and γ^- of particle motion. The invariant form of the integrals of motion $(\gamma^-)^2$ and $(\gamma^+)^2$ has the following form:

$$\left(\gamma^-\right)^2 + \left(\gamma^+\right)^2 = \left[\left(Q_t^+\right)^2 + \left(Q_t^-\right)^2\right]E^2 = 2\frac{1+\beta^2}{1-\beta^2} = 2\left(Q_\xi^+ + Q_\xi^+\right) - 2, \tag{80}$$

$$\left(\gamma^-\right)^2 - \left(\gamma^+\right)^2 = 2\left(Q_\xi^+ - Q_\xi^-\right) = 4\mathbf{n}\boldsymbol{\beta} Q_\xi^+ Q_\xi^- = 2\mathbf{n}\boldsymbol{\beta}\left(Q_\xi^+ + Q_\xi^-\right) = \frac{4\mathbf{n}\boldsymbol{\beta}}{1-\beta^2}. \tag{81}$$

We introduce invariant functions that have the form

$$g = \frac{1+\beta^2}{1-\beta^2} = \frac{\left(\gamma^-\right)^2 + \left(\gamma^+\right)^2}{2},\ h = \frac{\left(\gamma^-\right)^2 - \left(\gamma^+\right)^2}{4} = \frac{\mathbf{n}\boldsymbol{\beta}}{1-\beta^2}. \tag{82}$$

The functions g and h belong to the same class: their partial derivatives with respect to β are their eigenvalues and reflect the closure property of the Lorentz group:

$$\frac{dg}{d\beta} = 4hQ_\xi^+ Q_\xi^- = 4hE^2, \frac{dh}{d\beta} = gQ_\xi^+ Q_\xi^- = gE^2. \tag{83}$$

Taking the derivatives of the integrals of motion γ^- and γ^+ (see Eq. (18)) with respect to $\boldsymbol{\beta}$, we obtain

$$\mathbf{n}\frac{d\gamma^-}{d\boldsymbol{\beta}} = \frac{\gamma^+}{\left(Q_t^+\right)^2} = \frac{\left(\gamma^+\right)^2}{\left(Q_t^+\right)^2}\gamma^- = f_\beta\left(\gamma^-\right)\gamma^-,\ f_\beta\left(\gamma^-\right) = \frac{\left(\gamma^+\right)^2}{\left(Q_t^+\right)^2} = E^2. \tag{84}$$

Similarly, for γ^+ we have

$$\mathbf{n}\frac{d\gamma^+}{d\boldsymbol{\beta}} = f\left(\gamma_\beta^+\right)\gamma^+, f\left(\gamma_\beta^+\right) = -E^2. \tag{85}$$

As can be seen, for Eqs. (84) and (85) the following properties of the invariant form hold:

$$f_\beta(\gamma^+) + f_\beta(\gamma^-) = 0, \tag{86}$$

$$\mathbf{n}\frac{d\gamma^+}{d\boldsymbol{\beta}} + \mathbf{n}\frac{d\gamma^-}{d\boldsymbol{\beta}} = f_\beta\left(\gamma^-\right)\left(\gamma^- - \gamma^+\right) = 2\mathbf{nP}E^2, \tag{87}$$

$$\mathbf{n}\frac{d\gamma^-}{d\boldsymbol{\beta}} - \mathbf{n}\frac{d\gamma^+}{d\boldsymbol{\beta}} = f_\beta\left(\gamma^-\right)\left(\gamma^- + \gamma^+\right) = 2E^3. \tag{88}$$

From properties (5), (6), and (7) of the invariance of the energy E and momentum $\mathbf{P}$ with respect to $\boldsymbol{\beta}$, we have

$$\frac{d\mathbf{P}}{d\boldsymbol{\beta}} = E^3, \frac{dE}{d\boldsymbol{\beta}} = \boldsymbol{\beta}E^3 = \mathbf{P}E^2. \tag{89}$$

Taking the derivative of the integrals of motion (18), we obtain the invariant form

$$\frac{dE}{d\boldsymbol{\beta}} + \mathbf{n}\frac{d\mathbf{P}}{d\boldsymbol{\beta}} = \frac{d\gamma^-}{d\boldsymbol{\beta}}, \tag{90}$$

and substituting the values (89), (85) into Eq. (90) we obtain that the integrals of motion γ^- and γ^+ are invariant with respect to differentiation with respect to $\boldsymbol{\beta}$.

From properties (5), (6) and (7) of the invariance of energy and momentum with respect to time t, we obtain:

$$\mathbf{n}\frac{d\mathbf{P}}{dt} = P^2E. \tag{91}$$

Differentiating Eq. (18) with respect to time t and substituting the values (91), (45) and (19), we obtain the mutually invariant form (18). The fulfillment of the invariant form of Eqs. (19), (89) and (91) is also easy to check by differentiating the conservation law (6) with respect to time t or velocity $\boldsymbol{\beta}$.

Reverse energy of a relativistic particle. We introduce the inverse energy function as

$$E_g = \frac{1}{E} = \frac{1}{Q_\xi\gamma} = \frac{Q_t}{\gamma} = \sqrt{Q_t^+Q_t^-}. \tag{92}$$

The invariant form of the inverse energy has the form

$$E_g = \frac{Q_t}{\gamma} = Q_t^+\gamma^- = Q_t^-\gamma^+ = \sqrt{Q_t^+Q_t^-}. \tag{93}$$

From the properties obtained above, we introduce invariant relations using the reciprocal energy of the particle:

$$EE_g = 1. \tag{94}$$

$$P^2E_g^- = \left(Q_\xi^+Q_\xi^- - 1\right)Q_t^-\gamma^+ = Q_\xi^+\gamma^+ - Q_t^-\gamma^+. \tag{95}$$

$$P^2E_g^+ = \left(Q_\xi^+Q_\xi^- - 1\right)Q_t^+\gamma^- = Q_\xi^-\gamma^- - Q_t^+\gamma^-, \tag{96}$$

$$\mathbf{P}E_g = f\mathbf{n}\left(Q_\xi - 1\right)\gamma\frac{Q_t}{\gamma} = f\mathbf{n}\left(Q_\xi - 1\right)Q_t = f\mathbf{n}\left(1 - Q_t\right) = \boldsymbol{\beta}, \tag{97}$$

$$\mathbf{P}^+E_g^+ = \boldsymbol{\beta}^+ = \mathbf{P}^-E_g^- = \boldsymbol{\beta}^-, \tag{98}$$

$$\mathbf{P}^+E_g^- = \mathbf{n}\left(1 - Q_t^+\right) = \boldsymbol{\beta}^+,\ P^-E_g^+ = -\left(1 - Q_t^-\right) = \beta^-. \tag{99}$$

Pseudopotential energy of a relativistic particle. We introduce the pseudopotential energy function as

$$E_p = Q_t \gamma = Q_t^2 E. \tag{100}$$

$$EE_p = \gamma^2, \tag{101}$$

$$E^+ E_p^+ = E^- E_p^+ = \left(\gamma^+\right)^2, \tag{102}$$

$$E^+ E_p^- = E^- E_p^- = \left(\gamma^-\right)^2. \tag{103}$$

Reverse momentum of a relativistic particle. The reverse momentum of a particle has the form

$$\mathbf{nP}_g = \frac{1}{\mathbf{nP}} = \frac{Q_t}{\mathbf{n\beta}\gamma}, \tag{104}$$

and

$$\mathbf{nP}_g = \frac{1}{\mathbf{nP}} = \frac{Q_t}{\mathbf{n\beta}\gamma} = \frac{Q_t^+ \gamma^-}{\mathbf{n\beta}} = \frac{Q_t^- \gamma^+}{\mathbf{n\beta}}, \tag{105}$$

where $\mathbf{n\beta} = \mathbf{n\beta}^- = \mathbf{n\beta}^+$.

Decomposition into a spectrum in terms of γ^+ and γ^- of the squared momentum $\mathbf{P}^2$ and squared energy E^2 of the particle. The connections of the integrals of the motion of the particle γ^+ and γ^- through the energy of the particle has the form

$$\gamma^+ + \gamma^- = 2E. \tag{106}$$

Squaring Eq. (106) we obtain

$$\frac{\left(\gamma^+\right)^2 + \left(\gamma^-\right)^2}{4} + \frac{1}{2} = E^2. \tag{107}$$

Subtracting one from both sides of the equation, we have

$$\frac{\left(\gamma^+\right)^2 + \left(\gamma^-\right)^2}{4} - \frac{1}{2} = E^2 - 1 = P^2. \tag{108}$$

Using the invariant property of integrals of motion

$$Q_\xi^+ Q_\xi^- = \frac{\left(\gamma^+\right)^2 + \left(\gamma^-\right)^2}{4} + \frac{1}{2}, \tag{109}$$

we have

$$P^2 = Q_\xi^+ Q_\xi^- - 1 = \left(E-1\right)\left(E+1\right), \tag{110}$$

and

$$\left(1 - Q_\xi^+\right)\left(Q_\xi^- - 1\right) = \left(E-1\right)\left(E+1\right). \tag{111}$$

Adding $\gamma^+\gamma^- = 1$ to the left side of Eq. (111), we obtain

$$\left(\gamma^+ - Q_\xi^+ \gamma^+\right)\left(Q_\xi^- \gamma^- - \gamma^-\right) = \left(E-1\right)\left(E+1\right), \tag{112}$$

and using the invariance property

$$E = Q_\xi \gamma = Q_\xi^+ \gamma^+ = Q_\xi^- \gamma^-,$$

we have

$$(\gamma^+ - E)\,(E - \gamma^-) = (E - 1)(E + 1). \tag{113}$$

Revealing Eq. (113), we obtain a representation of the square momentum of the particle:

$$\frac{\gamma^{+}E + E\gamma^{-}}{2} = E^2 = P^2 + 1. \tag{114}$$

Dividing Eq. (114) by energy, we have

$$\frac{\gamma^{+} + \gamma^{-}}{2} = E. \tag{115}$$

Substituting into the left part of Eq. (115) the values of the invariant form of the energy of the particle

$$E = Q_\xi \gamma = Q_\xi^{+}\gamma^{+},$$

we obtain

$$\frac{\left(\gamma^{+}\right)^2 + 1}{2} = Q_\xi^{-}. \tag{116}$$

Similarly, substituting the values of the invariant form of the particle energy

$$E = Q_\xi \gamma = Q_\xi^{-}\gamma^{-}$$

into the left part of Eq. (115), we obtain

$$\frac{\left(\gamma^{-}\right)^2 + 1}{2} = Q_\xi^{+}, \tag{117}$$

Adding Eqs. (116) and (117), we have the following form:

$$\frac{\left(\gamma^{+}\right)^2 + \left(\gamma^{-}\right)^2}{2} + 1 = Q_\xi^{+} + Q_\xi^{-}, \tag{118}$$

and

$$\frac{\left(\gamma^{+}\right)^2 + \left(\gamma^{-}\right)^2}{2} = 2Q_\xi^{+}Q_\xi^{-} - 1 = P^2 + E^2. \tag{119}$$

Subtracting Eq. (117) from Eq. (116), we obtain

$$\frac{\left(\gamma^{-}\right)^2 - \left(\gamma^{+}\right)^2}{2} = Q_\xi^{+} - Q_\xi^{-} = 2\mathbf{n}\boldsymbol{\beta} Q_\xi^{+}Q_\xi^{-} = 2\mathbf{n}\boldsymbol{\beta} E^2, \tag{120}$$

Expressing $2\mathbf{n}\boldsymbol{\beta}$, we obtain

$$2\mathbf{n}\boldsymbol{\beta} = Q_t^{-} - Q_t^{+} = \frac{\left(Q_t^{-}\right)^2}{2} - \frac{\left(Q_t^{+}\right)^2}{2} = \frac{\left(Q_t^{-} - Q_t^{+}\right)\left(Q_t^{+} + Q_t^{-}\right)}{2}. \tag{121}$$

Converting Eq. (119), we obtain the value of the square of the particle energy:

$$E^2 = Q_\xi^{+}Q_\xi^{-} = \frac{\left(\gamma^{+}\right)^2 + \left(\gamma^{-}\right)^2}{2} + \frac{1}{2}. \tag{122}$$

Similarly, for the value of the square of the particle momentum, we obtain

$$P^2 = Q_\xi^{+}Q_\xi^{-} - 1 = \frac{\left(\gamma^{+}\right)^2 + \left(\gamma^{-}\right)^2}{2} - \frac{1}{2}. \tag{123}$$

Using the relations between the integrals of motion

$$Q_\xi^{+} + Q_\xi^{-} = 2Q_\xi^{+}Q_\xi^{-},\ Q_t^{+} + Q_t^{-} = 2, \tag{124}$$

it is easy to collect the particle momentum from the invariant forms of the integrals of motion Q_t and Q_ξ:

$$Q_\xi^+ + Q_\xi^- - \left(Q_t^+ + Q_t^-\right) = 2Q_\xi^+ Q_\xi^- - 2 = 2P^2. \tag{125}$$

Multiplying P^2 from Eq. (125) by $2\mathbf{n}\boldsymbol{\beta} = Q_t^- - Q_t^+$, we obtain

$$Q_\xi^+ - Q_t^- - Q_\xi^- + Q_t^+ = 2\mathbf{n}\boldsymbol{\beta}P^2. \tag{126}$$

Adding and subtracting Eqs. (125) and (126), we have

$$Q_t^- P^2 = Q_\xi^+ - Q_t^-,\ Q_t^+ P^2 = Q_\xi^- - Q_t^+. \tag{127}$$

The relation for the particle energy has the following invariant relations:

$$Q_t^- E^2 = Q_\xi^+,\ Q_t^+ E^2 = Q_\xi^-. \tag{128}$$

Representation of integrals of motion as a function $\gamma = \gamma\ (Q_t, \mathbf{P}, E, \mathbf{P}_g, E_g)$. From the problems on eigenfunctions and eigenvalues from the above paragraphs, it is convenient to represent the integrals of motion γ^+ and γ^- as invariant functions

$$\gamma = \left(1 - Q_t\right)^2 EP_g^2 - fQ_t\mathbf{nP}, \tag{129}$$

$$\gamma = E_g - fQ_t\mathbf{nP}. \tag{130}$$

The γ values (129) and (130) are expressed in terms of a one-to-one correspondence:

$$E_g + f\mathbf{nP}_g = f\frac{E_g\mathbf{nP}_g}{\gamma}, \tag{131}$$

$$\gamma^{-1} = E + f\mathbf{nP}. \tag{132}$$

Angular integral of motion θ

The functional dependencies $\gamma = \gamma(Q_t)$ and $Q_t = Q_t(\gamma)$ have been introduced above. We now define the integral angle, or angular integral of motion, as a function $\gamma = \gamma(\theta)$.

The integrals of motion from Eq. (18) with the application of Eq. (5) take the form

$$\gamma^+ = \cosh\theta - \sinh\theta = \exp(-\theta),\ \gamma^- = \cosh\theta + \sinh\theta = \exp(\theta). \tag{133}$$

Generalizing the Eqs. (133), the integral of motion γ has the following relation with the angular integral of motion θ

$$\gamma = \cosh\theta - f\sinh\theta = \exp(-f\theta). \tag{134}$$

Substituting the integral of motion (134) in the invariant form of the particle energy $E = Q_\xi\gamma$, we obtain

$$Q_t = 1 - f\tanh\theta, \tag{135}$$

where, in Eq. (135), Q_t^+ and Q_t^- take the forms

$$Q_t^+ = 1 - \tanh\theta,\ Q_t^- = 1 + \tanh\theta. \tag{136}$$

It is easy to show from Eq. (135) that the particle velocity has the form

$$\mathbf{n}\boldsymbol{\beta} = \tanh\theta. \tag{137}$$

Substituting the particle energy and momentum values from Eq. (5), it can be seen that

$$\mathbf{n}\boldsymbol{\beta} = \frac{\sqrt{E^2 - 1}}{E},\ \boldsymbol{\beta} = \frac{\mathbf{P}}{\sqrt{P^2 + 1}}, \tag{138}$$

thus, the functions $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{P}, E)$, $\mathbf{P} = \mathbf{P}(\boldsymbol{\beta})$ and $E = E(\boldsymbol{\beta})$ have a mutual functional dependence.

Multiplying Q_t^+ and Q_t^- from Eq. (136), we have

$$Q_t^+ Q_t^- = 1 - \tanh^2\theta,\ Q_\xi^+ Q_\xi^- = \cosh^2\theta. \tag{139}$$

Expressing from Eq. (135) Q_ξ, we obtain

$$Q_\xi = 0.5\cdot[\exp(2f\theta) + 1], \tag{140}$$

where

$$Q_\xi^+ = \frac{1}{2}\left[\exp(2\theta)+1\right],\ Q_\xi^- = \frac{1}{2}\left[\exp(-2\theta)+1\right]. \tag{141}$$

From Eq. (140) we obtain the invariant form of the integral of motion γ:

$$(\gamma^{-f})^2 = 2Q_\xi - 1 = \exp(2f\theta),\ \gamma^{-f} = \exp(f\theta), \tag{142}$$

where the choice of $f = +1$ or $f = -1$ is determined by the principle of relativity.

Representation of the space-time characteristics depending on the integral of motion θ. By representing the integral of motion (134) as a function differentiable with respect to θ, we obtain the eigenvalue of the function γ:

$$\gamma' = -f\exp(-f\theta) = -f\gamma. \tag{143}$$

From the representations of the principle of the particle motion for the space-time Lorentz coordinate, we have

$$\xi = t - f\frac{\mathbf{nr}}{c}, \tag{144}$$

where

$$t = 1 - \frac{f}{(1-Q_t)},\ \mathbf{nr} = \frac{1}{2}\ln\left[(1-Q_t)^2\right]; \tag{145}$$

using Eq. (135) we can be sure that the angular integral of motion has the following dependence:

$$t = 1 - \coth\theta,\ \mathbf{nr} = 0.5\ln(\tanh^2\theta). \tag{146}$$

Differentiating Eq. (146) with respect to θ, we obtain

$$\frac{dt}{d\theta} = \frac{1}{\sinh^2\theta},\ \mathbf{n}\frac{d\mathbf{r}}{d\theta} = \frac{1}{\cosh\theta\sinh\theta}, \tag{147}$$

where

$$\mathbf{n\beta} = \mathbf{n}\frac{d\mathbf{r}}{d\theta}\frac{d\theta}{dt} = \tanh\theta, \tag{148}$$

the value from Eq. (137).

Differentiating ξ from Eq. (144) with respect to θ, we obtain

$$\frac{d\xi}{d\theta} = \frac{\gamma}{\sinh^2\theta\cosh\theta} = \frac{\gamma}{P^2E} = \frac{Q_t}{\sinh^2\theta} = \frac{Q_t}{P^2}. \tag{149}$$

Using the differential form from Eq. (147) $\zeta = P^2 = \sinh^2\theta$, we have

$$\frac{d\xi}{dt} = \frac{d\xi}{d\theta}\frac{d\theta}{dt} = \frac{d\xi}{d\theta}\dot{\theta} = Q_t. \tag{150}$$

Lagrangian and Hamiltonian formalisms for a free particle in the representation θ. The Lagrangian depending on θ and $\zeta = \frac{d\theta}{dt}$ has the following representations:

$$L' = \frac{L}{mc} = -\sqrt{Q_t^+ Q_t^-} = -\frac{Q_t}{\gamma} = -\frac{1}{\cosh\theta} = -\frac{1}{\sqrt{\zeta+1}}. \tag{151}$$

The derivatives of the differential forms of θ and ζ from Eq. (151) have the form

$$\frac{\partial L'}{\partial\theta}=\frac{\sinh\theta}{\cosh^2\theta}=\frac{\mathbf{n}\boldsymbol{\beta}}{E}, \tag{152}$$

$$\frac{\partial L'}{\partial\zeta}=\frac{1}{2(\zeta+1)^{\frac{3}{2}}}=\frac{1}{2\cosh^3\theta}=\frac{1}{2E^3}, \tag{153}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial\zeta}\right)=-\frac{3}{2}\frac{P^3}{E^4}. \tag{154}$$

The Lagrangian function of the integral of motion Q_t and the coordinate $q = -f\mathbf{nr}/c$ from Eq. (146) has the following form in the representation θ:

$$\frac{\partial L'}{\partial Q_t}=-\frac{1-Q_t}{\sqrt{Q_t^+Q_t^-}}=-f\sinh\theta=-f\mathbf{nP}, \tag{155}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t}\right)=-f\left(\sqrt{Q_\xi^+Q_\xi^-}-\sqrt{Q_t^+Q_t^-}\right)=-f\tanh\theta\sinh\theta=-f\boldsymbol{\beta}\mathbf{P}, \tag{156}$$

$$\frac{\partial L'}{\partial q}=-f\left(\sqrt{Q_\xi^+Q_\xi^-}-\sqrt{Q_t^+Q_t^-}\right)=-f\tanh\theta\sinh\theta=-f\boldsymbol{\beta}\mathbf{P}. \tag{157}$$

It follows from Eqs. (155) and (157) that

$$\frac{\partial Q_t}{\partial q}=\mathbf{n}\boldsymbol{\beta}. \tag{158}$$

Also, from Eq. (135), we have a partial differential form

$$\frac{\partial\theta}{\partial Q_t}=-f\cosh^2\theta. \tag{159}$$

From Eqs. (158) and (159) we have the following differential form:

$$\frac{\partial\theta}{\partial q}=-f\cosh\theta\sinh\theta. \tag{160}$$

The Lagrange equation of the second kind as a function $L' = L'(q, t, Q_t, \theta, \zeta)$ has the following form:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial\zeta}\right)\frac{\partial\zeta}{\partial\theta}+\frac{d}{dt}\left(\frac{\partial\zeta}{\partial Q_t}\right)\frac{\partial L'}{\partial\theta}-\frac{\partial L'}{\partial\theta}\frac{\partial Q_t}{\partial q}=0. \tag{161}$$

From Eqs. (134), (140) and (50) we have that the Hamiltonian of a free particle is

$$H' = Q_\xi\gamma = E = \cosh\theta. \tag{162}$$

Substituting Eqs. (162) and (3) into the integral of motion (134), we obtain a generalized integral of motion in the form

$$\gamma = H' - f\frac{\partial H'}{\partial\theta}. \tag{163}$$

Due to the fact that γ^+ and γ^- are invariants of motion different in direction and from the principle of relativity, we obtain

$$\gamma^+ = H' - \frac{\partial H'}{\partial\theta},\ \gamma^- = H' + \frac{\partial H'}{\partial\theta}, \tag{164}$$

$$\gamma^+\gamma^- = (H')^2 - \left(\frac{\partial H'}{\partial\theta}\right)^2 = 1, \tag{165}$$

where Eq. (165) expresses the law of conservation of energy in Hamiltonian form.

Using Lagrangian and Hamiltonian formalisms, it is convenient to introduce the following equation describing the conservation law by ζ:

$$2\frac{\partial H'}{\partial \zeta} + L' = 0. \tag{166}$$

The classical action function for a particle that has no charge has the form [23]

$$\frac{\partial S'}{\partial t} = -\sqrt{Q_t^+ Q_t^-} = L'. \tag{167}$$

As is known, in the Hamilton – Jacobi equation [23], when substituting Eq. (167), we have the following form:

$$H' + \frac{\partial S'}{\partial t} = Q_\xi \gamma - \frac{Q_t}{\gamma} = \cosh\theta - \frac{1}{\cosh\theta} = \hat{H}', \tag{168}$$

where $\hat{H}$ is the displacement Hamiltonian.

Differentiating Eq. (168) with respect to θ, we have

$$\frac{\partial \hat{H}'}{\partial \theta} = \sinh(\theta) + \frac{\tanh(\theta)}{\cosh(\theta)} = \mathbf{nP} + \frac{\mathbf{n\beta}}{E} = \frac{\partial L'}{\partial \theta} - f\frac{\partial L'}{\partial Q_t}, \tag{169}$$

and

$$\frac{\partial \hat{H}'}{\partial \theta} - \frac{\partial L'}{\partial \theta} = \mathbf{nP}. \tag{170}$$

Representing Eq. (168) as a function $\hat{H} = \hat{H}(\zeta)$, we obtain

$$\hat{H}' = \sqrt{\zeta+1} - \frac{1}{\sqrt{\zeta+1}} = \frac{\zeta}{\sqrt{\zeta+1}} = \frac{P^2}{\sqrt{P^2+1}} = \boldsymbol{\beta}\mathbf{P}. \tag{171}$$

Differentiating Eq. (171) with respect to ζ, we have

$$\frac{\partial \hat{H}'}{\partial \zeta} = \frac{\zeta+2}{2(\zeta+1)^{\frac{3}{2}}} = \frac{\cosh^2\theta+1}{2\cosh^3\theta} = \frac{\partial L'}{\partial \zeta} - \frac{L}{2}. \tag{172}$$

The relationship between derivatives (169) and (172) has the following form:

$$\sinh(2\theta)\frac{\partial \hat{H}'}{\partial \zeta} = \frac{\partial \hat{H}'}{\partial \theta}, \tag{173}$$

where

$$\sinh(2\theta) = \frac{\partial \zeta}{\partial \theta} = -f\frac{1}{2}\frac{\partial \theta}{\partial q}. \tag{174}$$

Thus, it is established mutually expressible between γ = γ(θ) and θ = θ(γ), where the explicit relationship θ depending on γ has the form

$$\theta = \arctan h\left[f\left(\frac{1-\gamma^2}{1+\gamma^2}\right)\right]. \tag{175}$$

Conclusions

In this work, based on the integrals of motion $\gamma = \gamma(\mathbf{r}, t, Q_t, \theta)$ and $\theta = \theta(\mathbf{r}, t, Q_t, \gamma)$, new relativistic forms have been obtained, which can be used to analyze the motion of particles, both in electromagnetic and gravitational fields.

From the problem on eigenfunctions and eigenvalues, the integral of motion was shown to have a relativistic invariant form, when taking the derivative with respect to time *t* and velocity **β**. A mutually invariant relationship between γ = γ(θ) and θ = θ(γ) was derived.

The generalized integral of motion γ = γ(θ) from Eq. (163) was obtained using the Hamiltonian formalism.

Thus, in the future, to analyze the motion of a relativistic particle in electromagnetic fields, not only the invariant form γ from Eq. (1) should be taken into account, but also the contribution of the scalar φ and vector **A** potentials, that is, the integral of motion should be considered as a function of $\gamma = \gamma(Q_t, \mathbf{A}, \varphi)$.

Relations of mutually invariant differential forms with generalized forms of motion were obtained. The Lagrange equation of the second kind was derived as a function of $L' = L'(q, t, Q_t, \theta, \zeta)$. The Lagrangian formalism showed the mutual correspondence between $L' = L'(q, Q_t)$ and $L' = L'(\theta, \zeta)$.

The functions of reverse energy and reverse momentum, the functions of pseudopotential energy and pseudopotential momentum were introduced. Their connection with the integral of motion γ was shown.

Using these conclusions, it is further interesting to generalize such problems as the motion of a charged particle in electromagnetic fields of various configurations in the presence of stationary and variable electric and magnetic fields and to study relativistic effects that occur when particles are accelerated and decelerated by electromagnetic fields. Our findings will be also useful in the future for the development of such areas as double special relativity and gravitomagnetism.

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