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SUPERCONDUCTIVITY

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INTRODUCTION

This tutorial is based on a course of lectures on "Superconductivity," that the author delivered over the years for senior students of the Institute of Physics, Nanotechnology and Telecommunications (IFNIT) in Peter the Great St.Petersburg Polytechnic University.

The increased interest in the physics of superconductivity in recent years is due to the discovery in 1986 high-temperature superconductors, which on one hand made it possible for cooling of the superconducting transition to forgo expensive liquid helium but replace it with cheap liquid nitrogen, while on the other hand, raised hopes of observing superconductivity at room temperature.

Experimental and theoretical studies of the physics of superconductivity have not only laid the foundation for creating superconductors with necessary technical properties but also have led to a better understanding of different branches of physics. These studies highlighted a number of phenomena, not directly associated with a loss of resistance, such as Meissner and Josephson effects, Shapiro steps, magnetic flux quantization, macroscopic coherence of wave functions, etc. They have given rise to new physical images and concepts: Cooper pairs, Abrikosov vortices, intermediate state, Shubnikov phase –just to name a few.

When creating this course the author has aimed to introduce future physicists to these original ideas and images that have enriched not only solid state physics, but all branches of fundamental physics.

All measurements are presented using the International System of Units (SI). In some graphs, taken from the original works, there are also the units used in electromagnetism CGS, in particular Gauss and Oersted. Their conversion to the SI is based on the ratio of 1 G =10-4 Tl; 1 Oe=80A / m. It should be borne in mind that most of the formulas in the CGS system have a different appearance.

CHAPTER 1. SOME BASIC FACTS

§1.1. The absence of electrical resistance

In 1908, the Dutch physicist Heike Kamerlingh Onnes managed to liquefy the last inert gas - helium. This opened up to him the opportunity to study the properties of materials at temperatures near absolute zero. The most interesting results were obtained in the study of electrical resistance.

There were many gaps in the understanding of the mechanism of conductivity at that time. However, it was known that charge transfer is caused by the movement of electrons. The temperature dependences of the electrical resistance of many metals were measured. It was found that at room temperatures, the resistance is directly proportional to temperature. It was also possible to come to the conclusion that at lower temperatures the resistance falls more slowly. In principle, one could assume three possible options:

1. When the temperature decreases the resistance gradually decreases to zero (Figure 1.1, curve 1).

2. Resistance tends to some finite value (Figure 1.1, curve 2).

3. Resistance passes through a minimum, and at very low temperatures becomes infinite. (Figure 1.1, curve 3).

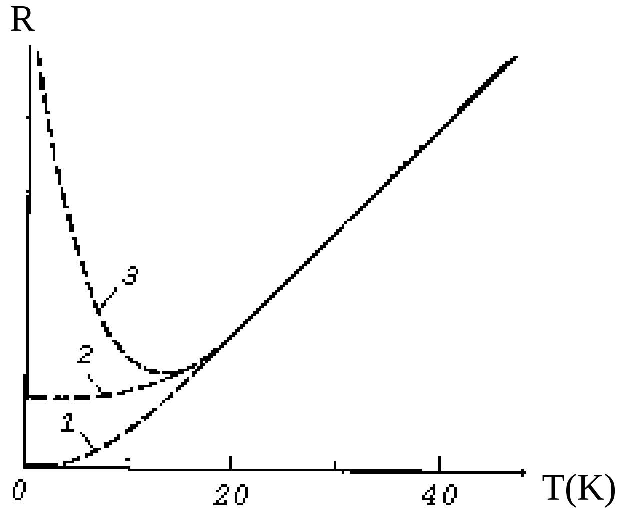


Fig. 1.1. The temperature dependence of the electrical resistance

The first option is based on the experimentally observed rapid decrease in resistance with cooling. The third option corresponds to the notion that at low temperatures, all electrons should have a foothold near their atoms and cease to be free. The second variant was confirmed by Onnes' experiments with different samples of platinum and gold (Fig.1.2), these metals at the time were in a sufficiently pure form. When the temperature approaches absolute zero the resistance tended towards the so-called residual value and depended on the purity of the sample. Onnes concluded that pure platinum and gold should have negligibly small resistance at temperatures close to absolute zero.

However, in 1911 when experimenting with mercury (it can be obtained in a purer form), he found that the observed effect had nothing to do with a gradual decrease in resistance with temperature - the change was abrupt (Figure 1.3). Onnes himself pointed out that the mercury moved to a new state and named it superconducting. The significance attached to this discovery, is evidenced by the fact that in 1913 Kamerlingh Onnes was awarded the Nobel Prize in Physics.

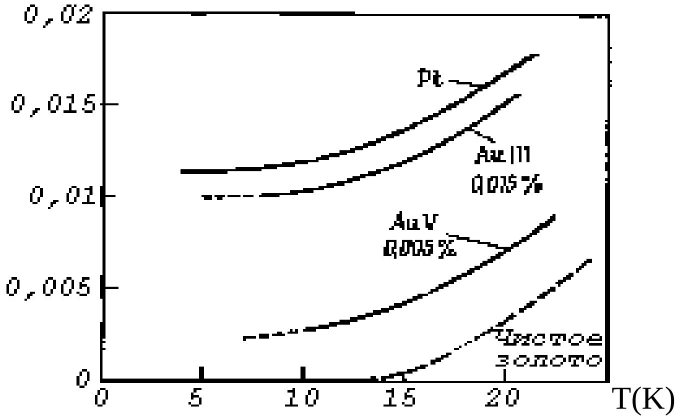


Fig. 1.2. The electrical resistance of various metals

Looking at Figure 1.3 brings forth a natural question: what is the value of the jump of resistance, in other words, to what extent it is correct to speak about the disappearance of electrical resistance?

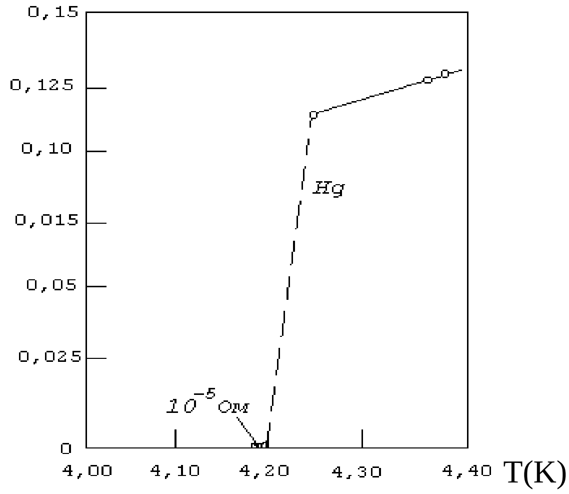


Fig. 1.3. The superconductivity of mercury

To answer this question it was necessary to find a sufficiently accurate method of measuring the resistance. In the first experiments, measurements were made on the basis of Ohm's law. Thus it was possible to consider only the fact that the resistance decreases abruptly, more than one thousand times, and becomes lower than the detection limit. In 1914 Onnes used a more precise method to measure extremely low resistance values. He measured the attenuation of the current in a superconducting ring. If the resistance exists, current should decrease with time due to the Joule losses.

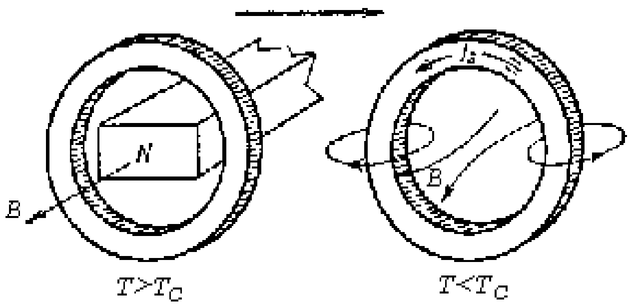


Fig. 1.4. The emergence of the persistent current in a superconducting ring.

The principle of the method is shown in Figure 1.4. Suppose that a ring of superconducting material (for example, lead) is in the normal state, i.e. it has a temperature above the transition temperature T_c. The magnet creates a magnetic field in a ring. Then the ring is cooled to a temperature at which it becomes superconducting. The magnetic field is not affected. Now remove the magnet. According to the law of electromagnetic induction an induction current arises in the ring. The speed of its decay allows one to find the magnitude of the resistance. A drop in the current of 1% per hour would correspond to a drop of 8 orders of magnitude in the resistance during the transition to the superconducting state. Nowadays, there are experiments with no change in the current for decades, which says that a drop in resistance is not less than 15 orders of magnitude. All these data allow us to legitimately assume that in the superconducting state, the electrical resistance really disappears.

Soon after the discovery of superconductivity in mercury, Onnes was able to show that other metals may become superconducting. Their transition temperatures turned out to be very low - a few Kelvins. For decades, scientists were searching for materials with higher transition temperatures. It turned out that a lot of metals, semiconductors and alloys have superconducting properties. However, the maximum found critical temperature was only 23 K - the alloy Nb_3Ge.

Theoretical studies began immediately after the discovery of the phenomenon. Scientists proposed different mathematical approaches to calculate the distribution of currents, magnetic field configuration, etc. But the microscopic theory explaining the nature of the phenomenon of superconductivity was created only 46 years after the discovery of the phenomenon. In 1957 American physicists Bardeen, Cooper and Schrieffer showed that at temperatures below the critical value, the conducting electrons are bonded in pairs and explained the nature of this bonding. Later we will discuss the general provisions of this theory - the so-called BCS theory, as well as consider other theoretical approaches which existed before its conception and continue to be useful today.

After the creation of the BCS theory, when the physical processes responsible for superconductivity became clear, experiments were begun for the creation of artificial materials with high transition temperatures. Substances with complex structures consisting of planes, one-dimensional filament structures and so forth were proposed. But all attempts were unsuccessful, although the problems of high-temperature superconductivity, as well as the creation of fusion reactors, was considered the most important applied problem of modern physics.

In 1986, when scientists had already begun to lose faith in the fact that high-temperature superconductivity could exist, there was an article by Johannes Georg Bednorz (Germany) and Karl Alexander Muller (Swiss) about the discovery of superconductivity in a new class of materials - ceramics - at a temperature of 35 K! It was a breakthrough; the critical temperature jumped 1.5 times after decades, when an increase of 0.1 K was considered a great success. The rapid investigation of a new class of substances ensued. Every month, each week brought new results: 40 K, 60 K, 90 K, 100 K. Nowadays, the reliable record critical temperature is 135 K.

One may ask why this phenomenon is called high temperature superconductivity, when temperatures lie between -150 and -180 C. In any case, for the existence of superconductivity materials have to be very much cooled. The important difference is that earlier cooling to the desired temperature could be achieved only by using helium, - as long as it remained liquid in desired temperature range. Helium is very expensive and its amount in nature is not very large. At temperatures above 77 K, i.e. the boiling point of nitrogen, liquid nitrogen can be used for cooling, and in nature there is a lot of nitrogen (let's recall the composition of the air), therefore it is very inexpensive.

We will mainly discuss issues concerning the ordinary (not high-temperature) superconductivity. The fact is that the theory of high-temperature superconductors (HTSC) is not yet established, due to the complexity of their crystalline structure. Most phenomena observed in ordinary superconductors, occur in high-temperature superconductors, so that the differences relate only to the temperature values. Therefore, the analysis of these phenomena in conventional superconductors allows one to enter into a wide array of problems and to understand the nature of the processes. HTSC and specific phenomena occurring in them will be discussed only in the last chapter of the book.

§1.2. Expelling the magnetic field from superconductors.

The magnetic properties of superconductors are as nontrivial as the electric ones. In 1933, Meissner and Ochsenfeld found that a superconductor in a magnetic field behave as a perfect diamagnet, inside which the magnetic induction is zero. In other words, the magnetic field is expelled from bulk superconductors. This is due to the appearance of the screening currents on the surface whose magnetic field completely compensates the external field throughout the volume of the sample. This phenomenon is called the Meissner effect.

At first glance it may seem that the perfect diamagnetism of superconductors is a consequence of zero resistance. Indeed, if we carry a sample in a magnetic field, as a result of electromagnetic induction, the induction currents arise. In normal metals they would decay with time because of Joule heating. In superconductors the resistance is zero and the currents do not decay with time and continue to provide further screening.

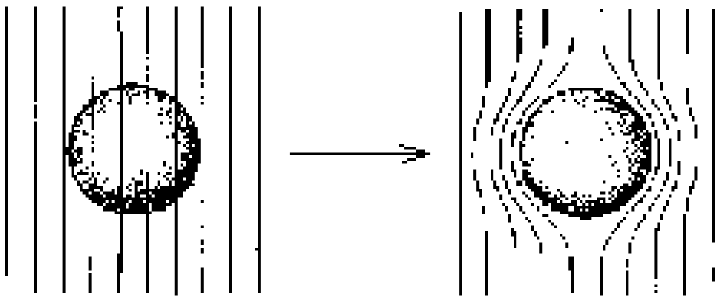


Fig. 1.5. Meissner effect in a superconducting ball, cooled in an external magnetic field

However, this explanation is not always applicable. Consider the case when the sample is carried into the magnetic field above the critical temperature, i.e., it does not possess superconducting properties. Then the lines of induction permeate it, as shown in Figure 1.5a. Now, if we cool the sample below the transition point T_C, the induction lines should be pushed out of it (Figure 1.5b). This important result can not be obtained by simply assuming the resistance to be zero. Ohm's law, $\vec{E} = \rho \vec{j}$, shows that when $\rho = 0$ an electric field is absent. From Maxwell equation $\partial \vec{B} / \partial t = rot \vec{E}$, it follows that the magnetic field should remain constant and can not be changed during the transition to the superconducting state. The Meissner effect contradicts this result, which gives reason to believe that the perfect diamagnetism and the lack of resistance are essentially two independent properties of the superconducting state.

§1.3. The destruction of superconductivity by a magnetic field

Superconductivity is destroyed by sufficiently strong magnetic field. The threshold, or critical magnetic field H_C, necessary to destroy superconductivity, depends on temperature. Figure 1.6 shows the dependence of the critical field on temperature for some superconductors. At the critical temperature, T_C, the critical field, H_C, is zero. With decreasing temperature it increases, and for the sample in the shape of a long cylinder is approximately described by the relation

$$H_C(T) = H_C(0)\left(1 - \frac{T^2}{T_C^2}\right) \qquad (1.1)$$

The difference between the free energy per unit volume in the normal and superconducting state can be obtained from the following considerations. The superconducting sample in an external magnetic field, H_e, less than critical H_C (the external field is the field generated by external sources in the absence of a superconducting medium), is in the Meissner state where, due to the screening currents, the external magnetic field is expelled from it. This means that, according to the principle of superposition, the magnetic field created by screening currents at all points is exactly equal to the external field and is directed opposite to it. The energy density of this field is $\mu_0 H_e^2/2$, and the total free energy per unit volume is $F_S + \mu_0 H_e^2/2$. When the external field is equal to the critical H_C, the sample returns to the normal state, because, at this point, the energies of the superconducting state and the normal state are equal, i.e., $F_N = F_S + \mu_0 H_C^2/2$, which implies

$$F_N - F_S = \mu_0 H_C^2/2 \qquad (1.2)$$

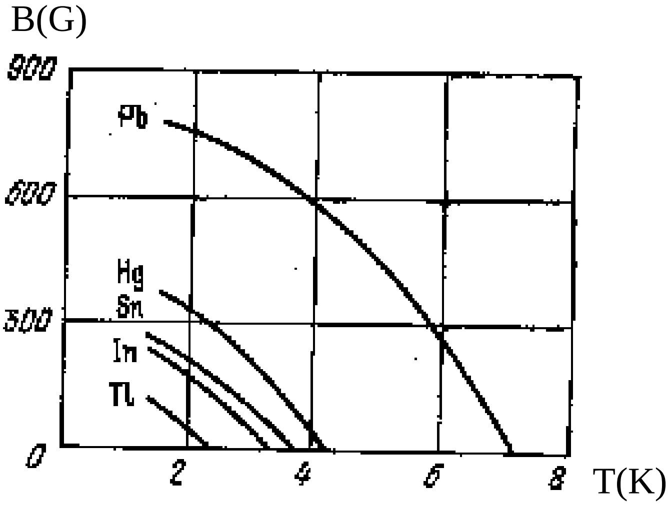


Fig. 1.6. Field dependence of the critical temperature for some superconductors.

§1.4. Type of phase transition

In the absence of the external magnetic field the superconducting transition is a phase transition of the second type, i.e., it occurs without the liberation or absorption of heat. At the same time, in full accordance with the theory of this transition, the specific heat at the transition point is discontinuous.

The situation is different if the process takes place in an external magnetic field. In this case, during the transition from the superconducting state to the normal some heat must be absorbed and vice versa. In other words, in a magnetic field a phase transition is of the first type.

§1.5. Three types of superconductors

The behavior of superconductors in a magnetic field allows one to divide them into three main types. It should be noted that behavioral differences in a magnetic field testify to essential distinctions in the physics of the microprocesses happening within samples.

In type I superconductors the Meissner state, when the magnetic field is pushed out from the volume of the superconductor and is other than zero only in a thin near-surface layer, exists up to some critical field H_C. If the external field exceeds this value, the sample passes into the normal state.

The effect of pushing out a magnetic field from a sample can be presented as follows. The shielding currents completely compensating an external magnetic field in a sample give it a magnetic moment. Formally we can speak about the magnetization, $\vec{M}$, being equal to the magnetic moment per unit volume of a sample. The magnetic induction in a sample is defined by the expression $\vec{B} = \mu_0(\vec{H}_e + \vec{M})$. Often the behavior of a superconductor is characterized by the dependence of magnetization M on an external magnetic field. Such a curve for a type I superconductor is shown in fig. 1.7.

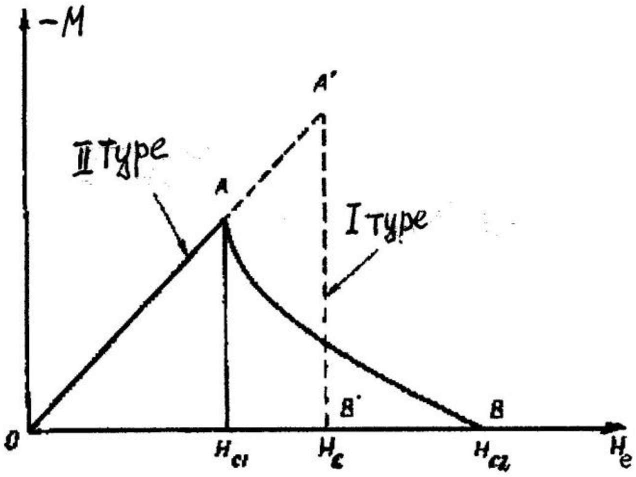


Fig. 1.7. Curves of magnetization of I and II type superconductors having a shape of a long cylinder in a longitudinal field

In the same figure, the continuous line represents a curve of magnetization of a type II superconductor. In this case, the value H_C corresponds to the equality of free energies of the normal and superconducting states, i.e. this is the value of the external field at which it would be energetically favorable for the sample to pass into its normal state. Comparing curves, we can see that in type II superconductors there exists some critical value H_{C1} (less than H_C) such that if the external magnetic field exceeds this value it starts penetrating the sample. With further increase of the magnetic field, the magnetic moment of the sample gradually decreases, i.e. the field gets into the sample more strongly. When the external field H_e achieves the value H_{C2}, greater than H_C, the magnetic moment becomes equal to zero, i.e. the external magnetic field completely suppresses superconductivity. Thus, it is possible to draw the conclusion that the behavior of type II superconductors isn't guided by simple energy reasons as type I superconductors are.

In type III superconductors, or as they are also called, rigid type II superconductors, the curve of magnetization has completely different appearance (a curve 2 in Fig. 1.8). The hysteretic character of the curve is obviously visible. After the removal of the external field the magnetic flux remains “frozen” in the sample.

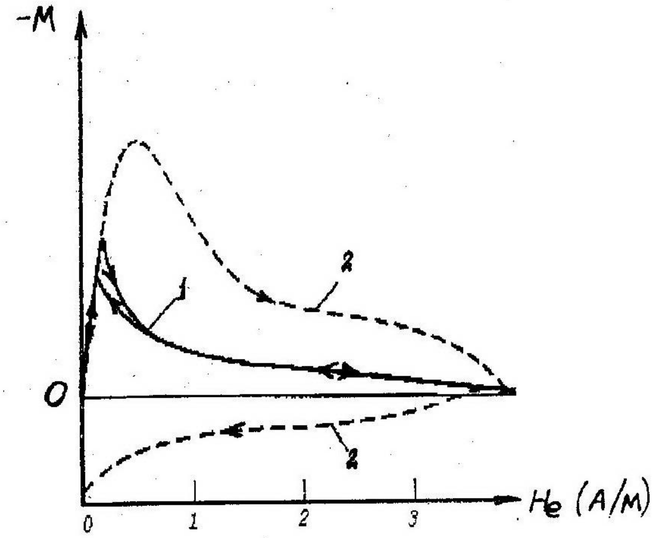


Fig. 1.8. Curves of magnetization of an alloy $Nb_{0,55}Ta_{0,45}$
1 - a well annealed sample, 2 - a sample with a big amount of structural defects.

Crystal lattices of type III superconductors contain a large amount of defects which obstruct the movement of vortex threads (we will in detail talk about them later). At careful annealing these defects can be eliminated and the curve of magnetization becomes almost reversible and corresponds to a type II superconductor (the curve 1 in Fig. 1.8).

§1.6. Energy gap

Various experiments, such as the tunneling effect, absorption of light and ultrasound, etc., show that upon transition of a substance to its superconducting state, a gap arises in its energy spectrum. The value of the gap is related to the critical temperature by the following approximate formula

$$E_g = 2\Delta \approx 3{,}5kT_c, \qquad (1.3)$$

where Δ is the so-called half-width of the gap.

We will discuss this situation in more detail.

At first, let’s consider a normal metal. In the main state at $T = 0$, electrons fill all states in Fermi sphere. In order to get to an excited state, it is enough to move one electron from an originally occupied state ($k \le k_F$) into an empty one ($k' > k_F$). Thus two quasi-particles are formed – an electron with a momentum $k' > k_F$ and a hole in the place where it was earlier. It is natural to measure the energy of excitations (quasi-particles) from Fermi energy:

$$\xi_{k'} = \frac{\hbar^2 (k'^2 - k_F^2)}{2m} \approx \frac{\hbar^2}{m} k_F (k' - k_F) \quad \text{at} \quad k' > k_F \qquad (1.4)$$

$$\xi_k = \frac{\hbar^2 (k_F^2 - k^2)}{2m} \approx \frac{\hbar^2}{m} k_F (k_F - k) \quad \text{at} \quad k < k_F \qquad (1.5)$$

If both moments lie close to the Fermi surface, the energy $\xi_k + \xi_{k'}$ necessary for their creation is small. In other words, in a metal an excitement can exist with an arbitrarily small energy. The dependence $\varepsilon_k(k)$, described by (1.4) and (1.5), is shown in Fig. 1.9 by straight lines.

In a superconductor the situation is different. Formulas (1.4) and (1.5) are already unsuitable. The energy necessary for the creation of a couple of excitations has to exceed some value called "the energy gap", and the energy of each of the arisen two excitations is described by a formula

$$\varepsilon_k = (\xi_k^2 + \Delta^2)^{1/2} \qquad (1.6)$$

and can't be less than a half-width of a gap Δ. The dependence $\varepsilon_k(k)$ described by (1.6) is shown in Fig. 1.9.

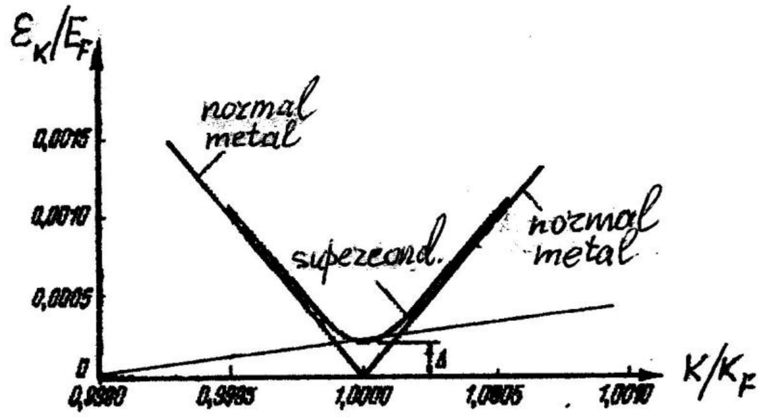


Fig. 1.9. Energy of excitements in normal and superconducting states as function of a wave vector.

Let us consider a crystal lattice with a mass, M. The superconducting current can be considered to be the collective movement of electronic gas in the lattice. It is possible to tell that the lattice moves with a velocity $\vec{v}$ relative the electronic gas. "Friction" will reduce this velocity, only if in this gas some excitements arise and the kinetic energy of the lattice turns into their energy. Let there be one excitement with energy E_k and momentum $\hbar\vec{k}$. From the conservation laws of energy and momentum we have

$$\frac{1}{2}Mv^2 = \frac{1}{2}Mv'^2 + E_k \ , \ M\vec{v} = M\vec{v}' + \hbar\vec{k} \qquad (1.7)$$

From these two formulae we obtain

$$0 = -\hbar\vec{k}\cdot\vec{v} + \frac{\hbar^2 k^2}{2M} + E_k \qquad (1.8)$$

Believing the mass of a crystal, M, infinitely large, we will come to the conclusion that there is a minimum critical value of velocity at which the condition (1.8) can be satisfied

$$v_c = \min\left(\frac{E_k}{\hbar k}\right) \quad (1.9)$$

If a gap exists within the spectrum of excitations then $E_k > 0$, and therefore $v_C > 0$. The inclined straight line in fig. 1.10 has the slope $\hbar k_F v_C / \varepsilon_F$. Thus, in a superconductor the currents with velocities less than v_C proceed without energy loss, i.e. without attenuation. Knowing v_C, we can calculate the critical current density. It can be considerable.

§1.7. One-particle tunneling

Research of the tunnel effect gives important information about the energy spectrum of carriers of current. The features of this spectrum in superconductors noted in the previous paragraphs couldn't but affect the tunnel characteristics. Analyzing the results of tunneling experiments, Norwegian physicist I. Giaever, in 1961, for the first time, proved the existence of a gap in an energy spectrum of superconductors, for which, in 1973, he was awarded a Nobel Prize in physics.

The technique is based on the supervision of a tunnel current through a thin nonconducting layer dividing two samples. The quantity of the electrons passing through a barrier depends on the number of electrons falling on the barrier, probabilities of tunneling and number of free states on another party of a barrier. We will exclude probability of tunneling from the analysis because it depends on barrier parameters, and not on the characteristics of the samples. The value of the tunneling current will then be defined by the density of occupied states on one side of the barrier and the density of free states on the other.

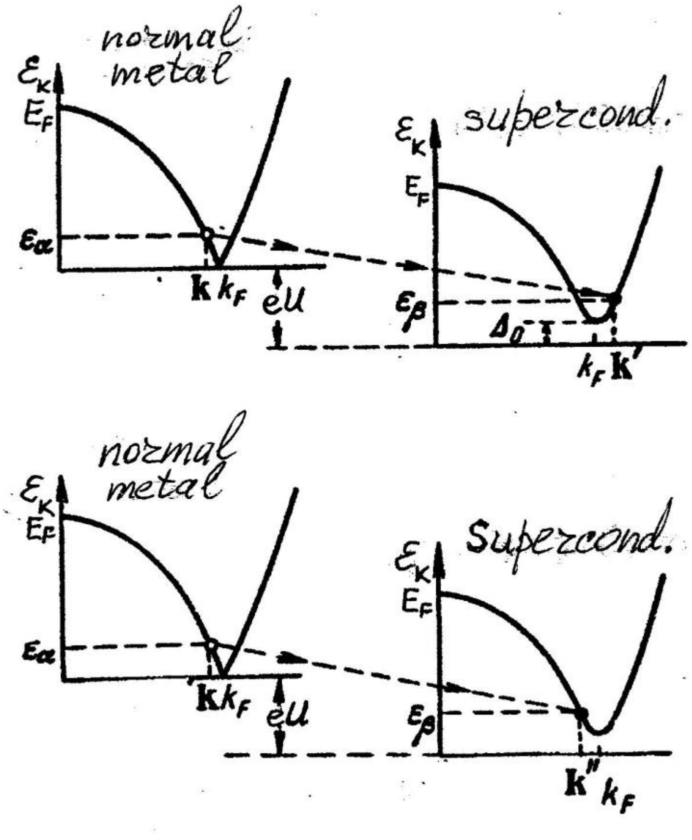


Fig. 1.10. One-particle tunneling between a normal metal and a superconductor.

In Fig. 1.10 we can see the one-particle tunneling between a normal metal and a superconductor when a voltage, U, is applied between them. We note that far from k_F the curve is defined by the formula $\varepsilon_k = \left| E_F - \hbar^2 k^2 / 2m \right|$. In the top drawing, an electron, which was in a state $\vec{k}\uparrow$ below the Fermi surface in a normal metal, tunnels through an oxide film into a state $\vec{k}'\uparrow$ above the Fermi surface in a superconductor. Thus, at the left there is a hole corresponding to the energy of the excitement, ε_α. Additionally, placing a particle in a state $\vec{k}'\uparrow$, we receive in the superconductor an excitement with the energy $\varepsilon_\beta = \sqrt{\xi_{k'}^2 + \Delta^2}$.

Another process is shown in the lower drawing, it coincides with the previous one except that the quasi-particle is initially located in a state $\vec{k}''\downarrow$ below the Fermi level.

Both of these processes can occur if energy is conserved: $\varepsilon_\alpha + \varepsilon_\beta = eU$, i.e. at $U > \Delta / e$. At the contact of two normal metals the current exists at arbitrarily small voltage.

The V-A characteristics for metal-metal and metal-superconductor are shown in Fig. 1.12 (curves 1 and 2).

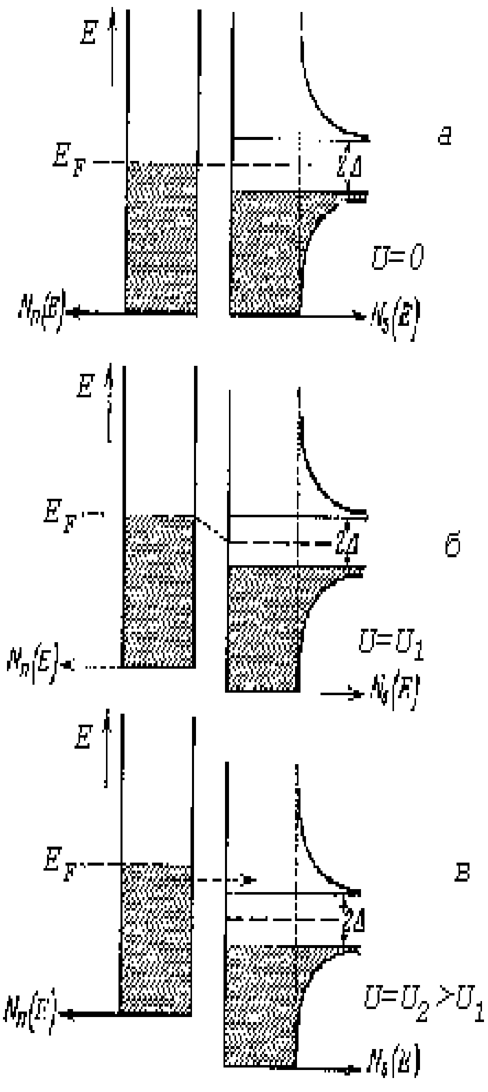


Fig. 1.11. Tunneling between a normal metal and a superconductor at T=0 (semiconductor model).

When considering the phenomenon of tunneling, "the semiconductor model" of the excitation spectrum is often useful (Fig. 1.11). However, it is necessary to use it with care since the states "above the energy gap" in this model are actually linear combinations of the quasi-particle states above and below the Fermi surface. As we will see later, the states of single electrons also exist inside the gap. In the case of semiconductors, one-particle states are absent inside the gap.

In this model, it is possible to discribe the events as follows. In Fig. 1.11a we can see a contact at zero voltage. The occupied states are shown by shading. The density of states is plotted on the horizontal axis. The equilibrium state is established at identical Fermi levels in both parts. The transition of electrons from one part into the other is absent. The general current is equal to zero. Up to voltage $U = \Delta/e$ the tunnel current is absent as electrons belonging to the normal metal can't find suitable states in the superconductor. At $U = \Delta/e$ the current abruptly increases on a vertical tangent (the curve 2 in Fig. 1.12). This sharp rise is caused by the high density of states within the superconductor. As the voltage increases the curve approaches that of the tunneling characteristic between two normal metals (the curve 1). At nonzero temperatures, in metal there is a quantity of electrons with energy higher than the Fermi level, plus the gap in the superconductor decreases. In this case the curve assumes an air of the curve 3 (Fig. 1.12).

The V-A graph for contact of two superconductors is schematically shown in Fig. 1.13.

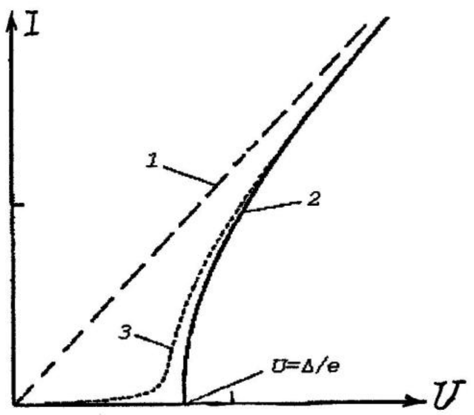


Fig. 1.12. Volt-Ampere characteristics of tunnel contacts.
1 - normal metal / normal metal; 2 - normal metal / superconductor, T=0;
3 - normal metal / superconductor, 0<T<T_c.

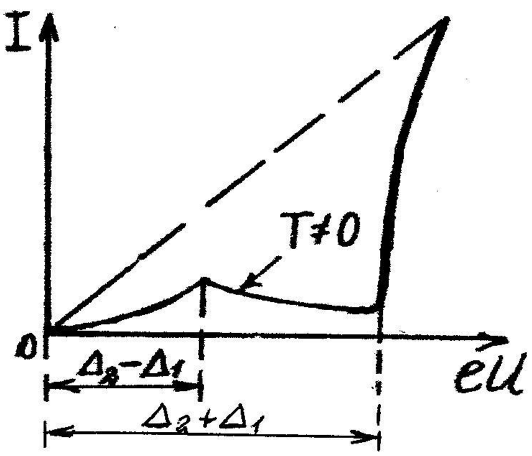


Fig. 1.13. Tunneling between two superconductors.

§1.8. Stationary and non-stationary Josephson effects

In the previous paragraph we considered tunneling of individual electrons through an insulating layer. But, as we will see later, in a superconductor electrons exist as Cooper pairs. Therefore, it is natural to assume that, in the case of a contact between two superconductors, pairs of electrons can tunnel through a rather thin layer of dielectric. B. Josephson was the first to consider this effect in 1962. For these works, in 1973, he was awarded the Nobel Prize. He showed that the tunneling of Cooper pairs becomes essential at a thickness of barrier of 10-20 angstrom. He also predicted some unusual and interesting phenomena taking place when electrons tunnel in pairs. Subsequently all his predictions were excellently confirmed by experiments. Apart from their basic role in the understanding of superconductivity, the Josephson effects (as it is accepted to call this group of phenomena) give opportunities for carrying out the most precise measurements. We will emphasize that they play a particularly important role in the processes occurring in the high temperature ceramic superconductors (HTSC) because, in them, Josephson contacts already exist naturally (contacts between granules). For this reason these substances are sometimes called Josephson media.

The stationary effect of Josephson is a percolation of not fading superconducting current through a thin insulating layer at a zero voltage at the contact. The magnitude of such a current can not exceed some critical value, I_C.

The non-stationary Josephson effect appears at a nonzero voltage, U_S, at the contact. In this case, a high-frequency alternating current percolates through the contact which frequency, ν, is proportional to the voltage:

$$\nu = \frac{2eU_S}{h} \qquad (1.10)$$

To understand a practical situation, consider the circuit represented in Fig. 1.14. When a constant superconducting current flows through the contact (stationary effect of Josephson) the voltage at the contact is equal to zero, i.e. all U_S falls on the resistance R. This situation can exist if the current (equal to U_S/R) does not exceed the critical value I_C. Thus, the stationary effect of Josephson takes place if $U_S < I_C R$. If $U_S > I_C R$, the generation of a high-frequency current begins. Then the mathematical description of the circuit becomes very difficult.

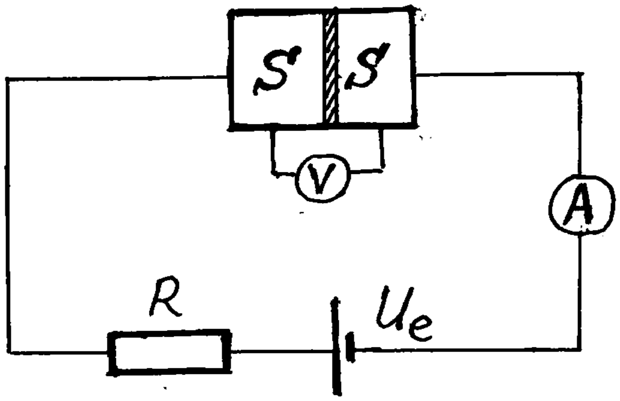


Fig. 1.14. The scheme for demonstration of Josephson effects.

§ 1.9. The magnetic flux quantization

Let us consider a superconducting ring. It is possible to generate a current within it by electromagnetic induction (for example, as shown in Fig. 1.4). This current will remain perpetually constant. It would seem that adjusting the magnetic field magnitude we can receive any value of the induced current. It would seem that the corresponding selection of a magnetic field magnitude gives the chance to receive any value of the induced current. However, it is not so. The current in the ring can only have discrete values. This situation was formulated clearly by F.London. He came to the conclusion that the magnetic flux penetrating a superconducting ring has to be equal to an integer of a so-called magnetic flux quantum, Φ_0. The situation is similar to the Bohr model of an atom in which the possible electronic states correspond to the values of angular moments equal to an integer of Planck constants.

The magnetic flux quantum, Φ_0, according to London, is equal to h/e, where h is the Planck constant, and e - the elementary charge. This conclusion was based on the assumption that an electric current is transferred by individual electrons. However, subsequently, it appeared that the current is transferred by Cooper pairs, i.e. particles with a charge of 2e. Therefore, the magnetic flux quantum is halved:

$$\Phi_0 = \frac{h}{2e} = \frac{\pi\hbar}{e} \approx 2 \cdot 10^{-15} Wb \qquad (1.11)$$

The predictions of London were excellently confirmed by experiment.

It is important to note that the condition of flux quantization formulated above is applicable regardless of, whether the magnetic flux penetrating the ring is created by an external magnetic field or by the current in the ring itself. In the presence of an external field, the superconducting currents in the ring will be distributed so that the total magnetic flux through the ring is equal to an integer of flux quanta.

The magnetic flux quantization, as well as the Josephson effects, considered in the previous paragraph, are a consequence of the so-called "phase coherence" of all Cooper pairs. From the point of view of quantum mechanics all pairs are in one quantum state, i.e. are coordinated among themselves in all physical parameters, in particular, in phases. This phase correlation extends over very large (practically unlimited) distances. Thus, all these effects are purely quantum phenomena, but, unlike the majority of such phenomena which manifest in a microcosm (atoms, molecules, etc.), they take place in macroscopic systems.

§1.10. Isotopic effect

In search of an explanation of the effect of superconductivity experimenters investigated the dependence of the critical temperature on various parameters. In particular, the question of, whether a crystal lattice influences superconductivity or if it is only dependent on its system of electrons, started being investigated in 1922 by Onnes. An ingenious idea was the basis for research: various isotopes of the same element, having different masses, have the same electronic structure. Therefore detection of the dependence of the critical temperature on a type of an isotope would prove that the lattice also participates in the creation of the phenomenon of superconductivity. The initial experiments didn't show such dependence, but as a result of development in physics, there appeared the possibilities of receiving, in sufficient quantities, isotopes with noticeably differing masses. In 1950, such a dependence was discovered by several groups of physicists. A change in the mass of atom of an isotope of mercury from 199.5 to 203.4 a.m.u. the critical temperature changed from 4.185 K to 4.146 K.

Already the elementary qualitative arguments made by Fröhlich allowed one to expect that the critical temperature had to be inversely proportional to the square root of an atom's mass:

$$T_c \sim M^{-0,5} \qquad (1.12)$$

Later the theory of BCS confirmed this result. However, accounting for thinner effects can lead to deviations from this formula.

The discovery of the isotopic effect has confirmed the influence of fluctuations of the lattice on superconductivity and has directed research in the theory of superconductivity to electron - phonon interactions that, finally, led to the creation of BCS theory.

§1.11. Application of superconductors

The questions of probable uses of superconducting materials began to be discussed practically right after the discovery of the phenomenon of superconductivity. Already Kamerling Onnes considered thatm, by means of superconductors, it was possible to create economic installations for receiving strong magnetic fields. However, actual uses of superconductors began in the late 1950s−early 1960s. Now, superconducting magnets of various sizes and shapes exist. Their application is beyond purely scientific researches and today they are widely used in laboratory practice, - accelerating equipment, tomographs, installations for the operated thermonuclear reaction. By means of superconductivity it has become possible to sufficiently increase the sensitivity of many measuring devices. Such devices are called SQUID (from English "Superconducting Quantum Interference Devices"). It is especially necessary to emphasize the introduction of SQUID in techniques and in modern medicine.

Today, the greatest application of superconductors is in the creation of strong magnetic fields. The modern industry makes, from type II superconductors, the various wires and cables used in the production of windings of superconducting magnets. This results in the creation of magnets capable of producing much stronger fields (more than 20 T) than their iron magnets counterparts. Superconducting magnets are also more economic. For example, for the maintaining a magnetic field of 100 kG in a 10 cm long copper solenoid (coil) with an internal diameter of 4 cm, an electric power not less than 5100 kW is required. And all the generated heat has to be taken away with the water cooling. It means that through the magnet, it is necessary to pump more than 1 m^3 of water per minute, and then still to cool water down. On the other hand, for a superconducting magnet to create a magnetic field of similar strength only the construction of the helium cryostat is necessary for the cooling of the windings, but that is a simple technical task.

Other advantage of superconducting magnets is that they can work autonomously, without external sources.

One more application of superconductors is the creation of bearings and supports without friction. If we place a superconducting sphere over a metal ring with current, due to the Meissner effect, on the surface of the sphere a superconducting current appears. As a result, some forces of repulsion between the ring and the sphere emerge, and the sphere can hover over the ring. A similar effect can be observed if a permanent magnet is placed over a superconducting ring. The creation, for example, of a new means of transport based on this effect is one possible application. It permits for the creation of a train on a magnetic pillow in which there are no energy losses due to friction on a road track. A 400 m long model of such a superconducting road was constructed in Japan in the 1970s. Calculations show that a train on a magnetic pillow will be able to gather speed of up to 500 km/h. Such a train will "hover" over the rails at a height of 2–3 cm, which will give it the chance to move with such speed.

Today, superconducting volume resonators with Q-factor of $5 \cdot 10^{11}$ are widely used.

The use of superconductivity can lead to the creation of superfast electronic computers. It is about so-called cryotrons – the switching superconducting elements. Such devices can easily be combined with the superconducting memorable elements. An important advantage of cryotrons in comparison with ordinary semiconductor devices is the absence of a need for energy in a steady state. After creation of Josephson contacts it was offered to replace cryotrons by them, and it appeared that time of switching of such systems makes about 10^{-12} seconds. It opens up wide prospects for creation of the most powerful computers.

The most promising areas for the wide use of high-temperature superconductors are in cryoenergetics and cryoelectronics engineering. In cryoenergetics a method of manufacturing sufficiently long (up to several kilometers) wires and cables based on bismuth HTSC materials has been developed. It is already enough for production of small engines with a superconducting winding, superconducting transformers, inductance coils etc.

In cryoelectronics the technique of producing SQUID films which, according to the characteristics, practically are not worse than helium analogs is developed. The technique of producing perfect magnetic screens from HTSC, particularly, for research in biomagnetic fields has been mastered. Many different electronic devices, such as antennas, transferring lines, resonators, filters, frequency mixers etc. are created with use of HTSC.

CHAPTER 2. THEORIES OF SUPERCONDUCTIVITY

§2.1. Bardin-Cooper-Schrieffer (BCS) theory

2.1.1. Main ideas and results.

As stated in chapter 1, the microscopic theory explaining the nature of the phenomenon of superconductivity was created in 1957 only, in 46 years after the discovery of superconductivity.

The main result on which this theory was based - the effect of grouping the electrons in pairs - was proposed by L. Cooper in 1956.

Cooper considered behavior of two electrons that are attracted to each other when all other electrons form the main state, i.e. like in a normal metal, according to the Pauli principle, fill the Fermi sphere.

Calculation showed that in this case the behavior of the two interacting electrons sharply differs from the behavior of these electrons if they are isolated from the others. In the presence of a filled Fermi sphere, at arbitrarily small attraction, these two electrons form the bound state which has a lower energy and is separated from the main state of the normal metal by an energy gap. The connected pair possesses the lowest energy (i.e. a gap is maximal) when the electrons have anti-parallel spins and equal, but oppositely directed momenta.

Bardin, Cooper and Schrieffer generalized Cooper's results for the case when <u>all</u> electrons are connected in pairs. Then each electron plays a double role. On one hand, owing to Pauli principle it creates restrictions on the possible values of wave vectors of other electrons that gives them the chance to be grouped in pairs. On the other hand, the electron itself is a part of one of the pairs.

Thus, for an explanation for the grouping of electrons in pairs it is necessary to find a possible reason for this attraction between electrons. The theoretical analysis showed that this attraction can be realized due to an exchange of phonons, i.e. due to the interaction of electrons with a crystal lattice. Such an interaction can be presented as follows. In the nodes of a lattice there are positively charged ions. An electron attracts them to itself. Thus, in the area surrounding the electron there is the polarization of the lattice which is expressed in a congestion of the positive charges. The second electron, which is nearby, is attracted to this build up, and therefore to the first electron. Considering a picture in dynamics, it is possible to say that one electron, in the process of its movement, creates a path on which it is favorable for a second electron to move. It explains why most obviously the effect is shown for oppositely moving electrons - each of them moves on the track left by the other one.

At first glance, there can be doubts as to what the polarization can do. It is possible to understand that it reduces a force of repulsion, but can it replace repulsion with an attraction? For an assessment of its opportunities we consider the interaction of a point charge with a not charged sphere. It is clear that the polarization connected with the redistribution of charges on the sphere will lead to an attraction. For compensation of this force it is necessary to place on the sphere some charge, of the same sign as the point charge. Since the charge of the sphere is less than this value, an attraction of two electric charges of one sign takes place!

According to the theory of BCS, the half-width of the energy gap at zero temperature is defined by the expression

$$\Delta(0) = 2\hbar\omega_D \exp\left(-\frac{1}{U \cdot N(E_F)}\right), \tag{2.1}$$

where $U>0$ – the potential of electron-lattice interaction, $N(E_F)$ - the density of electronic states at the level of Fermi, ω_D - the Debye frequency of the crystal.

From expression (2.1) it is clear, why the theory of superconductivity took so long to b created. This expression can't be expanded in a power series with reference to small interaction U. Therefore the perturbation theory, which is usually used at calculation of changes in an electronic energy spectrum, couldn't lead to the correct results, namely the emergence of a gap.

From (2.1) and (2.2) an interesting conclusion can be made. In normal metals, the stronger the interaction of electrons of conductivity with a lattice, i.e. the higher U, the higher the resistance. In superconductors, the greater U, the higher the critical temperature. Thus, the greater a metal’s resistance in its normal state, the more easily it passes into the superconducting state. However this regularity takes place only for metals with a comparable concentration of electrons.

From detailed calculations it follows that the critical temperature is connected with the half-width of an energy gap by the formula

$$3{,}5 k_B T_C = 2\Delta, \tag{2.2}$$

where k_B is the Boltzmann constant. (2.2) confirms the experimentally deduced equation (1.3).

The BCS theory shows that in order to create in a superconductor two unconnected electrons, i.e. two excited states, it is necessary to break a Cooper pair, i.e. to spend a minimum energy of 2Δ. It means that a minimum energy of one excitement (quasi-particle) is equal to Δ. Detailed calculation gives an expression for the energy of a quasi-particle with a momentum p to be

$$\varepsilon_p = \sqrt{\left(\frac{p^2}{2m} - \varepsilon_F\right)^2 + \Delta^2}, \tag{2.3}$$

where ε_F - Fermi energy.

Theoretically derived expression (2.3) coincides with formula (1.6) on the basis of which the existence of perpetual superconducting currents was proven in chapter 1. The same fact can be explained in a different way. Unlike electrons, which are fermions, i.e. having half-integer spins, a Cooper pair is a new particle having spin equal to zero. Particles with integer spins are called bosons and governed by Bose-Einstein statistics. For them, there is no Pauli ban. Moreover, all bosons seek to be in the same state. There is a so-called Bose-condensation - all Cooper pairs drop out in "condensate", i.e. have all identical parameters. In particular, all Cooper pairs have an identical momentum. It would seem that there is nothing special in this fact, because every pair contains two electrons with oppositely directed momenta and therefore the momentum of each pair is equal to zero. However the situation changes if the entire set of pairs starts moving, for example, in an electric field. All pairs seek to have an identical momentum. It means that none of them can be braked, thereby transferring energy to the lattice, i.e. the transfer of a charge through a lattice goes without resistance.

The BCS theory explains the electronic spectrum of superconductors, on the basis of which it is possible to predict practically all features of behavior of superconductors. It should be noted that in the case of "low-temperature" superconductivity there is not only qualitative, but also quantitative consistency of the theory and experiment. In the frame of this theory many properties of high-temperature superconductors can be explained as well, though the quantitative match is not so good.

2.1.2. Cooper effect. Cooper pairs.

Let us consider the interaction of two electrons at completely filled Fermi sphere.

We will apply the concept of quasi-particles used in normal metals to this case. Electronic states near the Fermi level are similar to usual particles. Therefore it is natural to count the energy from the Fermi level. We have already spoken about it in §1.6.

We write down Schrödinger's equation for two interacting quasi-particles with identical $|\vec{k}|$:

$$\left(H_0(\vec{r}_1) + H_0(\vec{r}_2) + U(\vec{r}_1;\vec{r}_2)\right)\Psi(\vec{r}_1;\vec{r}_2) = E\Psi(\vec{r}_1;\vec{r}_2) \qquad (2.4)$$

Here $H_0(\vec{r}_1)$ is the Hamiltonian of one free particle

$$H_0(\vec{r}_1)\psi_{\vec{k}}(\vec{r}_1) = |\xi_{\vec{k}}|\psi_{\vec{k}}(\vec{r}_1),$$

where the wave function for a free particle has the form

$$\psi_{\vec{k}}(\vec{r}_1) = V^{-1/2}\exp(i\vec{k}\vec{r}_1).$$

In the main state the total momentum and the spin have to equal to zero. Therefore we construct a wave function in the form

$$\Psi(\vec{r}_1;\vec{r}_2) = V^{-1/2}\sum_{\vec{k}} c_{\vec{k}}\psi_{\vec{k}\uparrow}(\vec{r}_1)\psi_{-\vec{k}\downarrow}(\vec{r}_2) \qquad (2.5)$$

Substituting (2.5) into (2.4), we obtain

$$2|\xi_{\vec{k}}|c_{\vec{k}} + \sum_{\vec{k}'} U_{\vec{k}\vec{k}'}c_{\vec{k}'} = Ec_{\vec{k}} \qquad (2.6)$$

We consider the simple model:

$$U_{\vec{k}\vec{k}'} = \begin{cases} -\lambda & if \quad k_F - \omega_D / v_F < k, k' < k_F + \omega_D / v_F \\ 0 & beyond\ the\ range \end{cases} \tag{2.7}$$

Introducing the notation: $$I = \sum_{|\vec{k}'| = k_F - \omega_D / v_F}^{|\vec{k}'| = k_F + \omega_D / v_F} c_{\vec{k}'} \tag{2.8}$$

we obtain $$c_{\vec{k}} = \frac{\lambda I}{2|\xi_{\vec{k}}| - E}. \tag{2.9}$$

Substituting (2.9) into (2.8), we come to an equation of self-consistence

$$I = \sum_{|\vec{k}'| = k_F - \omega_D / v_F}^{|\vec{k}'| = k_F + \omega_D / v_F} \frac{\lambda I}{2|\xi_{\vec{k}}| - E} \tag{2.10}$$

We are looking for the main state with <u>negative</u> energy. Having introduced the notation $E = -2\Delta$ and transforming the sum to the integral, we obtain

$$1 = \lambda N(E_F) \ln \frac{2\hbar\omega_D}{\Delta},$$

whereby we come to the relation

$$\Delta = 2\hbar\omega_D \exp\left(-\frac{1}{\lambda N(E_F)} \right) \tag{2.11}$$

We see that $\Delta \neq 0$ at any force of attraction.

<u>2.1.3. Energy spectrum.</u>

In the Cooper effect, two interacting electrons differ from all others as they change state and are grouped in pair. Other electrons remain in their initial state. In fact, it is necessary to consider reorganization of the states of all electrons. Each of them, on one hand, owing to Pauli's principle, creates restrictions on possible values of wave vectors of the other electrons, which gives them the chance to be grouped in pairs. On the other hand, this electron itself is a part of one of the pairs.

For calculation of the energy spectrum we will use the method of secondary quantization, i.e. we will apply the occupation-number representation. We should minimize the free energy. We will calculate the energy of the main state for the case of an attraction between electrons. To avoid the additional condition of constancy of number of particles, we will count their energies from the chemical potential μ, i.e. $\xi_{\vec{k}} = \frac{\hbar^2 k^2}{2m} - \mu$.

As a model Hamiltonian of N electrons in a volume of V we will accept

$$H = \sum_{\vec{k},\sigma} \xi_{\vec{k}} a^+_{\vec{k},\sigma} a_{\vec{k},\sigma} + \frac{1}{2V} \sum_{\vec{k},\vec{k}',\sigma} U_{\vec{k}\vec{k}'} a^+_{\vec{k}',\sigma} a^+_{-\vec{k}',-\sigma} a_{-\vec{k},-\sigma} a_{\vec{k},\sigma}, \tag{2.12}$$

where a^+ and a - operators of the creation and annihilation, $U_{\vec{k}\vec{k}'}$ - the matrix element of the energy of interaction of two electrons. The first term is the self energy of the electrons. The second one is the energy of interaction between electrons caused by the exchange of virtual phonons. In each term of the sum a pair of electrons with opposite spins (σ and $-\sigma$) and momenta ($\vec{k}$ and $-\vec{k}$) is destroyed and another pair with $\vec{k}'$ and $-\vec{k}'$ is created.

The terms, differing only in values of spin, give an identical contribution in (2.12), therefore we can write down

$$H = 2\sum_{\vec{k}} \xi_{\vec{k}} a^{+}_{\vec{k},1/2} a_{\vec{k},1/2} + \frac{1}{V}\sum_{\vec{k},\vec{k}'} U_{\vec{k}\vec{k}'} a^{+}_{\vec{k}',1/2} a^{+}_{-\vec{k}',-1/2} a_{-\vec{k},-1/2} a_{\vec{k},1/2} \qquad (2.13)$$

The task of finding the ground state and the spectrum of excitations for the system with this Hamiltonian can be solved in different ways: Bogolyubov transformation, summation of Feynman diagrams, method of spin analogy, etc. We will solve it by means of Bogolyubov canonical transformation - we will define quasi-particle operators $A_{\vec{k}0}, A_{\vec{k}1}$ by the following relations:

$$a_{\vec{k},1/2} = u_{\vec{k}} A_{\vec{k}0} + v_{\vec{k}} A^{+}_{\vec{k}1} \qquad a_{-\vec{k},-1/2} = u_{\vec{k}} A_{\vec{k}1} - v_{\vec{k}} A^{+}_{\vec{k}0} \qquad (2.14)$$

where $u_{\vec{k}}$ and $v_{\vec{k}}$ are real functions, symmetric with respect to the transformation $\vec{k} \to -\vec{k}$.

For all Fermi operators the commutation relations for anti-commutators have to be satisfied: $\{a^{+}_i a_k\} = \delta_{ik}$, $\{a_i a_k\} = 0$, $\{a^{+}_i a^{+}_k\} = 0$, and similarly for A. All of them are satisfied, if the functions $u_{\vec{k}}$ and $v_{\vec{k}}$ meet the conditions:

$$u^2_{\vec{k}} + v^2_{\vec{k}} = 1 \qquad (2.15)$$

To check it, let us calculate, for example, the anti-commutator:

$$\left\{a_{\vec{k},1/2} a^{+}_{\vec{k},1/2}\right\} = u^2_{\vec{k}} \underbrace{\left\{A_{\vec{k}0} A^{+}_{\vec{k}0}\right\}}_{=1} + v^2_{\vec{k}} \underbrace{\left\{A^{+}_{\vec{k}1} A_{\vec{k}1}\right\}}_{=1} + u_{\vec{k}} v_{\vec{k}} \underbrace{\left\{A^{+}_{\vec{k}1} A^{+}_{\vec{k}0}\right\}}_{=0} + u_{\vec{k}} v_{\vec{k}} \underbrace{\left\{A_{\vec{k}1} A_{\vec{k}0}\right\}}_{=0} = u^2_{\vec{k}} + v^2_{\vec{k}} = 1.$$

Then (2.13) will be transformed to the form:

$$H = E_0 + H^0_0 + H_1 + H_2$$

where $$E_0 = 2\sum_{\vec{k}} \xi_{\vec{k}} v^2_{\vec{k}} - \frac{1}{V}\sum_{\vec{k},\vec{k}',\sigma} U_{\vec{k}\vec{k}'} u_{\vec{k}'} v_{\vec{k}'} u_{\vec{k}} v_{\vec{k}} \qquad (2.16)$$ -

a constant, not depending on Fermi-operators and corresponding to the energy of the ground state;

$$H^0_0 = \sum_{\vec{k}} [\xi_{\vec{k}} (u^2_{\vec{k}} - v^2_{\vec{k}}) - \frac{2u_{\vec{k}} v_{\vec{k}}}{V} \sum_{\vec{k}} U_{\vec{k}\vec{k}'} u_{\vec{k}'} v_{\vec{k}'}](A^{+}_{\vec{k}0} A_{\vec{k}0} + A^{+}_{\vec{k}1} A_{\vec{k}1}) \qquad (2.17)$$ -

the diagonal part of the Hamiltonian;

$$H_1 = \sum_{\vec{k}} [2\xi_{\vec{k}} u_{\vec{k}} v_{\vec{k}} + \frac{1}{V}(u^2_{\vec{k}} - v^2_{\vec{k}}) \sum_{\vec{k}'} U_{\vec{k}\vec{k}'} u_{\vec{k}'} v_{\vec{k}'}](A^{+}_{\vec{k}0} A^{+}_{\vec{k}1} + A_{\vec{k}1} A_{\vec{k}0}) \qquad (2.18)$$ -

the off-diagonal part of the Hamiltonian containing the product of two Fermi-operators. The operator H_2 contains the product of four new Fermi operators. In the study of low-energy excited states it can be omitted.

So far the functions $u_{\vec{k}}$ and $v_{\vec{k}}$ were arbitrary, on condition of (2.15). We will choose them so that the operator in (2.18) becomes zero. For this purpose it is sufficient to require the equality

$$2\xi_{\vec{k}} u_{\vec{k}} v_{\vec{k}} = -\frac{1}{V}(u^2_{\vec{k}} - v^2_{\vec{k}}) \sum_{\vec{k}'} U_{\vec{k}\vec{k}'} u_{\vec{k}'} v_{\vec{k}'} \qquad (2.19)$$

It is possible to show that this equality, when (2.15) is satisfied, is at the same time a condition of a minimum of energy of the main state (2.16).

We introduce the notation: $\Delta_{\vec{k}} \equiv -\frac{1}{V}\sum_{\vec{k}'} U_{\vec{k}\vec{k}'} u_{\vec{k}'} v_{\vec{k}'}$. (2.20)

Then it is possible to express from (2.15) and (2.19) required $u_{\vec{k}}$ and $v_{\vec{k}}$ through $\xi_{\vec{k}}$ and $\Delta_{\vec{k}}$:

$$u_{\vec{k}}^2 = \frac{1}{2}\left[1 + \frac{\xi_{\vec{k}}}{\sqrt{\Delta_{\vec{k}}^2 + \xi_{\vec{k}}^2}}\right]; \qquad v_{\vec{k}}^2 = \frac{1}{2}\left[1 - \frac{\xi_{\vec{k}}}{\sqrt{\Delta_{\vec{k}}^2 + \xi_{\vec{k}}^2}}\right] \tag{2.21}$$

Substituting (2.21) into (2.20), we find the nonlinear equation defining $\Delta_{\vec{k}}$:

$$\Delta_{\vec{k}} = -\frac{1}{2V}\sum_{\vec{k}'} \frac{U_{\vec{k}\vec{k}'}\Delta_{\vec{k}'}}{\sqrt{\Delta_{\vec{k}'}^2 + \xi_{\vec{k}'}^2}} \tag{2.22}$$

Substituting (2.20) and (2.21) into (2.17), it is possible to transform the diagonal part of a Hamiltonian to give

$$H_0^0 = \sum_{\vec{k}} \sqrt{\xi_{\vec{k}}^2 + \Delta_{\vec{k}}^2}\,(A_{\vec{k}0}^+ A_{\vec{k}0} + A_{\vec{k}1}^+ A_{\vec{k}1}) \tag{2.23}$$

Thus, owing to the interaction of electrons with each other the spectrum of elementary excitations has the form:

$$\varepsilon_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + \Delta_{\vec{k}}^2} \tag{2.24}$$

Each value of the quasi-momentum $\vec{k}$ corresponds to two types of excitations relating to the operators of creation $A_{\vec{k}0}^+$ and $A_{\vec{k}1}^+$.

Changing the single-particle spectrum, due to the interaction, is determined by the value $\Delta_{\vec{k}}$ that is the root of the equation (2.22).

Let us turn to the study of this equation. It has a trivial solution $\Delta_{\vec{k}} = 0$ corresponding to the normal state. Consider other options for the simplest case

$$U_{\vec{k}\vec{k}'} = \begin{cases} -\lambda & if \quad k_F - q < k, k' < k_F + q \\ 0 & beyond\ the\ range \end{cases} \tag{2.25}$$

In this case, it follows from (2.22) that within the specified interval, the value $\Delta_{\vec{k}}$ is also constant ($\Delta_{\vec{k}} = \Delta$), and the equation (2.22) becomes:

$$1 = \frac{\lambda}{2V}\sum_{\vec{k}} \frac{1}{\sqrt{\Delta^2 + \xi_{\vec{k}}^2}} \tag{2.26}$$

We replace the sum with an integral according the rule $\sum_{\vec{k}} ... = V(2\pi)^{-3}\int ... d^3k$. Believing chemical potential to be equal to the Fermi energy, $\mu = E_F = \frac{\hbar^2 k_F^2}{2m}$, we obtain

$$\xi_{\vec{k}} = \frac{\hbar^2(k^2 - k_F^2)}{2m} \approx \hbar^2 k_F \underbrace{(k - k_F)}_{\delta}/m$$

Further we substitute $d^3\vec{k} = 4\pi k_F^2 dk$, and the equality (2.26) takes the form:

$$1 = \frac{\lambda k_F^2}{4\pi} \int_{-q}^{q} \left[\Delta^2 + \left(\frac{\hbar^2 k_F}{m} \delta \right)^2 \right]^{-1/2} d\delta \qquad (2.27)$$

Calculating the integral and solving the equation for Δ, we obtain

$$\Delta = \frac{2\hbar^2 k_F q}{m(\exp(1/\lambda N(E_F)) - 1)} \qquad (2.28)$$

where $N(E_F) = \dfrac{mk_F}{2\pi^2\hbar^2}$ is the density of electronic states at the Fermi level (without taking into account spin).

The maximum change in the electron wave vector corresponds to the maximum (Debye) frequency of the virtual phonon ω_D: $q = \omega_D / v_F$. In the weak-coupling approximation ($\lambda \cdot N(E_F) << 1$) we finally obtain

$$\Delta(0) = 2\hbar\omega_D \exp(-1/\lambda N(E_F)) \qquad (2.29)$$

It becomes clear why the theory of superconductivity could not be created on the basis of the perturbation theory for the accounting of interaction. The perturbation theory gives the corrections to the energy in the form of degrees of small interaction energy λ, and the obtained value Δ tends to zero as $\exp(-1/\lambda N(E_F))$, and for small values of λ can not be expanded in a power series.

To clarify the physical meaning of Δ we will find the ground-state energy E_0. Substituting (2.20) and (2.21) to (2.16), we obtain

$$E_0 = \sum_{\vec{k}} \frac{\xi_{\vec{k}}(\sqrt{\xi_{\vec{k}}^2 + \Delta_{\vec{k}}^2} - \xi_{\vec{k}}) - \Delta_{\vec{k}}^2/2}{\sqrt{\xi_{\vec{k}}^2 + \Delta_{\vec{k}}^2}} \qquad (2.30)$$

In the case of the trivial solution $\Delta_{\vec{k}} = 0$, corresponding to the normal state, we have $E_0 = 0$. If, however, $\Delta_{\vec{k}} \neq 0$, then $E_0 < 0$. Thus, this solution is energetically more favorable than the normal state. Replacing the sum by an integral and calculating it, we find that the energy decreases by $N(E_F)\Delta^2/2$.

When $\Delta_{\vec{k}} \neq 0$ the functions $u_{\vec{k}}$ and $v_{\vec{k}}$ are both different from zero, therefore, the new Fermi operators A_0^+ and A_1^+ correspond to the creation of new elementary excitations (quasi-particles) each of which is a superposition of the electron and hole states. Values $u_{\vec{k}}$ and $v_{\vec{k}}$ characterize probabilities of different states: $u_{\vec{k}}^2$ is the probability that, when the excitations are absent, the electron states with $\vec{k}$ and $-\vec{k}$ are not

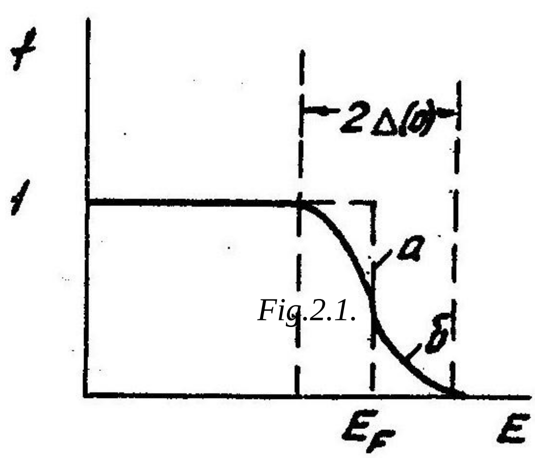


Fig.2.1.

simultaneously occupied, and $v_{\vec{k}}^2 = \frac{1}{2}\left[1 - \frac{\xi_{\vec{k}}}{\sqrt{\Delta_{\vec{k}}^2 + \xi_{\vec{k}}^2}}\right]$ - that both are occupied.

It allows for the question of how the electrons are distributed by momentum and energy to be answered. Indeed, upon the transition from the normal state to the superconducting one in a sample, there is the same quantity of electrons, but they don't fill the Fermi sphere any more.

Figure 2.1 shows the graphs

a) the probability of filling the one-electron states with energy E in the normal metal,

b) $v_{\vec{k}}^2$ - the probability that the one-electron states $\vec{k}$ and $-\vec{k}$ are filled in the ground state of a superconductor.

Graph 2.1b shows that, as mentioned in §1.7, the one-electron states in the gap are occupied. The energy gap exists in the energy of elementary excitations, but not in the one-electron energy states.

We find the density of states in the spectrum of excitations of the superconductor, i.e., the number of states per unit energy. The states of excitements that existed in the normal metal are reordered because of the gap. Since the quantity of states remains constant, we can write down $N_n d\xi_{\vec{k}} = N_S d\varepsilon_{\vec{k}}$, hence, using (2.24) and taking into account that the density of states in the normal metal near the Fermi level is constant, we obtain

$$N_S(\varepsilon) = N(E_F)\frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}} \qquad (2.35)$$

This expression was used in plotting the density of states in the superconductor in Fig. 1.11.

These results explain the process of single-particle tunneling, shown in Fig. 1.10.

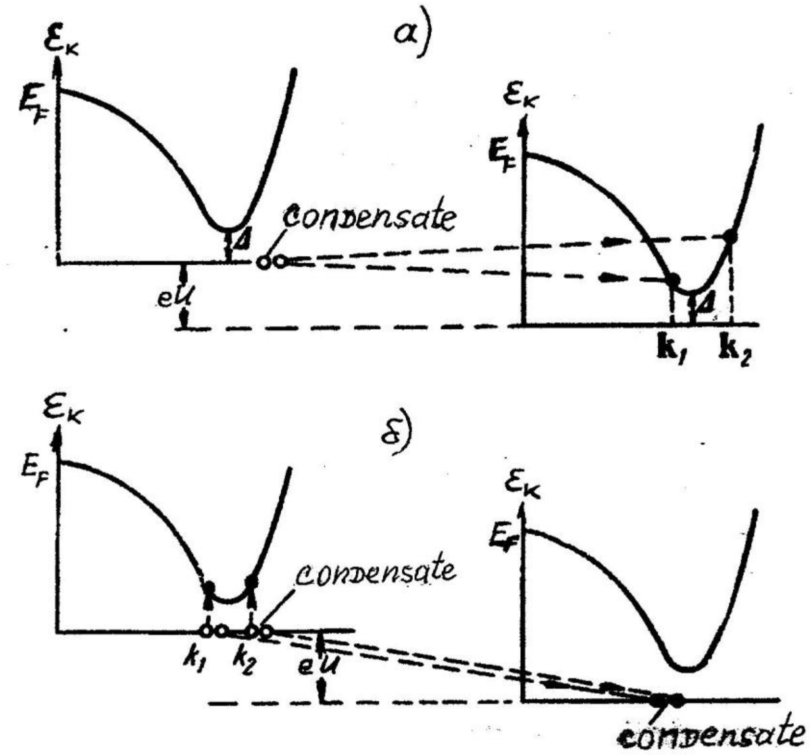


Figure 2.2. Tunneling of electron pairs.

But pairs of electrons can tunnel as well. On the scheme of Fig. 2.2 the superconducting electrons (which are parts of Cooper pairs) are located at zero level. After the breaking of a pair, two excitements (two quasi-particles) are formed. All pairs form so-called "condensate" (no real condensate is present; this term is introduced only in the energy space). In fig. 2.2 possible processes of tunneling of pairs of electrons between two superconductors are represented. The first process is shown in fig. 2.2a: a pair of superconducting electrons leaves the left superconductor, without leaving excitations in it, and occupies quasi-particle states in the right one. In the second process (fig. 2.2b) two superconducting electrons leave the left sample, however two quasi-particles remain in it. Electrons tunnel into the right sample where they drop out in the condensate so in the right superconductor excitements don't arise.

As mentioned above, we have neglected the operator H_2, containing the product of four Fermi quasi-particle operators. This is acceptable for a small quantity of excitations. At nonzero temperatures, the operator H_2 must be taken into account. That is why the notation $\Delta(0)$ in (2.29) indicates that the determined value of Δ corresponds to $T=0$. Rigorous calculation leads to expressions

$$2\xi_{\vec{k}} u_{\vec{k}} v_{\vec{k}} = (u_{\vec{k}}^2 - v_{\vec{k}}^2)\Delta \qquad (2.36)$$

$$\Delta = \lambda \sum_{\vec{k}'} u_{\vec{k}'} v_{\vec{k}'} (1 - n_{\vec{k}0} - n_{\vec{k}1}) \qquad (2.37)$$

Formula (2.37) shows that the value of the gap depends on the number of quasi-particles and their energy distribution.

Let us calculate the temperature dependence. Quasi-particles are distributed according to Fermi-Dirac law:

$$n_{\vec{k}} = \frac{1}{\exp(\varepsilon_{\vec{k}} / k_B T) + 1} \qquad (2.38)$$

Substituting (2.36) and (2.38) into (2.37) and converting to the integral, we obtain

$$1 = \frac{\lambda g(E_F)}{2} \int_0^{\hbar\omega_D} \frac{th \dfrac{\sqrt{\xi^2 + \Delta^2}}{2k_B T}}{\sqrt{\xi^2 + \Delta^2}} d\xi \qquad (2.39)$$

The dependence of the half-width of the energy gap on temperature obtained as a result of calculations is given in fig. 2.3.

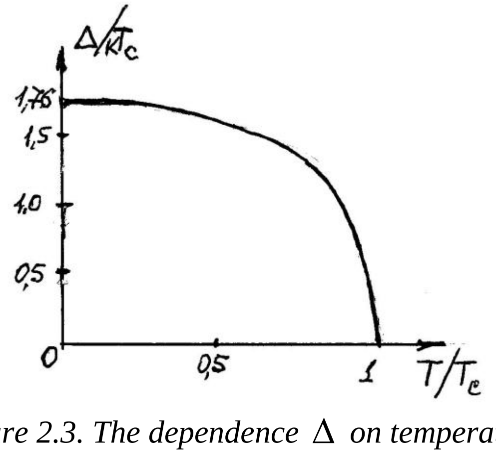


Figure 2.3. The dependence Δ on temperature.

We considered the situation when electrons interacted only through the exchange of virtual phonons, which provided an attraction, and did not take into account the Coulomb repulsion. Rigorous calculation shows that the Coulomb repulsion does not very effectively prevent the manifestation of superconductivity. In particular, it can sometimes happen that, even at resulting repulsion, superconductivity, nevertheless, is retained.

The explanation can be found if we consider that the electron moves leaving behind a positively charged "track", which disappears in a finite time.

§2.2. Magnetic field inside superconductors

The BCS theory explains the reasons for the restructuring of the energy spectrum of the metal, causing it to become superconducting. This theory is not directly involved in the calculation of the distribution of the fields and currents in a superconducting sample, depending on its shape and the character of the external magnetic field. In the following sections various theoretical descriptions, in which the superconductor is the macroscopic environment, are stated. In these descriptions, the main equations come from a condition of a minimum of some thermodynamic potential. These equations allow for calculation of the coordinate dependence of magnetic fields, superconducting currents and the value of the energy gap.

The present section contains some data on the electrodynamics of a continuous medium which will be used later. In particular, concepts of the macroscopic characteristics of a magnetic field are introduced, and the possibility of use of the various thermodynamic potentials is justified.

As a characteristic of the magnetic field in a superconductor, we can take the microscopic field strength $\vec{h}$ at each point. In a steady state this field satisfies the Maxwell equations $div\vec{h}=0$ and $rot\vec{h}=\vec{j}_S$ where $\vec{j}_S$ is the superconducting current density. When the currents are macroscopic quantities (for example the Meissner state), we can allocate them as separate sources of the field, while neglecting the diamagnetism of the substance, i.e., considering the magnetic permeability of the medium equal to unity. In this case, the magnetic induction is linked to the strength of the microscopic field by the relation $\vec{B}=\mu_0\vec{h}$.

In those situations where there are microscopic currents and the field varies considerably over short distances, it makes sense to forgo the microscopic analysis and move on to the macroscopic characteristics of the field. These are the macroscopic induction, $\vec{B}$, and macroscopic field strength, $\vec{H}$. The vector of magnetic induction in the medium is defined as the average by the volume (smaller than the typical sample size, but larger than the characteristic length of the field changes) of the field strength multiplied by the magnetic permeability of a vacuum: $\vec{B}=\mu_0\left\langle\vec{h}\right\rangle$. At points where there are no external sources of a field (a wire, a coil, etc.), the vector $\vec{B}$ satisfies the equations

$$div\vec{B}=0, \quad rot\vec{B}=\mu_0\left\langle\vec{j}_S\right\rangle, \tag{2.40}$$

where $\left\langle\vec{j}_S\right\rangle$ is the average density of the superconducting current.

We define the vector of macroscopic field strength, $\vec{H}$, by the relation

$$\vec{B} = \mu_0(\vec{H} + \vec{M}), \quad (2.41)$$

where $\vec{M}$ is the magnetization vector equal to the magnetic moment per unit volume of the sample. Then the vector of field strength $\vec{H}$ satisfies the equation

$$rot\vec{H} = 0. \quad (2.42)$$

It can be shown that the field strength $\vec{H}$ can be found as a partial derivative of the free energy density with regard to magnetic induction:

$$H = \frac{\partial F}{\partial B} \quad (2.43)$$

In a vacuum, we have the relations

$$\vec{H} = \vec{h}, \quad \vec{B} = \mu_0\vec{H} = \mu_0\vec{h} \quad (2.44)$$

From (2.42) and the first equation of (2.40), it follows that on the interface of two different environments the following boundary conditions have to be satisfied

$$B_{1n} = B_{2n}, H_{1\tau} = H_{2\tau} \quad (2.45)$$

If the sample has the shape of a long rod and placed in an external field, $\vec{H}_e$, parallel to its axis (by the external field, we mean a field generated by an external source in the absence of a superconducting medium), the magnetic field, $\vec{H}$, near the surface outside of the sample equals the external field. From the condition of continuity of the tangential components of the vector $\vec{H}$ (2.45) it follows that the field $\vec{H}$ inside the sample at the border is equal to $\vec{H}_e$ as well. Then from (2.42) it follows that everywhere in the sample the field $\vec{H}$ is uniform and equal to $\vec{H}_e$. It explains the use of the relation $\vec{B} = \mu_0(\vec{H}_e + \vec{M})$ instead of (2.41) in the curves in Figures 1.7. and 1.8.

In thermodynamics, it is proven that the magnetic field distribution at the given external currents corresponds to the minimum not of the free energy F, but of the Gibbs thermodynamic potential G, related to the free energy by the formula

$$G = F - \int \vec{B}\vec{H}dV \quad (2.46)$$

The integral in (2.46) is calculated over the entire space, i.e. over the area outside the sample as well. But as shown above, in the case of samples, infinite along the direction of the external field, the field strength at all points of the sample is equal to the external, i.e. the energy of the field in this region does not depend on the current distribution in the sample. Therefore, this part of the energy can be excluded from consideration in minimizing the Gibbs potential, thus the integration in (2.46) has to be carried out only over the sample volume.

We emphasize once again that the case when the sample has the shape of a rod, endless along the direction of the external field, is special. As stated above, at this geometry the magnetic field strength at any point outside of the sample is equal to the external, i.e. to the field strength which would be at this point in the absence of the sample. Therefore, carrying out the integration in the Gibbs potential before its minimization we can restrict ourselves to the volume of the sample.

Furthermore, as shown above, in this case the macroscopic magnetic field strength $\vec{H}$ at all points within the sample is also equal to the external field $\vec{H}_e$. In other words, the field is uniform throughout the space. This fact often simplifies the search for solutions. In later consideration we will deal most often with this particular geometry. In cases of other geometries the possibility of excluding from consideration the integral over the region outside of the sample will be analyzed specifically. The same applies to the assertion of the homogeneity of the field.

§2.3. Equation of F. and G. London

In 1935, brothers F. and G. London showed that when all fields and currents are weak and slowly change in space, the condition of minimum of free energy leads to a simple relation between fields and currents.

Brothers London based their theory on the "two-fluid" model of a superconductor proposed a year earlier by Gorter and Cazimir. This model assumes the existence of two types of electrons within the superconductor - "normal" with concentration $n_n(T)$ and "superconducting" with concentration $n_S(T)$. The total concentration of the conducting electrons is given by $n = n_n + n_S$. The concentration of superconducting electrons decreases with increasing temperature and vanishes at $T = T_C$. When $T \to 0$ it tends to the total concentration of electrons. The superconducting current is provided by the perpetual movement of superconducting electrons, while the normal ones behave in the usual way.

We will consider a pure metal with a parabolic conduction band; the effective mass of the electrons equals to m. Free energy is as follows:

$$F = \int F_S dV + E_k + E_m, \tag{2.47}$$

where F_S is the energy of electrons per unit volume in the condensed state in a system at rest, E_m and E_k are the magnetic and the kinetic energy connected with persistent currents. The drift velocity of electrons $\vec{v}$ at the point $\vec{r}$ is associated with the current density $\vec{j}_S$

$$n_S e\vec{v}(\vec{r}) = \vec{j}_S(\vec{r}), \tag{2.48}$$

where e – the electron charge, n_S – the concentration of "superconducting" electrons.

The kinetic energy can be written as:

$$E_k = \frac{1}{2}\int n_S m v^2 dV, \tag{2.49}$$

where the integral is taken over the sample volume. Expression (2.49) is valid provided that the velocity $\vec{v}(\vec{r})$ is a slowly changing function of the coordinates.

The energy associated with a magnetic field $\vec{h}(\vec{r})$ is

$$E_m = \frac{\mu_0}{2}\int h^2 dV. \tag{2.50}$$

We consider the magnetic permeability of the medium to be equal to 1, and take into account the superconducting currents explicitly.

The field $\vec{h}(\vec{r})$ is related to the current density by the Maxwell equation:

$$rot\vec{h} = \vec{j}_S \tag{2.51}$$

Using (2.49)-(2.51), we present the free energy in the form

$$F = F_0 + \frac{\mu_0}{2}\int (h^2 + \lambda_L^2 \left|rot\vec{h}\right|^2)dV \tag{2.52}$$

where $F_0 = \int F_S dV$, and a quantity λ_L, having the dimension of length, is defined as follows:

$$\lambda_L = \sqrt{\frac{m}{\mu_0 n_S e^2}} \tag{2.53}$$

Note that the Londons believed that the current is carried by individual electrons. At $T = 0$ all the electrons are "superconducting" and $n_S = n$. In fact, the electrons are grouped in pairs, so the values m, e and n_S in (2.53) should be changed by $2m, 2e, n/2$. It is easily seen that the formula (2.53) will not change, only n_S will be replaced by n. For simple metals such as Al, Sn etc., in which the mass m is close to the mass of a free electron, we find $\lambda_L \approx 0.05$ microns.

Let us consider a sample, infinite in the direction of the applied magnetic field. As shown in §2.2, in this case it is possible to minimize the integral calculated over only the sample volume without taking into account the area outside it.

Strictly speaking, as mentioned above, at constant external current producing the field, it is necessary to minimize the Gibbs potential. However, it can be shown that in this situation the minimization of the free energy leads to the same result. Later we will see that the London equation can be obtained in a more general case of other considerations.

We minimize F from (2.52) with respect to the distribution of the field $\vec{h}(\vec{r})$. If we change the field $\vec{h}(\vec{r})$ by $\delta\vec{h}(\vec{r})$ the energy F will get the increment δF:

$$\delta F = \mu_0 \int (\vec{h}\cdot\delta\vec{h} + \lambda_L^2 rot\vec{h}\cdot rot\delta\vec{h})dV = \mu_0\int(\vec{h} + \lambda_L^2 rot\,rot\vec{h})\cdot\delta\vec{h}dV \tag{2.54}$$

(We have integrated the second term by parts, using the formula of vector analysis $div(\vec{a}\times\vec{b}) = \vec{b}\cdot rot\vec{a} - \vec{a}\cdot rot\vec{b}$ for $\vec{a} = \delta\vec{h}, \vec{b} = rot\vec{h}$). Consequently, the configuration of the field inside the sample, giving minimum to the free energy, must satisfy the equation

$$\vec{h} + \lambda_L^2 rot\,rot\vec{h} = 0 \tag{2.55}$$

Using Maxwell's equation (2.51) we can write the equation (2.55) in the form

$$rot\vec{j} = -\frac{\vec{h}}{\lambda_L^2} \tag{2.56}$$

Equation (2.55) or (2.56) is called the London equation. Together with the Maxwell equation (2.51), it allows one to find the distributions of the field and currents. From it, in particular, it follows that the field inside a superconductor can not be uniform: $\vec{h}(\vec{r}) \neq const$. Let us consider some more consequences of the London equation.

2.3.1. The Meissner effect

Now we apply the London equation (2.55) to the problem of the magnetic field penetration into a superconductor. Choose a simple geometry: the sample surface coincides with the plane xy, the region $z<0$ is empty (Fig.2.4). Then the field strength $\vec{h}$ and the superconducting current density $\vec{j}_S$ depend only on z.

In addition to equation (2.55), there are, as usual, the Maxwell equations

$$rot\vec{h} = \vec{j}_S \qquad (2.57)$$

$$div\vec{h} = 0 \qquad (2.58)$$

From formula (2.56) it follows that $h_z = 0$ (as $\vec{j}$ is independent of x and y).

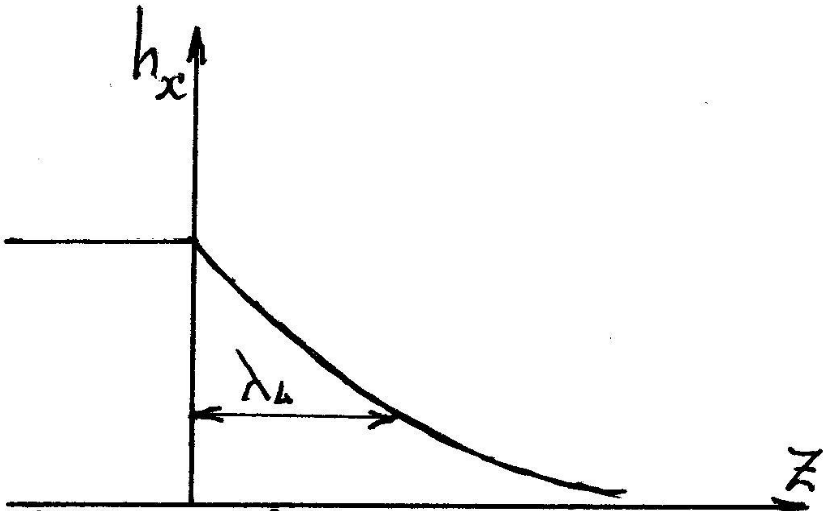


Fig.2.4. Penetration of a weak magnetic field into a superconductor.

We choose the x axis along the direction of the field $\vec{h}$. Equation (2.58) is automatically satisfied, and equation (2.57) shows that the current density $\vec{j}_S$ is directed along the y axis:

$$\frac{dh}{dz} = j_S \qquad (2.59)$$

Finally, equation (2.55) takes the form

$$\frac{d^2h}{dz^2} = \frac{h}{\lambda_L^2} \qquad (2.60)$$

The solution, finite inside the superconductor, is exponentially decreasing:

$$h(z) = h(0)\cdot\exp(-z/\lambda_L) \qquad (2.61)$$

i.e. the field penetrates the superconductor only at a depth of λ_L. This result, obtained for the half-space, can be easily generalized to the case of a sample of arbitrary shape. The depth of penetration, λ_L, is small, so it can be said that a weak magnetic field does not penetrate the macroscopic sample, or in other words, the magnetic field is expelled from the sample. As was mentioned earlier, this result was found in the experiments of Meissner and Ochsenfeld in 1933, before the creation of the London theory.

2.3.2. Thin films in the longitudinal magnetic field

Consider a sample in the shape of a thin film (Figure 2.5). It is infinite along the axes x and y, so that edge effects can be neglected. The field inside the sample is described by the same equation:

$$\frac{d^2h}{dz^2} = \frac{h}{\lambda_L^2} \quad (2.62)$$

Now, however, the boundary conditions are

$$h(d/2) = h(-d/2) = h_0 \quad (2.63)$$

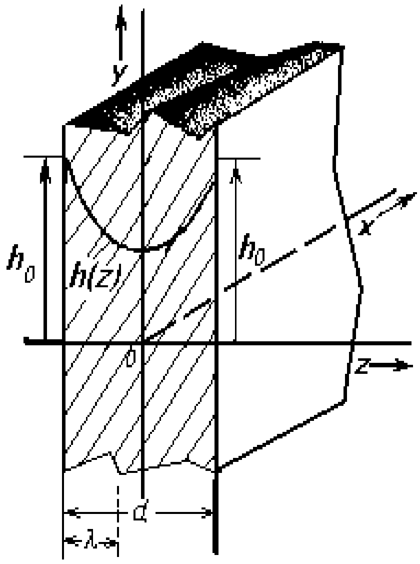


Figure 2.5. The distribution of the magnetic field inside the thin superconducting film.

The general solution can be written as:

$$h(z) = C_1 \cdot \exp(-z/\lambda_L) + C_2 \cdot \exp(z/\lambda_L) \quad (2.64)$$

From the symmetry of the problem on z it follows that $C_1 = C_2$. Satisfying the boundary conditions (2.63), we finally obtain

$$h(z) = h_0 \frac{ch(z/\lambda_L)}{ch(d/2\lambda_L)} \quad (2.65)$$

Dependence (2.65) is shown in Figure 2.5. By decreasing the film thickness, d, the efficiency of weakening of the field decreases. When $d << \lambda_L$ the field almost uniformly permeates the superconducting film. The energy expended on ejecting the external field is small, and therefore, the external magnetic field required for the destruction of superconductivity is more than its value for a bulk superconductor.

§2.4. The Pippard relation

In deriving the London equation we assumed that the velocity $\vec{v}(\vec{r})$, or, which is the same, the superconducting current density $\vec{j}(\vec{r})$, changes slowly in space. Let us specify what is meant by "slowly."

In a condensed system velocities of two electrons - 1 and 2 - are correlated if the distance between them is less than a certain value. We denote it ξ_0. Our conclusion is valid if the velocity

$\vec{v}(\vec{r})$ varies slightly over distances of the order of ξ_0. To evaluate ξ_0, we note that the scope of the essential values of the momenta of the electrons is given by the inequality

$$E_F - \Delta < \frac{p^2}{2m} < E_F + \Delta \qquad (2.66)$$

where E_F - the Fermi energy, Δ - the half-width of the energy gap.

The corresponding variation in momentum is $\Delta p = 2\Delta / v_F$ where $v_F = p_F / m$ - Fermi electron velocity. Because of the uncertainty relation we estimate the width of the corresponding wave packet to be $\delta x \approx \hbar / \Delta p$, which allows us to introduce the characteristic length, called the coherence length of the superconductor (sometimes referred to as the Cooper pair length):

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta} \ . \qquad (2.67)$$

(Factor $1/\pi$ is introduced for reasons of convenience).

From the London equation it follows that the characteristic length of the changes of the field, currents and velocity is the length λ_L, so this equation is valid only on the condition $\lambda_L >> \xi_0$.

In the simple (non-transitive) metals the penetration depth λ_L is small (hundreds $\overset{\circ}{A}$). And the Fermi velocity is high (thousands of km/s), therefore the parameter ξ_0 is large (for Al $\xi_0 \approx 10^4 \overset{\circ}{A}$). It means that the London equation is not suitable to describe the process in such metals. In this case, the London equation (2.56) has to be replaced by another one, that is more complicated. Its form was proposed by Pippard. We will call such superconductors the type I superconductors. Those superconductors, to which the London equation is applicable, i.e., the condition $\lambda_L >> \xi_0$ is valid, are called type II superconductors. We will discuss it in detail later.

Historically, it happened that in the 20 years following the discovery of the Meissner effect, experiments were carried out mainly on type I superconductors, and only later the study of type II superconductors began. It is interesting to note that the theory developed in the reverse order: the theory of London was established in 1935, and its modifications to the type I superconductors were offered by Pippard but before 1953.

To complete the picture it is worth mentioning the superconducting alloys in which the coherence length ξ_0 and the penetration depth λ_L depend on the mean free path of the electrons. As it decreases ξ_0 decreases and λ_L increases. Therefore it often turns out that the addition of impurities into a type I superconductor turns it into a type II superconductor.

Pippard's idea can be explained as follows.

Using instead of the field strength $\vec{h}(\vec{r})$ vector potential, $\vec{A}(\vec{r})$, associated with the field by

$$rot\vec{A} = \vec{B} = \mu_0 \vec{h} \qquad (2.68)$$

we can write the London equation (2.56) $rot\vec{j} = -\vec{h}/\lambda_L^2$ in the form

$$\vec{j} = -\frac{\vec{A}}{\mu_0 \lambda_L^2} \qquad (2.69)$$

We note that (2.68) defines the vector potential $\vec{A}(\vec{r})$ ambiguously. In the derivation of (2.69), we chose it so that $div\vec{A} = 0$ (it is called the London calibration).

The relation (2.69) is applicable only when both $\vec{j}$ and $\vec{A}$ vary slowly in space. In general, it can be assumed that the current density, $\vec{j}(\vec{r})$, at a point, $\vec{r}$, depends on the vector potential, $\vec{A}$, in all adjacent points $\vec{r}\,'$ satisfying the condition $|\vec{r} - \vec{r}\,'| < \xi_0$. Pippard proposed the following phenomenological expression:

$$\vec{j}(\vec{r}) = C\int \frac{\left(\vec{A}(\vec{r}\,') \cdot \vec{R}\right)\vec{R}}{R^4} \exp(-\frac{R}{\xi_0}) dV', \text{ where } \vec{R} = \vec{r} - \vec{r}\,' \qquad (2.70)$$

Later, on the basis of the microscopic BCS theory it has been shown that the exact relationship between the current and the field is very similar to (2.70), but the mathematical expression is much more complicated. Therefore, the approximate result of Pippard still has not lost its value.

§2.5. Ginzburg – Landau theory

2.5.1. Basic equations

London theory is applicable only to those systems in which the concentration of the superconducting charge carriers is constant throughout the sample volume. In 1950, Soviet physicists V.L. Ginzburg and L.D. Landau published a theory that does not require a constant concentration n_S. This theory is based on the fact that the transition from the normal (N) to the superconducting (S) state is a phase transition of type II, i.e. not accompanied by the release or absorption of heat.

The theory of such transitions was created by Landau earlier. A parameter, called the order parameter, was introduced, which in the new phase (in this case - in the superconducting phase) is equal to zero at $T = T_C$ and increases with decreasing temperature. As such an order parameter in the superconductor Ginzburg and Landau considered the wave function $\psi(\vec{r})$, so that $|\psi(\vec{r})|^2$ equals the concentration of superconducting carriers.

Note that the conditions of applicability of the Ginzburg-Landau theory is the proximity of the temperature of the sample to the critical temperature.

a) Let us first consider the simplest case when there is no magnetic field and the parameter ψ is independent of the coordinates. Since the value of ψ in the superconducting phase should gradually drop to zero when the temperature approaches T_C, the free energy F_S near T_C can be expanded in a power series in $|\psi|$:

$$F_S = F_N + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \dots \qquad (2.71)$$

where F_N is the free energy of the normal state.

Let us discuss expression (2.71).

At $T = T_C$, i.e. when $\psi = 0$, the energy of the superconducting state is equal to the energy of the normal, and it explains the appearance of the term F_N in (2.71). The absence of a linear term in $|\psi|$ is due to the symmetry consideration; we will not discuss this situation and just take it on faith.

We now show that the coefficient α should equal zero at $T = T_C$. Since at $T < T_C$ the energy of the superconducting state must be less than the energy of normal, the coefficient α should be negative. At $T > T_C$, i.e. after transition, it should be positive. So, at the transition point it becomes zero and in the lowest order in ($T - T_C$) we obtain

$$\alpha = k(T - T_C) \qquad (2.72)$$

The coefficient k is positive.

The point of transition must be stable, i.e. at $\alpha = 0$, the function $F(|\psi|)$ should have minimum at $|\psi| = 0$. Thus, the third order term must be equal to zero, and the fourth-order term should be positive. Hence it follows that the coefficient β is positive.

Near T_C we keep only two terms of the expansion. In this case the coefficients α and β are quite simply related to the thermodynamic critical field, B_C, and the equilibrium density of the Cooper pairs at arbitrarily large distance from the border $n_S(\infty) = |\psi_\infty|^2$. Indeed, taking into account (1.2) $F_N - F_S = \mu_0 H_C^2/2$, we obtain the following relation

$$F_S - F_N = \alpha|\psi_\infty|^2 + \frac{1}{2}\beta|\psi_\infty|^4 = -\frac{1}{2\mu_0}B_C^2 \qquad (2.73)$$

The second equation for α and β is obtained from the condition of minimum F_S at equilibrium, i.e. $\dfrac{\partial F_S}{\partial(|\psi|^2)} = 0$, whence we have

$$\alpha + \beta|\psi_\infty|^2 = 0 \qquad (2.74)$$

Solving the system of equations (2.73), (2.74), we find

$$\alpha = -\frac{1}{\mu_0}\frac{B_C^2}{n_S(\infty)} \qquad (2.75)$$

$$\beta = \frac{1}{\mu_0}\frac{B_C^2}{n_S^2(\infty)} \qquad (2.76)$$

b) Now we assume that the order parameter varies slowly from point to point. One can show that in this case the magnetic field effect on free energy F_S will manifest itself in the addition of two terms: $\dfrac{1}{2m'}\left|(-i\hbar\vec{\nabla} - e'\vec{A})\psi\right|^2$ and $\dfrac{B^2}{2\mu_0}$, where $m' = 2m$ and $e' = 2e$ - the mass and the charge of the particle, i.e. Cooper pair; $\vec{A}$ - the vector potential of the magnetic field. Note that when creating their theory the authors, not knowing about the existence of Cooper pairs, thought that the charge carriers were unpaired electrons and believed $m' = m$ and $e' = e$.

The first of these terms is usually obtained in quantum mechanics from the kinetic energy when replacing the momentum of a particle by the generalized momentum in a magnetic field. It describes the energy of the superconducting currents, as well as the energy associated with the spatial inhomogeneity of the distribution of the Cooper pairs. The second term corresponds to the energy of the magnetic field.

As mentioned in §2.2, the magnetic field distribution at the given external currents should be found of the minimum condition not for the free energy, F, but for the Gibbs thermodynamic potential, G. To find the value G_S of the sample one should subtract from the total value of potential $G = (F_S - BH)$ the potential of the external field $\frac{B_e^2}{2\mu_0} - B_e H_e = -\frac{B_e^2}{2\mu_0}$. Let the sample be a core, endless along the external field (§2.2). In this case, the macroscopic magnetic field $\vec{H}$ is equal to the external field $\vec{H}_e$. Taking this into account, we get

$$G_S = F_N + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{(B - B_e)^2}{2\mu_0} + \frac{1}{2m'}\left|(-i\hbar\vec{\nabla} - e'\vec{A})\psi\right|^2 \qquad (2.77)$$

The physical meaning of the term $\frac{(B - B_e)^2}{2\mu_0}$ is that it corresponds to the energy necessary for the magnetic field, which in the absence of the superconductor equals $\vec{B}_e$, accepts the current value $\vec{B}$. When $\vec{B} = 0$ (Meissner phase) this term is equal to the full magnetic energy.

The Gibbs potential of the sample is obtained by integrating (2.77) over the whole volume. Minimizing the resulting expression by ψ and $\vec{A}$ by means of the variational method and taking into account (2.40), we obtain the two Ginzburg-Landau equations:

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m'}\left|-i\hbar\vec{\nabla} - e'\vec{A}\right|^2\psi = 0 \qquad (2.78)$$

$$\vec{j}_S = \frac{e'i\hbar}{2m'}(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi) - \frac{e'^2}{m'}|\psi|^2\vec{A} \qquad (2.79)$$

where ψ^* is the complex conjugate function of ψ.

From the requirement of vanishing of the current component perpendicular to the border superconductor-vacuum (or insulator), we get the following boundary conditions for this system of equations

$$\vec{n}(-i\hbar\vec{\nabla} - e'\vec{A})\psi = 0 \qquad (2.80)$$

where $\vec{n}$ is the normal to the boundary. In the case of the superconductor-normal metal the right side of (2.80) takes the form $i\lambda\psi$ where λ is a nonzero real constant.

Solving the system of equations (2.78) - (2.80), together with the Maxwell equations, one can find $\psi(\vec{r})$ and $\vec{j}(\vec{r})$, and then $\vec{B}(\vec{r})$.

We note that (2.79) is identical to the expression for the current density, introduced in quantum mechanics. Thus, the applicability of the second Ginzburg-Landau equation is not limited by a condition of closeness to the critical temperature.

2.5.2. Two characteristic lengths

We will show that the Ginzburg - Landau equations (2.78) - (2.79) contain two characteristic lengths.

a) First, consider the case where the magnetic fields and currents are absent. Select the calibration, in which the function $\psi(\vec{r})$ is real and, for simplicity, we confine ourselves to the one-dimensional case. Then (2.78) is noticeably simplified:

$$-\frac{\hbar^2}{2m'}\frac{d^2\psi}{dx^2}+\alpha\psi+\beta\psi^3=0 \qquad (2.81)$$

This equation has two obvious solutions: 1) $\psi=0$, relating to a normal state; 2) $\psi=\psi_0$, where $\psi_0^2=-\alpha/\beta$, describing the ordinary superconducting state. The second solution exists and corresponds to a lower energy when $\alpha<0$; i.e. when $T<T_C$. However, we would like to review the solutions of a more general type, such as when, under the influence of an external factor, the order parameter $\psi(x)$ at some point has a value other than ψ_0. How does $\psi(x)$ behave in the vicinity of this point?

Let us turn to the reduced variable $f=\psi/\psi_0$ and introduce the notation $\frac{\hbar^2}{2m'|\alpha|}=\xi^2(T)$ where the parameter $\xi(T)$ has the dimension of length. Equation (2.81) takes the form

$$-\xi^2(T)\frac{d^2 f}{dx^2}-f+f^3=0 \qquad (2.82)$$

It follows that the parameter $\xi(T)$ is the natural unit of measurement of distance, at which the function f can change. We will call it the coherence length at a given temperature, T. It can be shown that for pure metals

$$\xi(T)=0{,}74\left(1-\frac{T}{T_C}\right)^{-0{,}5}\xi_0 \quad . \qquad (2.83)$$

From this expression it is seen that, at temperatures near T_C, the order parameter varies little over distances of the order of the pair length ξ_0.

b) The second characteristic length appears when we consider electromagnetic effects, for example, when calculating the depth of penetration of the weak magnetic field.

If the external field is small, in the first order in the field h, the parameter $|\psi|^2$ can be replaced by its equilibrium value $|\psi_0|^2$ in the absence of the field. Calculating rotor (curl) of both sides of the second Ginzburg-Landau equation (2.79) and taking into account that $rot\vec{h}=\vec{j}_S$ and $rot\vec{A}=\mu_0\vec{h}$ we arrive at the equation of London type (because ψ_0 does not depend on the coordinates)

$$rot\,\vec{j} = -\frac{e'^2}{m'}\psi_0^2\mu_0\vec{h} \qquad (2.84)$$

Comparing (2.84) with the London equation (2.56), we obtain an expression for the characteristic length

$$\lambda(T) = \sqrt{\frac{m'}{\mu_0 n_S e'^2}} \qquad (2.85)$$

Note that the depth λ is proportional to ψ_0^{-1}, i.e. $(1-T/T_C)^{-0,5}$. For a pure metal the BCS theory gives

$$\lambda(T) = \frac{1}{\sqrt{2}}\left(1-\frac{T}{T_C}\right)^{-0,5}\lambda_L(0), \qquad (2.86)$$

where $\lambda_L(0) = \sqrt{\frac{m}{\mu_0 n e^2}}$ (2.87) is the London penetration depth at $T=0$.

Since, as mentioned above, the applicability of the second Ginzburg-Landau equation is not limited to a condition of proximity to the critical temperature, the obtaining of (2.84) and (2.85) can be regarded as an independent derivation of the London equation, the applicability of which is determined only by the condition of independence of the module of order parameter on coordinates. This result is more general than that obtained in §2.3.

c) We have found two characteristic lengths, $\xi(T)$ and $\lambda(T)$, governing the behavior of a superconductor near its critical temperature. Both values are proportional to $(T_C - T)^{-0,5}$, so their ratio $\kappa = \frac{\lambda(T)}{\xi(T)}$, which is called the Ginzburg-Landau parameter of the substance, is of special interest.

Depending on the value of κ the superconductors are divided into two types:

$\kappa < 1$ (i.e. $\lambda < \xi$) - type I superconductors;

$\kappa > 1$ (i.e. $\lambda > \xi$) - type II superconductors.

Later we will see that the exact boundary corresponds to the value $\kappa = 1/\sqrt{2}$.

2.5.3. Problems with constant amplitude of the order parameter

We now use the Ginzburg-Landau equations to solve some specific problems. First, consider the simplest case when the amplitude of the order parameter $|\psi|$ is the same at all points of the sample. We have already encountered this situation during the calculation of the penetration depth of the weak magnetic field into the bulk sample. Now we'll investigate the case of another kind. We consider thin samples (film, wire, etc.), in which arbitrary change of ψ over the thickness is disadvantageous, since it would lead to a sharp increase of the term $|\nabla\psi|^2$ in the expression for the Gibbs potential. The field strength $\vec{h}$ and current density $\vec{j}$ are not assumed to be weak, so the amplitude $|\psi|$ remains constant, but not necessarily equal to its unperturbed value $|\psi_0|$.

2.5.3.1. The critical current in a thin film

Let us consider a current with density $\vec{j}$ flowing along the x axis in a film of thickness d, as shown in Fig. 2.6a.

Let the film be thin, i.e. $d << \xi(T)$ and $d << \lambda(T)$. The first inequality provides constant amplitude over the film thickness, and the second one - the constancy of the current density. The equations are considerably simplified.

Indeed, we put

$$\psi = |\psi| \exp(i\varphi(\vec{r})), \qquad (2.88)$$

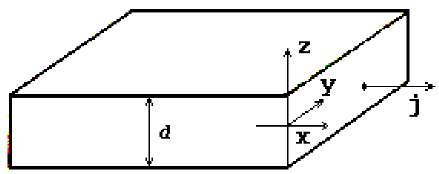
where the amplitude $|\psi|$ does not depend on $\vec{r}$.

The expression for the current density (2.79) can be written as

$$j = \frac{e'}{m'}|\psi|^2 (\hbar \frac{\partial \varphi}{\partial x} - e' A_x) = e'|\psi|^2 v, \qquad (2.89)$$

where

$$v = \frac{1}{m'}(\hbar \frac{\partial \varphi}{\partial x} - e' A_x) \qquad (2.90)$$



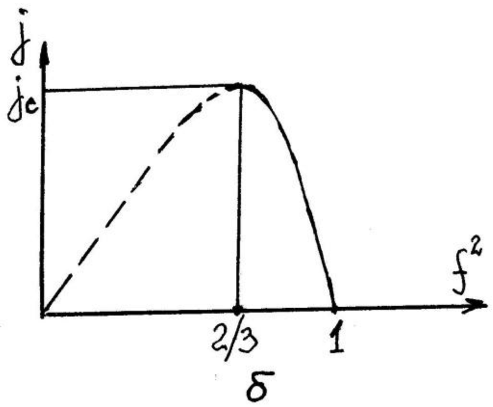


Figure 2.6. The calculation of the critical current of the thin film.

The quantity v - is the velocity of "particles" in the state with a wave function ψ.

The Gibbs potential (2.77) also takes the simple form

$$G_S = F_N + |\psi|^2 (\alpha + \frac{1}{2}\beta|\psi|^2 + \frac{1}{2}m'v^2) + \frac{(B - B_e)^2}{2\mu_0} \qquad (2.91)$$

From the condition of minimum of (2.91) relative to $|\psi|^2$ we obtain

$$(\alpha + \beta|\psi|^2 + \frac{1}{2}m'v^2) = 0 \qquad (2.92)$$

Assuming $|\psi| = \psi_0 \cdot f$, where $\psi_0^2 = -\dfrac{\alpha}{\beta}$, and excluding the velocity v from (2.89) and (2.92), we obtain

$$j = e'\psi_0^2 \left(\frac{2|\alpha|}{m'} \right)^{1/2} f^2 (1 - f^2)^{1/2} = e'\psi_0^2 \frac{\hbar}{m'\xi(T)} f^2 (1 - f^2)^{1/2} \quad (2.93)$$

The relationship between j and f^2 is shown in Figure 2.6b. When j increases from zero the function f decreases from the initial value of 1 to 0.8 at $j = j_C = 2e'\psi_0^2 \dfrac{\hbar}{3\sqrt{3}m'\xi(T)}$. When $j > j_c$ there are no solutions, that is, the film is in the normal state. At the transition point the parameter f jumps from 0.8 to 0. The value j_c is called the critical current density of the film.

From a physical point of view, the existence of the critical current density can be easily explained. The current creates a magnetic field that penetrates the sample. At a certain value of current density the magnetic field at some points exceeds the critical value and the sample can no longer remain superconducting in the entire volume. In the future, the critical currents in superconductors will be discussed in detail.

2.5.3.2. Little and Parks effect

Let us consider a superconducting film deposited on an insulating substrate in the shape of a cylinder of radius R (Figure 2.7). The thickness of the film $d << R$. The uniform magnetic field is applied along the axis of the cylinder.

We find the superconducting transition temperature as a function of the applied magnetic field. As before, we assume that $d << \xi(T)$ and $d << \lambda(T)$, and the amplitude $|\psi|$ within the film is constant. As before, we write $\psi = |\psi| \exp(i\varphi(\vec{r}))$, where the amplitude $|\psi|$ does not depend on $\vec{r}$. The density of the Gibbs potential is given by (2.91), and the velocity – by the formula $\vec{v} = \dfrac{1}{m'}(\hbar\vec{\nabla}\varphi - e'\vec{A})$.

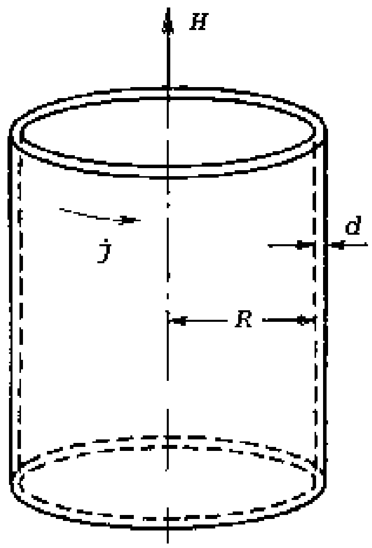


Fig.2.7. The experiment of Little and Parks.

Let us find the dependence of the velocity $\vec{v}$ on the field B. Consider the circulation $\oint \vec{v}\cdot d\vec{l}$, where the integration is carried out along the circumference of the cylinder (radius R). Integrating expression (2.90), obtain

$$\oint \vec{v}\cdot d\vec{l} = 2\pi R v = \frac{\hbar}{m'}[\varphi] - \frac{e'}{m'}\oint \vec{A}d\vec{l} \qquad (2.94)$$

where $[\varphi]$ - the phase change due to a complete revolution around a cylinder. It follows from unambiguity of the function ψ that $[\varphi] = 2\pi n$, where n – any integer. The second term in (2.94) is proportional to the integral

$$\oint \vec{A}d\vec{l} = \int rot\vec{A}\cdot d\vec{\sigma} = \int \vec{B}\cdot d\vec{\sigma} = \Phi = \pi R^2 B \qquad (2.95)$$

i.e. to the magnetic flux inside the cylinder. Consequently,

$$v = \frac{\hbar}{m'R}\left(n - \frac{\Phi}{\Phi_0}\right), \qquad (2.96)$$

where $\Phi_0 = \dfrac{2\pi\hbar}{e'} = 2\cdot 10^{-15}$ Wb – the magnetic flux quantum.

At a fixed magnetic field B the flux Φ is also fixed, and the velocity, in agreement with (2.96), can have an infinite set of discrete values. However from (2.91) it follows that the Gibbs' potential has a minimum only at those values of n at which the modulus of the velocity is minimal. Thus,

$$v = \min\left(\frac{\hbar}{m'R}\left|n - \frac{\Phi}{\Phi_0}\right|\right) \qquad (2.97)$$

i.e. v is the periodic function of a field B with the period $\dfrac{\Phi_0}{\pi R^2}$ (for example, at R=0.8 mkm the period equals 10^{-3} T). Knowing the velocity, we will find the value $|\psi|$ minimizing the Gibbs' potential (see (2.92)):

$$|\psi|^2 = \beta^{-1}(-\alpha - 0{,}5m'v^2) \qquad (2.98)$$

The solution exists only at $-\alpha > 0{,}5m'v^2$. The temperature of transition T_H can be found of a condition $-\alpha = 0{,}5m'v^2$, i.e. T_H is the periodic function of the field B with the period $\dfrac{\Phi_0}{\pi R^2}$. As α is proportional to ($T - T_C$), it is possible to tell that the curve of dependence $T_H(H)$ consists of a number of parabolic arches. The greatest shift of transition temperature takes place at $v = \hbar/2m'R$

$$(T_c - T_H)_{\max} = 0{,}55T_C\left(\frac{\xi_0}{2R}\right)^2 \qquad (2.99)$$

From a physical point of view, the Little and Parks effect can be explained as follows. If the magnetic flux of an external field through an opening of the superconductor isn't equal to an integer of magnetic flux quanta, in accordance with the law of magnetic flux quantization, there has to be a superconducting current in the film, by its field bringing the value of a full magnetic flux to an integer of quanta Φ_0. Emergence of this current leads to an increase in the internal energy due to the kinetic energy of moving Cooper pairs and the energy of the magnetic field created by the current. Therefore the transition to a normal state will happen at a lower temperature: the stronger the currents, i.e. the greater the difference between the external field flux and an integer of quanta Φ_0, the lower the transition temperature.

2.5.4. Variation of amplitude of the order parameter in space

2.5.4.1. Formation of nuclei of superconductivity in a sample

Let us place a superconductor into a strong magnetic field so that the superconductivity is destroyed and the magnetic field in the sample is uniform. We will gradually reduce the field. When the field achieves some value H_{C2}, superconducting areas will start to be formed within the sample. We will show that the field H_{C2} is not equal to the critical field H_C, it can be either more or less.

In the area of formation of the nuclei the amplitude of the order parameter $|\psi|$ is small which allows one to linearize the Ginzburg-Landau equation (2.78)

$$\frac{1}{2m'}\left(-i\hbar\vec{\nabla}-e'\vec{A}\right)^2\psi=-\alpha\psi \qquad (2.100)$$

Also, we will assume that $rot\vec{A}=\mu_0\vec{H}_e$, where $\vec{H}_e$ is a uniform external field. It is admissible because superconducting currents are proportional to $|\psi|^2$ and in the linear approach the amendments to a field caused by them are negligible. The same fact allows for the exclusion, at a minimization of Gibbs's potential, of the integral over the area outside the sample (see 2.5.1) which confirms the acceptability of the Ginzburg-Landau equations in this case.

Equation (2.100) formally coincides with the Schrödinger equation for a particle with charge e' and mass m' in a uniform magnetic field. In an infinit environment such a particle moves along the field with a constant speed and revolves in the xy plane with a frequency

$$\omega_c=\frac{e'B}{m'} \qquad (2.101)$$

The energy levels corresponding to the eigenfunctions of equation (2.100) have the form

$$-\alpha=\frac{1}{2}m'v_z^2+(n+\frac{1}{2})\hbar\omega_c, \qquad (2.102)$$

where n is a non-negative integer. The greatest value of H_e, i.e. ω_C, at a given α corresponds to the case $v_z=0, n=0$, whence $-\alpha=0{,}5\hbar\omega_c$. From this we find H_{c2}. It is possible to show that

$$H_{c2}=\kappa\sqrt{2}H_c\ . \qquad (2.103)$$

Let us discuss the formula (2.103).

At $\kappa>1/\sqrt{2}$, i.e. $H_c<H_{c2}$, nuclei of superconductivity can be formed in the thickness of the sample at fields $H_c<H<H_{c2}$. In this state the field can't be pushed out completely from a sample because at $H>H_c$ the full effect of Meissner is energetically unfavorable. In this magnetic field range a special, so-called mixed state, is established in the sample. This state is typical for type II superconductors and we will discuss it in detail further.

At $\kappa < 1/\sqrt{2}$, i.e. $H_c > H_{c2}$, at reduction of the field, at first the value H_c is reached at which the full Meissner effect takes place, in other words, we have a type I superconductor.

Thus, the division of superconductors into two types can be made depending on the value of the parameter κ: for type I superconductors $\kappa < 1/\sqrt{2}$, at $\kappa > 1/\sqrt{2}$ we have type II superconductors.

CHAPTER 3. MAGNETIC PROPERTIES OF SUPERCONDUCTORS.

As was said in chapter 1, depending on their behavior in a magnetic field superconductors can be divided into three main types. The present chapter is devoted to consideration of the microprocesses occurring in type I and type II superconductors and highlighting their differencies.

§3.1. Type I superconductors. Intermediate state.

In type I superconductors the Meissner state when the magnetic field is pushed out from the volume of a superconductor and is other than zero only in a thin near-surface region, takes place up to some critical field, H_C. If the external field exceeds this value, the sample passes into the normal state. Thus, the dependence of the magnetic induction in a sample on the intensity of the external magnetic field has the form shown in fig. 3.1.

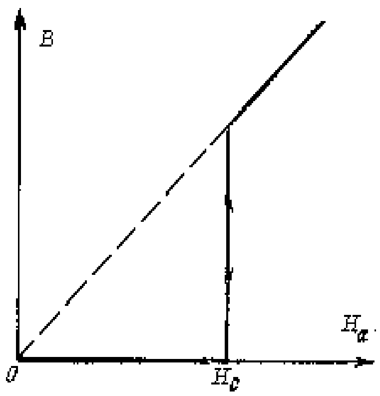


Fig. 3.1. Magnetic field in a sample.

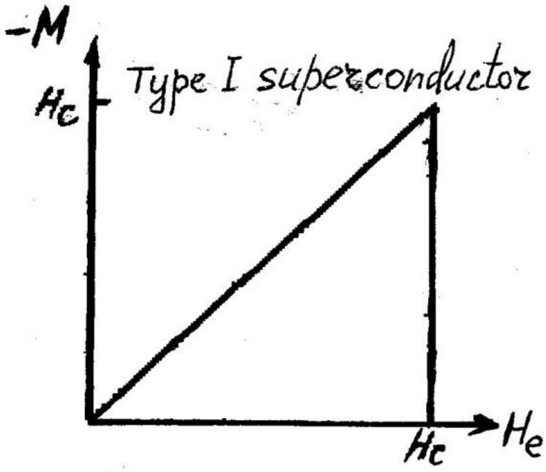


Fig. 3.2. A curve of magnetization of a sample having a core shape in a longitudinal field.

The effect of expelling a magnetic field from a sample can be presented as follows. The shielding currents completely compensating an external magnetic field in a sample give it a magnetic moment. Though, strictly speaking, the internal areas of the sample don't possess a magnetization, it is formally acceptable to speak about the magnetization, $\vec{M}$, being equal to the magnetic moment per unit volume. As was said in §2.2, it allows for the concept of the magnetic field strength, $\vec{H}$, in the sample, using the relation $\vec{B} = \mu_0(\vec{H} + \vec{M})$. Often the magnetization curve, i.e. the dependence of magnetization M on an external magnetic field, is used to characterize the behavior of a sample in a magnetic field. Such a dependence for a type I superconductor derived from the graph of Fig. 3.1 is shown in Fig. 3.2.

As it was shown in §2.2, if the sample has the shape of a long core and is placed into an external field $\vec{H}_e$ parallel to its axis, the field $\vec{H}$ is uniform and equal to $\vec{H}_e$ everywhere in the sample. This fact explains the use of $\vec{H}_e$ instead of $\vec{H}$ in the formula $\vec{B} = \mu_0(\vec{H} + \vec{M})$ to draw Figures 3.1 and 3.2.

In the Meissner state, the magnetic induction $\vec{B}$ in the superconductor is equal to zero, and the macroscopic intensity of the magnetic field, $\vec{H}$, is equal to the external field, i.e. is other than zero. Then the relation $\vec{B} = \mu\mu_0\vec{H}$ can be fulfilled, only if $\mu = 0$. Thus, a superconductor in the Meissner phase is an ideal diamagnet with $\mu = 0$.

The threshold, or critical, magnetic field, H_C, necessary for the destruction of superconductivity depends on temperature. At the critical temperature T_C the critical field is equal to zero. With a decrease of temperature the value H_C increases, which is approximately described by the ratio (see 1.1)

$$H_C(T) = H_C(0)\left(1 - \frac{T^2}{T_C^2}\right) \qquad (3.1)$$

Using expressions $\frac{\hbar^2}{2m'|\alpha|} = \xi^2(T)$ (see the line before (2.82)), $\alpha = -\frac{1}{\mu_0}\frac{B_C^2}{n_S(\infty)}$ (see (2.75)) and $\lambda^2(T) = \frac{m'}{\mu_0 n_S e'^2}$ (see (2.85)), it is easy to come to an expression for H_C

$$H_C = \frac{\hbar}{2\sqrt{2}\mu_0 e\lambda(T)\xi(T)} \qquad (3.2)$$

The difference between free energies per unit volume in the superconducting and normal states of a sample is equal to $F_N - F_S = \mu_0 H_C^2/2$ (see (1.2)).

All this concerns samples having the shape of a long core, placed in a field, parallel to their axis (Fig. 3.3). If one were to neglect the influence of the ends, such a geometry provides equality of values of a field on the entire surface of the sample.

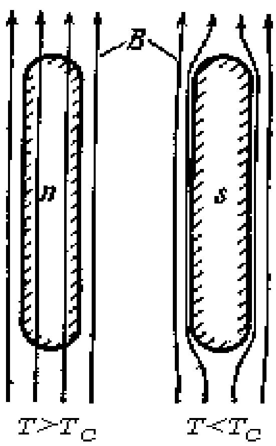


Fig. 3.3. Expulsion of a magnetic field from a sample having a core shape when cooling in a magnetic field.

Let us consider a less trivial case, for example, a superconducting sphere of radius a placed in a uniform external magnetic field $\vec{H}_e$ (Fig. 3.4). If the field H_e is less than $2H_C/3$, the lines of magnetic induction are forced out from the sample. Distribution of the field outside the sphere is defined by equations

$$div\vec{h} = 0; \quad rot\vec{h} = 0 \qquad (3.3)$$

and boundary conditions $h \to H_e$ at $r \to \infty$, $h_n\big|_{r=a} = 0$,

where $h_n\big|_{r=a}$ is a field component, normal to the surface of the sphere, r – the distance from the center of the sphere. The second of the written-down boundary conditions follows from the Meissner effect, i.e. from the fact that magnetic induction lines can't get into a superconducting sphere.

The solution for the area outside the sphere has the form

$$\vec{h} = \vec{H}_e + H_e \frac{a^3}{2} \nabla\left(\frac{\cos\theta}{r^2}\right) \qquad (3.4)$$

A field component, parallel to the sphere surface, is equal to

$$h_\tau\big|_{r=a} = \frac{3}{2} H_e \sin\theta \qquad (3.5)$$

At the poles of the sphere, Q and Q', the field is equal to zero, on the equator ($\theta = \pi/2$) the tangential component is maximum and equal to $3/2H_e$. When the external field H_e reaches the value $2/3H_C$, the field at the equator becomes equal to H_C. Therefore, in the range $H_C > H_e > 2/3H_C$ some areas of the sphere pass into the normal state. The other parts of the sphere do not lose superconductivity (if the entire sample passed into the normal state, the field at any point would equal $H_e < H_C$, i.e. superconductivity would appear again).

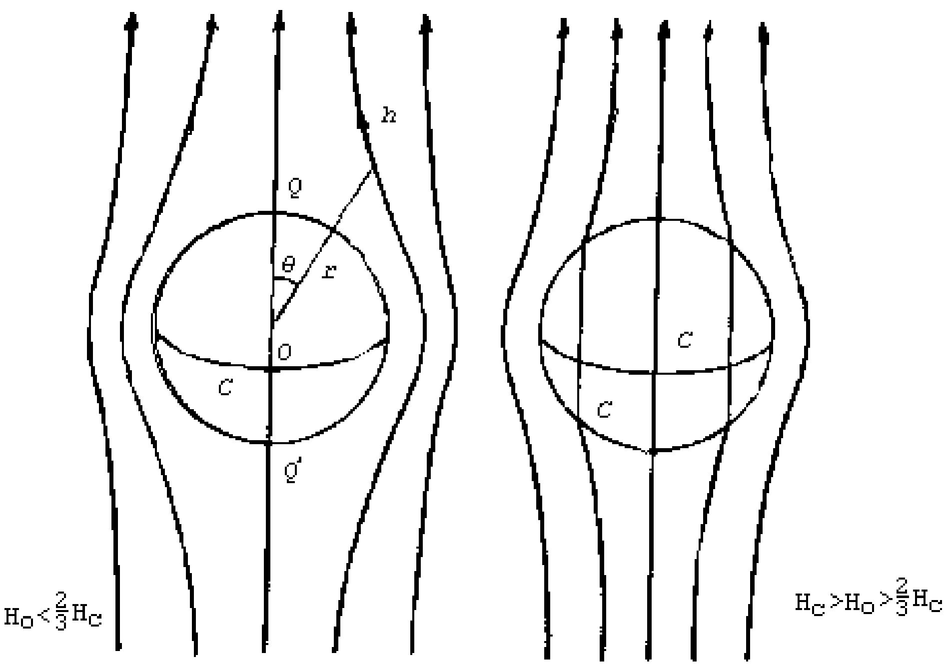


Fig. 3.4. The distribution of a magnetic field near the superconducting sphere.

It could seem that the area near the equator has to pass into the normal state while the central part remains superconducting. Let us show that this can't happen. In this case the field at the interface of superconducting and normal phases has to be critical. With increasing distance from the sphere's axis the field decreases and, therefore, the corresponding areas have to remain superconducting. That contradicts the initial assumption.

It is possible to show that the entire volume inside the sphere will be in a so-called intermediate state, in which normal and superconducting microscopic areas alternate. In the general case of bodies of arbitrary shape, not necessarily all the volume has to be in an intermediate state. There can remain as well areas of purely superconducting and normal states adjoining to an area of an intermediate state, but only not in direct contact with each other.

We note that the range of fields in which there exists the intermediate state depends on the geometrical shape of the sample. For example, if the sample has the shape of an ellipsoid extended (flattened) in the direction of the field, on its equator the field will less (more) differ from the external field and the intermediate state will begin at other values of an external field. In particular, in the case of a thin plate in the field $\vec{H}_e$, perpendicular to its planes, the intermediate state will exist at arbitrarily small value of the field $H_C > H_e > 0$. If in any field, other than zero, any macroscopic part of the plate remained completely superconducting, as a result of pushing out the field on the edge of this area would be very high which inevitably would lead to a transition to the normal state.

The thicknesses of the normal and superconducting layers in an intermediate state are small, therefore at the solution of many tasks it is possible to ignore the microscopic structure of layers and to operate only with the relative volume of S-areas ρ, and also the macroscopic quantities B and H. We will calculate values of these quantities in an intermediate state.

The density of the free energy is equal to

$$F = F_N - \rho \frac{\mu_0 H_C^2}{2} + (1-\rho)\frac{\mu_0 h_N^2}{2} \quad (3.6)$$

where h_N is the field strength in normal areas.

Here the second term represents the energy of condensation in the superconducting areas (see (1.2)), and the third - magnetic energy in normal areas. We neglected the energy of the interfaces between normal and superconducting areas, and also the terms considering a distortion of power lines near the film's surface (for macroscopic bodies these terms are negligible). The magnetic induction by definition is proportional to the average field strength in a point and is equal to

$$B = \mu_0 \langle h \rangle = \mu_0 (1-\rho) h_N + \mu_0 \rho \cdot 0 = \mu_0 (1-\rho) h_N \quad (3.7)$$

In variables ρ and $\vec{B}$ the density of free energy is

$$F = F_N - \rho \frac{\mu_0 H_C^2}{2} + \frac{B^2}{2(1-\rho)\mu_0} \quad (3.8)$$

The free energy is minimal at fixed values of induction $\vec{B}(\vec{r})$ at each point and the temperature T. We write down the expression for the density of thermodynamic potential of Gibbs having a minimum at the fixed temperatures and currents in the coils creating the external field:

$$G = F - BH = F_N - \rho\frac{\mu_0 H_C^2}{2} + \frac{B^2}{2(1-\rho)\mu_0} - BH \quad (3.9)$$

The entire Gibbs potential is equal to the integral over the volume of all space. But as is highlighted previously, the distortion of the field near the surface of the plate is negligible; therefore, at minimization of the Gibbs potential it is possible to be limited to integration (3.9) only over the sample volume.

1) The minimization of Gibbs potential on ρ gives

$$B = \mu_0(1-\rho)H_C \quad (3.10)$$

Comparing (3.10) with (3.7), we come to $h_N = H_C$.

2) As a result of minimization on B we obtain

$$B = \mu_0(1-\rho)H \quad (3.11)$$

We could derive this ratio in another way, having substituted into (3.9) the value of H, found from (2.43), and then having minimized (3.9) on B (in accordance with (2.43)).

From the comparison of (3.11) with (3.10) we receive $H = H_C$, i.e. the value of the field H is constant and equal to H_C in the entire volume of the sample.

Having written down this fact in the form $H_C^2 = H^2$ and having taken a gradient from both parts, we will use the formula $grad(\vec{a}\cdot\vec{b}) = \vec{a}\times rot\vec{b} + \vec{b}\times rot\vec{a} + (\vec{a}\vec{\nabla})\vec{b} + (\vec{b}\vec{\nabla})\vec{a}$ to obtain $0 = \vec{\nabla}H^2 = 2(\vec{H}\vec{\nabla})\vec{H} + 2\vec{H}\times rot\vec{H}$. Since inside the sample $rot\vec{H} = 0$, then $(\vec{H}\vec{\nabla})\vec{H} = 0$, i.e. the vector $\vec{H}$ doesn't change along the line of magnetic induction, therefore, all these lines are straight lines. Generalizing this result, it is possible to tell that for any structure of the intermediate state the borders of the phases have to be parallel to the magnetic field (see Fig. 3.4).

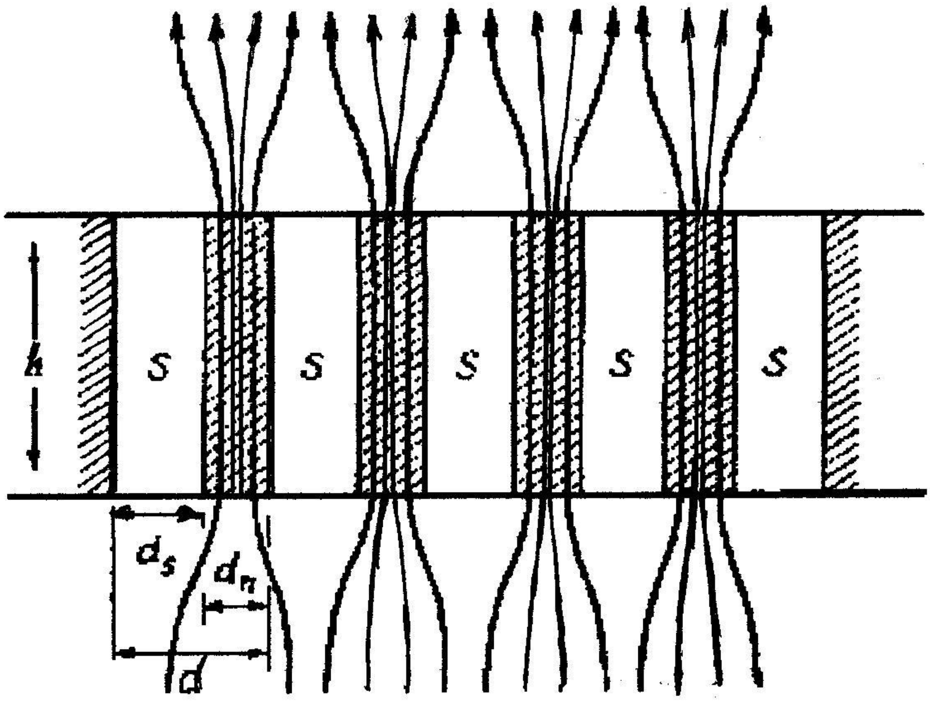


Figure 3.5. The distribution of magnetic induction lines in a plate located in a magnetic field perpendicular to its plane.

Let us consider the case of a flat plate in a field perpendicular to its plane. The distribution of normal (N) and superconducting (S) areas is shown in Figure 3.5: areas N and S form layers perpendicular to the plane of the drawing. Magnetic induction lines pass only through N. The magnetic induction at the interfaces should be equal to $\mu_0 H_C$, and inside S-regions – to zero.

For such a simple configuration the part of the volume occupied by S-regions $\rho = d_S/(d_S + d_N)$ can be found just from the flux conservation. Away from the film the field is uniform, $h = H_e$, and the magnetic flux is $\mu_0 S H_e$ (S - the surface area of the film). In the film this flux passes through the area of the N-region equal to $S(1-\rho)$, and the field in the N-areas, as shown above, is uniform and equal to H_C. Consequently, $\mu_0 S H_e = \mu_0 S(1-\rho) H_C$ whence

$$\rho = 1 - \frac{H_e}{H_C} \tag{3.12}$$

The greater the external field, the smaller the fraction of the volume occupied by the S-regions. Figure 3.6 shows a picture of an intermediate state of a thin film.

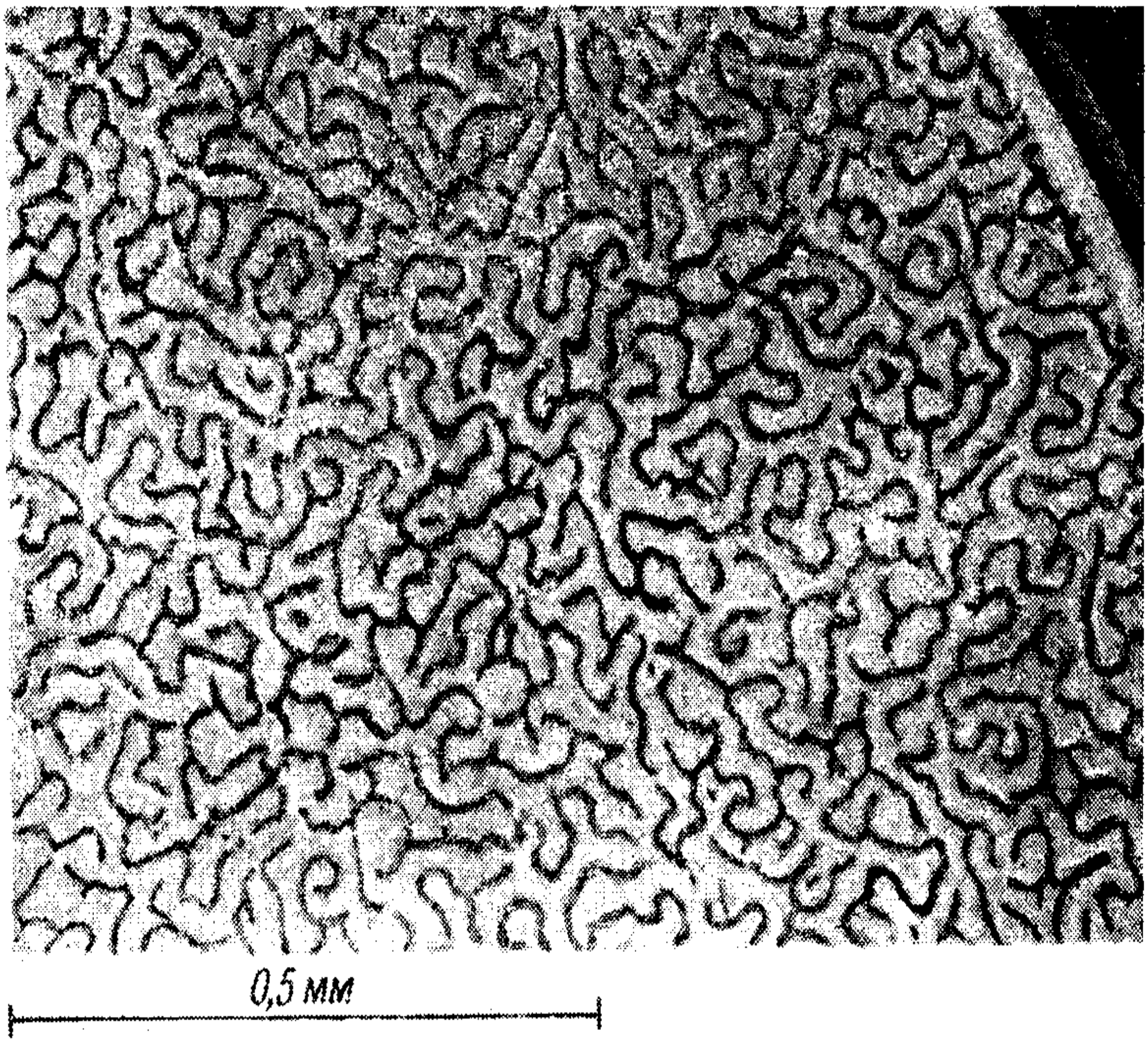


Figure 3.6. Photo of the structure of the intermediate state of the thin film. The thickness of the film is 7 microns. Dark areas are superconducting regions.

§3.2. The energy of the boundary between the phases.

In chapter II it has been shown that the external magnetic field required for the destruction of superconductivity in a superconducting film, having a thickness smaller than the penetration depth, is higher than the corresponding values for the bulk superconductor. Therefore, it would seem, it is energetically favorable for the sample to split into thin layers of alternating S and N, and thus it could remain in that state at $H_e > H_C$. However, it does not happen and at $H_e = H_C$ the sample goes into the normal state. It means that splitting into thin layers is energetically unfavorable. The reason for this is that the formation of an interface is associated with an additional energy, which is positive for type I superconductors. Later we will see that this energy can be negative (in type II superconductors).

Let us consider the interface energy in detail. Figure 3.7 schematically shows a boundary between normal and superconducting phases.

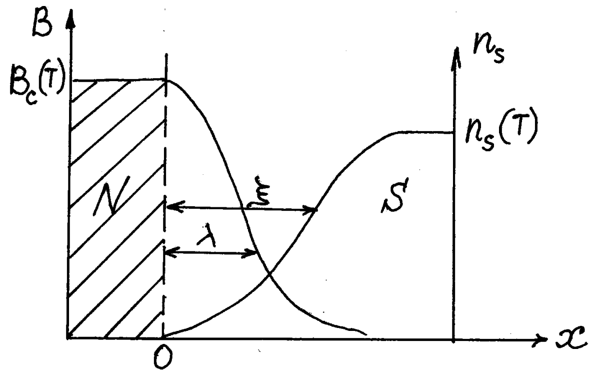


Figure 3.7. The distribution of the magnetic field and the density of Cooper pairs near the boundary of the normal and superconducting phases.

In the normal region (x<0), the magnetic field is higher than or equal to the critical value, and in the superconductor decreases to zero at the London depth λ. In the interior of the superconductor ($x \to \infty$), the density of Cooper pairs is equal to its equilibrium value. The Ginzburg-Landau theory (§2.5) shows that the density of Cooper pairs can not change abruptly, the characteristic length of change is the coherence length ξ. Therefore, in a superconductor the density of Cooper pairs and the magnetic field changes as shown in Figure 3.7. Let us assume that in the given superconductor

$$\xi > \lambda . \qquad (3.13)$$

The energy of the interface is determined by the distinction of the picture near the interface from the situation where immediately to the right of the boundary the field is zero, and the density of superconducting pairs is equal to the equilibrium density. Let us find the energy E_B associated with the expulsion of the magnetic field and the energy, E_C, released by the condensation of Cooper pairs.

In the normal region $E_B = E_C = 0$, and in the depth of the superconductor $E_B = E_C = \mu_0 H_C^2 V/2$ (see Eq. (1.2)), where V is the volume. In the border area, both energies do not reach their full values. The positive energy of pushing out the field is smaller than it would be in the case of full pushing by the value of $\Delta E_B = S\lambda\mu_0 H_C^2/2$, where S is the area of the border. The negative energy of pairs condensation is also reduced (by modulus), since in the boundary layer the concentration is less than the equilibrium value. The reduction of the condensation energy is equal to $\Delta E_C = S\xi\mu_0 H_C^2/2$.

ΔE_B is gained and ΔE_C is lost. Therefore, the additional energy required for the formation of the border is

$$\Delta E_C - \Delta E_B = (\xi - \lambda) S\mu_0 H_C^2/2 \qquad (3.14)$$

and in the case $\xi > \lambda$ is positive. Therefore, the formation of boundaries is energetically unfavorable.

From (3.14) it follows that the sign of the interface energy is determined by the ratio between the London length λ and the coherence length ξ.

§3.3. The magnetic properties of type II superconductors.

From the previous discussion it follows that in the case $\xi < \lambda$ the creation of the interface must be connected with an energy gain. Therefore, we should expect that the magnetic field can penetrate a superconductor already at the field $H < H_C$, while there are irregularities in the spatial distribution of the magnetic field and the density of Cooper pairs.

It turns out that the condition $\xi < \lambda$ can be fulfilled in any case if we reduce the mean free path of the electrons. The fact is that, when it decreases, the penetration depth λ slightly increases, and the coherence length ξ rapidly decreases. The mean free path can be easily reduced by doping a superconductor with foreign metals. The electrons are scattered by the atoms of impurities and their mean free path is reduced.

Superconductors with $\xi < \lambda$ are called type II superconductors. They are characterized by the following macroscopic properties.

1. For the cylinder placed in a longitudinal magnetic field, the Meissner effect occurs up to a value H_{C1} which is significantly less than H_C.

2. When $H > H_{C1}$ the induction lines penetrate the sample, but only partially. This is the case for the field $H_{C1} < H < H_{C2}$. The field H_{C2} is higher than H_C and in some cases is very high.

3. When $H > H_{C2}$ the macroscopic sample does not push out the flux. However, even in this case, the superconductivity is not completely destroyed: in the field region $H_{C2} < H < H_{C3}$ on the cylinder surface there remains a superconducting layer with a thickness of the order less than a micron. The physical reason for the presence of such a layer is as follows: a small superconducting region can more easily be formed near the sample surface like an air bubble is more easily formed on the bottom of the glass of lemonade than anywhere inside.

Changing of the fields H_{C1}, H_{C2}, H_{C3} with temperature is shown in Figure 3.8.

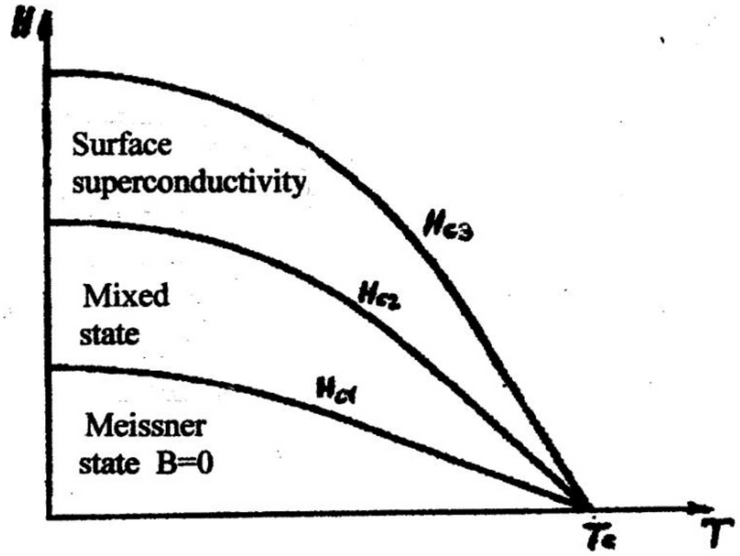


Figure 3.8. The phase diagram for a type II superconductor in the shape of a long cylinder.

Consider the range $H_{C1} < H < H_{C2}$ in which the flux penetrates partially. For the first time the existence of such a region has been demonstrated by Shubnikov in 1937. Therefore, sometimes it is called the Shubnikov phase. We will call this area the vortex or the mixed state.

Typical plots of $B(H)$ for type I and type II superconductors are shown in Figure 3.9. As it was mentioned earlier, the behavior of the sample in a magnetic field is often described by a magnetization curve, that is, the dependence of the magnetization M on the external magnetic field strength H_e. In Figure 3.10 such graphs derived from the Figure 3.9 are shown. Note that if the values H_C are the same for both materials, then the shaded area of curvilinear triangles on Figure 3.10 are equal.

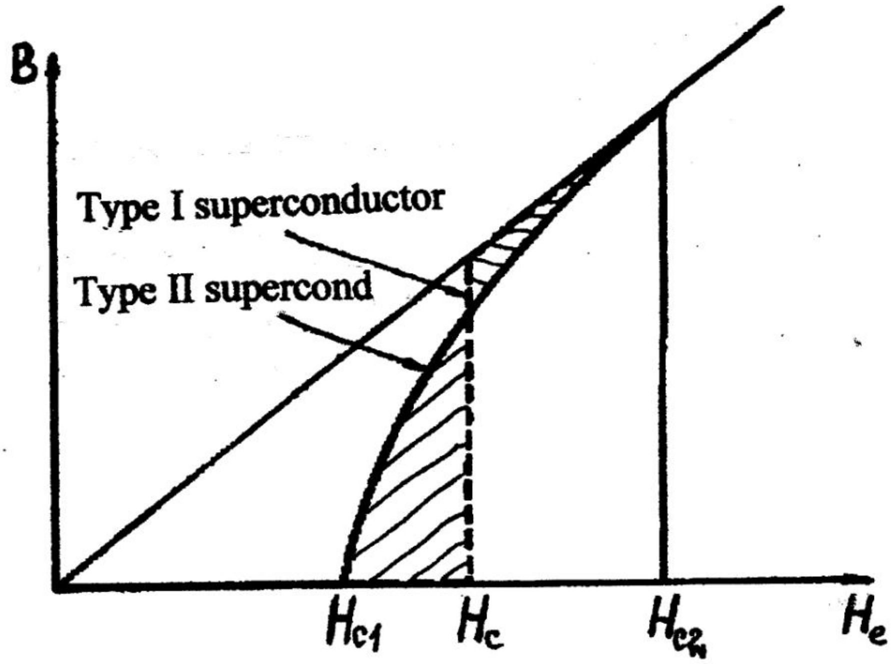


Figure 3.9. The dependence of the induction B on the applied field H_e for type I and type II superconductors in the shape of a long cylinder.

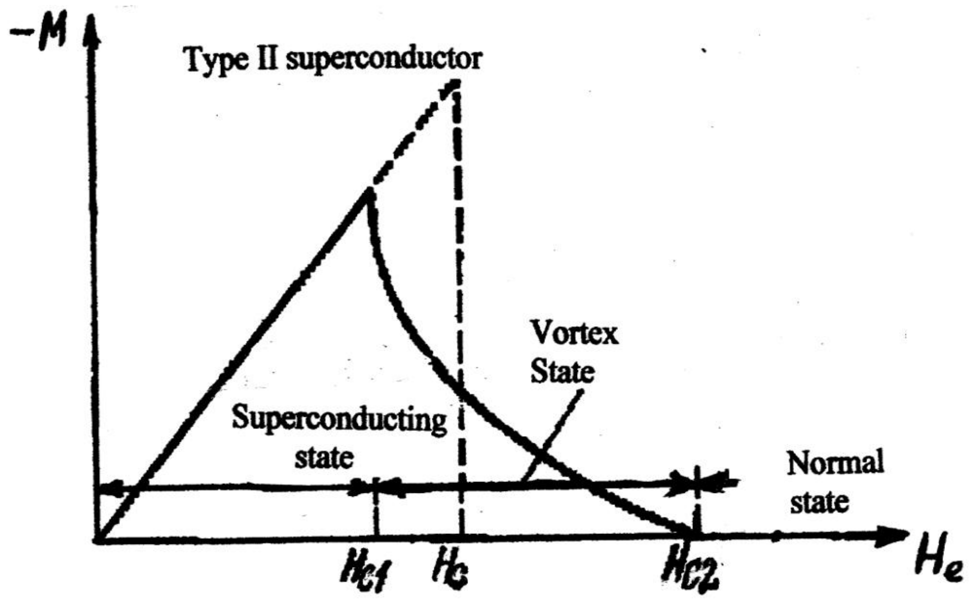


Figure 3.10. Reversible magnetization curves of type I and type II superconductors in the shape of a long cylinder.

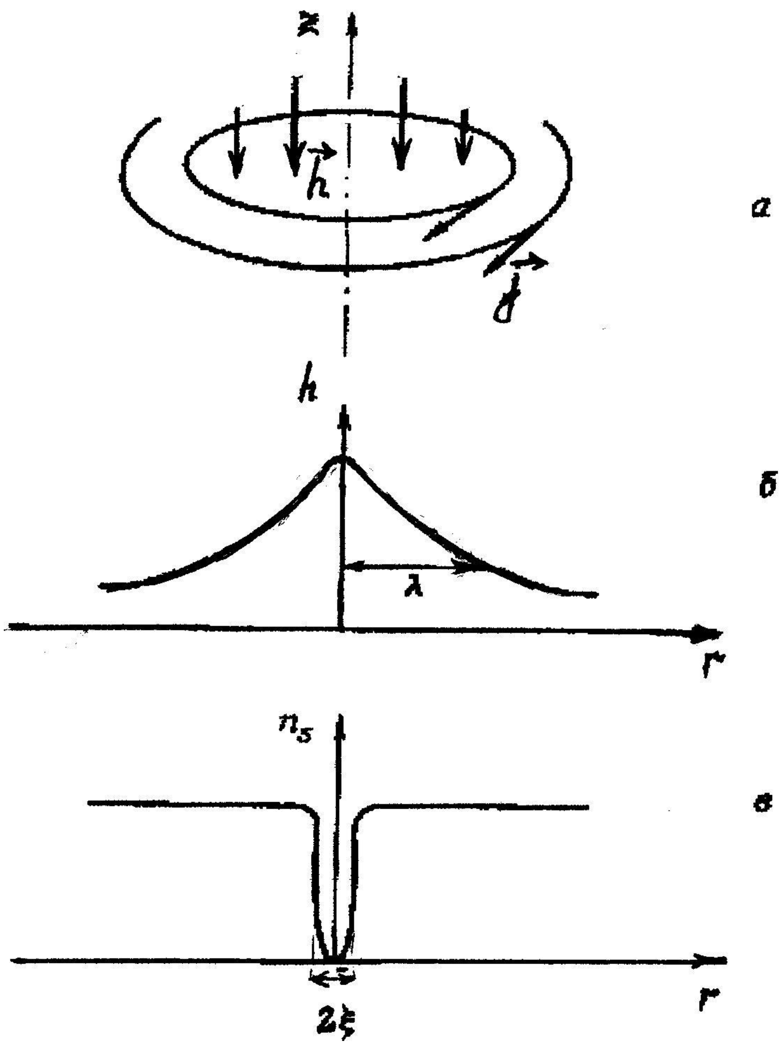


Figure 3.11. The structure of an isolated vortex filament: a – the configuration of the field and the currents, b – the dependence of the magnetic field on the distance to the filament axis, c - the density of Cooper pairs in the core region of the filament.

As mentioned above, when $\xi < \lambda$, it is advantageous to be split up into a large number of microscopic regions with a characteristic size of the order of ξ. Two types of regions are most likely: layers of small thickness and filaments of small diameter. Theoretical calculations show that a filamentary structure has less energy. For the first time such structures were discussed by Onsager and Feynman in connection with the phenomenon of the superfluidity of helium. In 1956, A.A.Abrikosov generalized this approach to the case of superconductivity.

The structure of an isolated vortex filament is shown in Figure 3.11. The thread has a rigid skeleton of the radius ξ in which the density of superconducting electrons falls to zero when approaching the center. The magnetic field lines exist not only in the core region. The field has maximum on the axis of the thread and extends from it to a distance of the order of λ. We will show that the value of the magnetic flux associated with one thread is equal to one flux quantum (see also section 1.9).

The equation for the current density has the form

$$\vec{j}_S = \frac{e'i\hbar}{2m'}(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi) - \frac{e'^2}{m'}|\psi|^2\vec{A} \qquad (3.15)$$

It should be noted once again that, although (3.15) is one of the Ginzburg-Landau equations (see. (2.43)), its applicability is not limited by the proximity of the temperature to T_C, because, as mentioned earlier, it is the general expression for the current density in quantum mechanics.

Substituting $\psi(\vec{r}) = |\psi(\vec{r})| \cdot \exp(i\varphi(\vec{r}))$ into (3.15), we obtain

$$\vec{j} = \frac{e'}{m'}|\psi|^2(\hbar\vec{\nabla}\varphi - e'\vec{A}) \qquad (3.16)$$

Let us integrate (3.16) over a circle around the axis of the vortex with a radius $r >> \lambda$. At these distances the current density j can be considered as being equal to zero. It should be taken into account that, due to the axial symmetry, the modulus of the current density is the same at all points of the circle, we obtain

$$\hbar[\varphi] = e'\oint \vec{A}\cdot d\vec{l} = e'\int rot\vec{A}\cdot d\vec{S} = e'\int \vec{B}\cdot d\vec{S} = e'\Phi \qquad (3.17)$$

where $[\varphi]$ is the entire phase change, Φ - the magnetic flux through the loop. From the unambiguity of the function ψ we can infer that $[\varphi] = 2\pi n$, where n is an arbitrary integer. The minimum value of n is 1, so the minimum magnetic flux associated with the thread is equal to one quantum of magnetic flux $\Phi_0 = h/2e = 2\cdot 10^{-15}$ Wb. In principle, the vortex can contain any integer number of quanta Φ_0, but for reasons of minimum energy it prefers to be disintegrated into several vortices with one quantum Φ_0 each. Indeed, we will see that the energy of the vortex is proportional to the square of the magnetic flux. This means that the energy of a vortex with n quantum Φ_0 is n times larger than the energy of n vortices with one Φ_0 each.

3.3.1. The properties of the isolated vortex filament

Let us consider in detail the structure of a single vortex filament in the case $\xi << \lambda$.

Since in this case the core of the vortex is small, when calculating the energy we ignore its input, i.e. the change in the energy of the superconducting phase condensation. Then the free energy per unit length of an individual thread is equal to

$$F = \frac{\mu_0}{2}\int (h^2 + \lambda^2 \left| rot\vec{h} \right|^2) dV \qquad (3.18)$$

where the integral is taken over the area $r > \xi$.

From the condition of minimum for F we obtain (beyond the core) the London equation

$$\vec{h} + \lambda^2 rotrot\vec{h} = 0 \qquad (3.19)$$

Inside the core, strictly speaking, it would be necessary to apply a more complicated equation, but, since the radius of the core is small, we can replace the existing singularity by the two-dimensional δ-function:

$$\vec{h} + \lambda^2\, rotrot\vec{h} = \frac{\vec{\Omega}}{\mu_0}\delta(\vec{r}) \qquad (3.20)$$

where $\vec{\Omega}$ - a vector directed along the thread.

We will show that the modulus of the vector $\vec{\Omega}$ is equal to the flux quantum Φ_0. Let us integrate (3.20) over the area of the circle of radius r centered on the axis:

$$\int \vec{h} d\vec{\sigma} + \lambda^2 \oint rot\vec{h} \cdot d\vec{l} = \Omega / \mu_0 \qquad (3.21)$$

If the radius of the selected circle is much larger than λ $(r >> \lambda)$, the currents along the path, and hence all the contour integrals can be neglected. Then we find that the modulus of vector $\vec{\Omega}$ is equal to the magnetic flux associated with the thread, i.e. $\Omega = \Phi_0$.

Let us solve equation (3.20), together with the Maxwell equation $div\vec{h} = 0$.

It is easy to find the value of the current density $\vec{j} = rot\vec{h}$ in the area $\xi < r << \lambda$. If the integration path lies in this region, then in equation (3.21) the first term can be neglected, since only a small part of the total flux Φ_0 passes through the circuit. Then we obtain $\lambda^2 2\pi r \left| rot\vec{h} \right| = \Phi_0 / \mu_0$ or

$$j = \left| rot\vec{h} \right| = \frac{\Phi_0}{2\pi\lambda^2 \mu_0}\frac{1}{r}, \text{ at } \xi < r << \lambda \qquad (3.22)$$

Considering $\left| rot\vec{h} \right| = -\frac{\partial h}{\partial r}$ and integrating, we obtain

$$h = \frac{\Phi_0}{2\pi\lambda^2 \mu_0}\left(\ln\frac{\lambda}{r} + const \right), \text{ at } \xi < r << \lambda \qquad (3.23)$$

To calculate the constant let us find the exact solution of (3.20). In cylindrical coordinates, this equation has the form (for $r > 0$)

$$h''+\frac{1}{r}h'+\frac{h}{\lambda^2}=0 \qquad (3.24)$$

The solution of this equation decreasing when $r\to\infty$ has a form

$$h=C\cdot K_0\left(\frac{r}{\lambda}\right),\text{ at } \xi<r \qquad (3.25)$$

where K_0 is the Bessel (Hankel) function of zero order of imaginary argument.

Coefficient C in (3.25) and $const$ in (3.23) can be found by combining (3.23) and (3.25), resulting in (3.23) and (3.25) taking the form

$$h=\frac{\Phi_0}{2\pi\lambda^2\mu_0}\ln\frac{\lambda}{r} \qquad \xi<r<<\lambda \qquad (3.26)$$

$$h=\frac{\Phi_0}{2\pi\lambda^2\mu_0}K_0\left(\frac{r}{\lambda}\right) \qquad \xi<r \qquad (3.27)$$

The asymptotic solution of (3.27) at $r>>\lambda$ takes the form

$$h=\frac{\Phi_0}{2\pi\lambda^2\mu_0}\sqrt{\frac{\pi\lambda}{2r}}\cdot\exp\left(-\frac{r}{\lambda}\right) \qquad (3.28)$$

Knowing the expression for the field, we can find the energy F per unit length of the filament. Using the formula of vector analysis $div(\vec{a}\times\vec{b})=\vec{b}\cdot rot\vec{a}-\vec{a}\cdot rot\vec{b}$ and equation (3.19), we obtain the expression

$$div(\vec{h}\times rot\vec{h})=(rot\vec{h})^2-\vec{h}\cdot rotrot\vec{h}=(rot\vec{h})^2+h^2/\lambda^2\,. \qquad (3.29)$$

Substituting (3.29) into (3.18) and applying the Gauss theorem, we obtain

$$F=\frac{\mu_0\lambda^2}{2}\int div(\vec{h}\times rot\vec{h})dV=\frac{\mu_0\lambda^2}{2}\int(\vec{h}\times rot\,\vec{h})\,d\vec{\sigma}\,, \qquad (3.30)$$

where the integration in the last integral is taken over the surface of the core, i.e. of the cylinder with a radius ξ. Since on the core surface the vectors $\vec{h}$, $rot\vec{h}$ and $d\vec{\sigma}$ are mutually perpendicular, and their modules, according to (3.22) and (3.26), are constant, they can be taken out from the integral in (3.30). Then we get

$$F=\frac{\mu_0\lambda^2}{2}h(\xi)j(\xi)\cdot 2\pi\xi=\frac{\Phi_0^2}{4\pi\lambda^2\mu_0}\ln\frac{\lambda}{\xi}\,. \qquad (3.31)$$

Taking into account the core energy gives the final expression for the energy per unit length of the thread

$$F=\frac{\Phi_0^2}{4\pi\lambda^2\mu_0}\left(\ln\frac{\lambda}{\xi}+\varepsilon\right), \qquad (3.32)$$

where $\varepsilon\approx 0{,}1$.

3.3.2. Interaction of vortex filaments

Let us consider two threads parallel to the z axis and passing at z = 0 through the points $\vec{r}_1 = (x_1, y_1)$ и $\vec{r}_2 = (x_2, y_2)$. The resulting magnetic field distribution is described by the equation

$$\vec{h} + \lambda^2\, rot rot \vec{h} = \frac{\vec{\Phi}_0}{\mu_0}\left[\delta(\vec{r} - \vec{r}_1) + \delta(\vec{r} - \vec{r}_2)\right], \quad (3.33)$$

which is a generalization of (3.20).

The solution is a superposition of the two fields $\vec{h}(\vec{r}) = \vec{h}_1(\vec{r}) + \vec{h}_2(\vec{r})$ generated by each of the threads individually

$$\vec{h}_i(\vec{r}) = \frac{\vec{\Phi}_0}{2\pi\lambda^2\mu_0} K_0\left(\frac{|\vec{r} - \vec{r}_i|}{\lambda}\right) \quad (3.34)$$

To find the free energy of the system it is necessary to calculate the integral (3.30) over the surface of the cores of both vortices that is not as easy as it was for a single thread. To calculate the energy of interacting of two vortices per unit length we should deduct from the energy of the system the own energy of the threads (3.31) that in the case $\xi << \lambda$ gives the following expression

$$U_{12} = \frac{\mu_0\lambda^2}{2}\int(\vec{h} \times rot\vec{h})\, d\vec{\sigma} - 2F = \frac{\Phi_0^2}{2\pi\lambda^2\mu_0} K_0\left(\frac{|\vec{r}_2 - \vec{r}_1|}{\lambda}\right) > 0 \quad (3.35)$$

The positive interaction energy U_{12} corresponds to the mutual repulsion of the threads. For large distances, r_{12}, between the threads ($r_{12} >> \lambda$) the interaction energy U_{12} decreases as $\sqrt{\frac{1}{r_{12}}} \cdot \exp\left(-\frac{r_{12}}{\lambda}\right)$.

Writing (3.35) in the form $U_{12} = \Phi_0 h_1$, we can find the force acting in the *x* direction, for example, on the second vortex $f_2 = -\partial U_{12} / \partial x_2 = \Phi_0 j_{1y}(\vec{r}_2)$, as $rot\, \vec{h} = \vec{j}$. Writing the equation in the vector form, we get

$$\vec{f}_2 = \vec{j}_1(\vec{r}_2) \times \vec{\Phi}_0 \quad (3.36)$$

Summarizing (3.36) for an arbitrary lattice of vortices, we obtain the expression for the force acting on the vortex

$$\vec{f} = \vec{j}(\vec{r}) \times \vec{\Phi}_0, \quad (3.37)$$

where $\vec{j}(\vec{r})$ is the full current density from all other vortices (including even the density of the transport current) at the point of location of the axis of the vortex.

3.3.3. The magnetization curve of type II superconductors.

Suppose that the sample has the shape of a long core and is placed into an external field $\vec{H}_e$ parallel to its axis. If the field is sufficiently small, threads, if they exist, are rarely located, and their interaction can be neglected. Then the Gibbs potential per unit volume (1 m^2 x 1 m) is equal to

$$G = F_S + n_L F - BH \tag{3.38}$$

where n_L - the number of threads per 1 m^2, F - the energy per unit length of the thread (3.31).

It has repeatedly been said that, for the considered geometry, the magnetic field $\vec{H}$ at all points, both outside and inside the sample, is equal to the external field $\vec{H}_e$. Therefore, instead of $\vec{H}$ in the formulas we will write $\vec{H}_e$.

Since each filament has a magnetic flux Φ_0, the induction equal to the magnetic flux per unit area is given by

$$B = n_L \Phi_0 \tag{3.39}$$

which allows us to write the Gibbs potential in the form

$$G = F_S + B\left(\frac{F}{\Phi_0} - H_e\right) \tag{3.40}$$

If $H_e < \frac{F}{\Phi_0}$ the most favorable energy situation corresponds to $B = 0$; the magnetic field is expelled from the sample (the Meissner effect). If, however, $H_e > \frac{F}{\Phi_0}$ the advantageous situation is $B \neq 0$. Thus, the critical field H_{C1} is

$$H_{C1} = \frac{F}{\Phi_0} = \frac{\Phi_0}{4\pi\lambda^2\mu_0}\ln\frac{\lambda}{\xi} \tag{3.41}$$

Compare the value of the field H_{C1} with the field $H_C = \frac{\hbar}{2\sqrt{2}\mu_0 e\lambda(T)\xi(T)}$ (see 3.2). The ratio of these fields is equal

$$\frac{H_{C1}}{H_C} = \frac{1}{\sqrt{2}}\frac{\xi}{\lambda}\ln\frac{\lambda}{\xi} \sim \frac{\xi}{\lambda} \tag{3.42}$$

and may be very small.

The magnitude of the field H_{C2}, in which the small superconducting areas are starting to be formed in the sample, as it has been shown in §2.5.4, is govern by the Ginzburg-Landau parameter $\kappa = \lambda(T)/\xi(T)$:

$$H_{c2} = \kappa\sqrt{2}H_c = \frac{\Phi_0}{2\pi\mu_0\xi^2(T)} \tag{3.43}$$

From (3.43) it is clear that the field H_{C2} corresponds to the situation when the cores of the vortices begin to overlap.

The value H_{C3} is associated with a surface superconductivity and is determined by the creation of superconducting nuclei at the surface of the sample. Calculation based on the Ginzburg-Landau equations leads to the following expression

$$H_{c3} = 1{,}7H_{c2} = 2{,}4\kappa H_C \tag{3.44}$$

The equilibrium density of the threads in the sample can be found from the condition of minimum of Gibbs potential for a large number of threads. Consider the case when the external field slightly exceeds the critical value. Abrikosov showed that the minimal Gibbs potential corresponds to the periodic structure. Detailed calculations show that this is a triangular lattice (see Fig.3.12). For a small excess of H_e over H_{C1} the density of the vortex filaments is low, so it is necessary to take into account only the interaction of the nearest neighbors (see 3.35)

$$G = F_S + n_L(F + \frac{Z}{2}U_{12}) - BH = F_S + B\left(H_{C1} - H_e + \frac{1}{2}Z\frac{\Phi_0}{2\pi\lambda^2\mu_0}K_0\left(\frac{d}{\lambda}\right)\right) \quad (3.45)$$

where Z - the number of nearest neighbors (in a triangular lattice, Z = 6), K_0 - Bessel function (Hankel) of zero order of imaginary argument, d – the distance between adjacent vortices. Given that the area of each triangle $S = d^2\sqrt{3}/4$ contains a flux of $3 \cdot \Phi_0 / 6 = \Phi_0 / 2$ (in the triangular plane lattice each node is divided by 6 cells – see Fig. 3.12) from the formula $B = \Phi / S$ we can find $d = \sqrt{\dfrac{2\Phi_0}{B\sqrt{3}}}$.

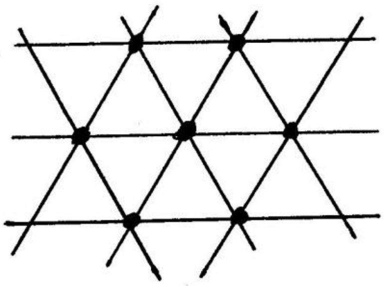


Figure 3.12.

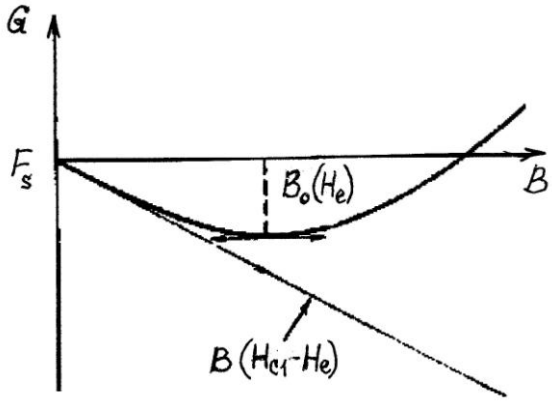


Figure 3.13. The dependence of the thermodynamic potential G on the induction B.

The dependence $G(B)$ is shown in Figure 3.13. For a small excess of H_e over H_{C1} the last term in (3.45), corresponding to the interaction, is small and the slope is negative. With an increase of the induction B the interaction is growing, but rather slowly. This is due to the fact that when $d > \lambda$ the value of $K_0\left(\frac{d}{\lambda}\right)$ takes the form $K_0\left(\frac{d}{\lambda}\right) \approx \exp\left(-\frac{d}{\lambda}\right) = \exp\left(-1.07\sqrt{\frac{\Phi_0}{B\lambda^2}}\right)$. Therefore, for small B the interaction is small. However, at high B the contribution of this term is predominant, which leads to the growth of the function $G(B)$. Therefore, at a certain value $B_0(H_e)$ the function $G(B)$ reaches a minimum. This value will be the equilibrium value of the induction in the field $\vec{H}_e$.

The theoretical curve $M(H_e)$ for the vortex model is shown in Figure 3.14 (solid curve). At $H_e = H_{C1}$ it has an infinite slope. The dashed curve relates to a laminar model. The crosses show the results of the experiment.

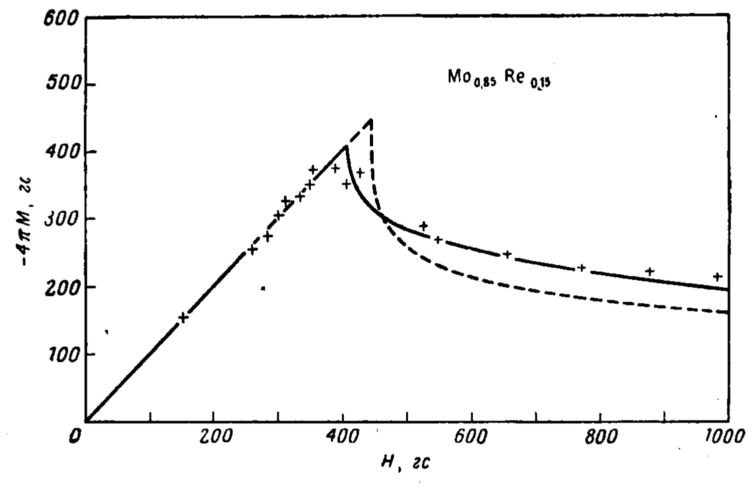


Figure 3.14. The magnetization curves of the type II superconductor having a shape of a long cylinder: solid line - vortex model, dashed - laminar model, crosses - the experimental values.

3.3.4. Laminar structure

Let us find the Gibbs potential and critical field for the structure shown in Figure 3.15. It is a system of thin equidistant N-layers. We denote by d the period of the structure and assume that the layers are perpendicular to the x-axis. Within the N-layers the superconductivity is suppressed, and in superconducting areas it is characterized by the usual value of the density of superconducting electrons.

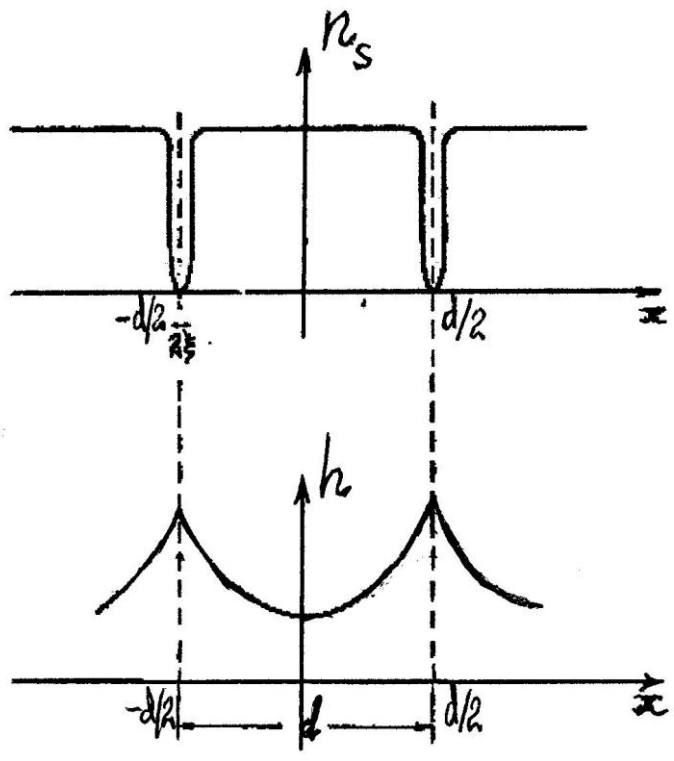


Figure 3.15. The structure of the Shubnikov phase in a laminar model.

Suppose, as before, that $\xi << \lambda$. We place the origin of coordinates in the middle between the neighboring N-layers. Field $h(x)$ is parallel to the z axis and everywhere, except the narrow N-layers, satisfies the equation

$$h = \lambda^2 \frac{d^2 h}{dx^2} \qquad (3.46)$$

The solution of equation (3.46) has the form

$$h(x) = H_m \frac{ch(x/\lambda)}{chP} \qquad (3.47)$$

where $P = d/2\lambda$, H_m is the field within the N-layers.

Let us find the value of the magnetic induction

$$B = \mu_0 \langle h \rangle = \mu_0 \frac{2}{d} \int_0^{d/2} H_m \frac{ch(x/\lambda)}{chP} dx = \mu_0 H_m \frac{thP}{P} \qquad (3.48)$$

The free energy density in the S-regions, according to (3.18) is equal to

$$F_1 = \frac{2}{d} \frac{\mu_0}{2} \int_0^{d/2} \left[h^2 + \lambda^2 \left(\frac{dh}{dx} \right)^2 \right] dx = \frac{\mu_0 H_m^2}{2} \frac{thP}{P} \qquad (3.49)$$

We must add to this quantity the energy of creating the N-layers (per 1 m^3)

$$F_2 \approx \frac{\mu_0 H_C^2}{2} \frac{2\xi}{d} = \frac{\mu_0 H_C^2}{2} \frac{1}{P\kappa} \qquad (3.50)$$

where $\kappa = \lambda/\xi$ - the Ginzburg-Landau parameter.

To move to the Gibbs potential we need to subtract $BH = \mu_0 H H_m \frac{thP}{P}$:

$$G_{ламин} = \frac{\mu_0}{2} \left(H_m^2 \frac{thP}{P} + \frac{H_C^2}{P\kappa} - 2HH_m \frac{thP}{P} \right) \qquad (3.51)$$

Taking into account the comments to (3.38), in formulas we will write H_e instead of H.

Minimizing G by H_m, we find that the minimum is reached at $H_m = H_e$

$$G_{\min} = \frac{\mu_0}{2P} \left(\frac{H_C^2}{\kappa} - H_e^2 thP \right) \qquad (3.52)$$

At $H_e < H_C / \sqrt{\kappa}$ the minimum corresponds to $P = \infty$, i.e. there is a full Meissner effect.

When $H_e > H_C / \sqrt{\kappa}$ the minimum is reached at a finite value of P.

Thus, the critical field in the laminar model is equal to

$$H_{C1_{ламин}} = \frac{H_C}{\sqrt{\kappa}} \qquad (3.53)$$

Comparing this value with the critical field for the threads H_{C1} from (3.41)

$$H_{C1} = \frac{1}{\sqrt{2}} \frac{\ln\kappa}{\kappa} H_C \qquad (3.54)$$

we come to the conclusion that in this case (when $\xi << \lambda$, i.e. $\kappa >> 1$) the critical field for the filaments is less than for the laminar structures. From the analysis it follows that in the range $H_{C1} < H_e < H_C / \sqrt{\kappa}$ the laminar state is a Meissner one, and the energy of the vortex state in this range is less than that of the Meissner one. In other words, the vortex state is energetically more favorable than the laminar state.

We can show that this is true as well in the field range $H_e > H_C / \sqrt{\kappa}$, i.e. throughout the entire range of fields $H_e > H_{C1}$ the vortex state is the most energetically favorable.

CHAPTER 4. CRITICAL CURRENTS IN SUPERCONDUCTORS

In §1.4 it was shown that there exists a critical velocity of Cooper pairs, and hence a critical current density. When the current density is less than the critical value the system of Cooper pairs can not interact with the lattice. If the current density is greater than the critical value the Cooper pairs are being destroyed and the superconductivity disappears.

The problems related to the critical currents are essential for the technical applications of superconductors. While type II superconductors retain superconductivity in strong magnetic fields, for their technical applications it is just important that they could carry, without losses, currents of sufficiently high values. As we will see, this problem can be solved in type III superconductors.

§4.1. Critical currents in type I superconductors

The simplest case, from a geometrical point of view, is a wire of circular cross section, through which a current flows. When the current is weak, the wire should be in the Meissner phase, i.e. the magnetic induction $\vec{B}$ within the sample is zero. It follows that inside the sample the current can not flow, because it would create a magnetic field inside the superconductor. Therefore, the current flows only in a thin surface layer, into which the magnetic field can penetrate. These currents, in order to be distinguished from screening ones, will be called the transport currents.

Figure 4.1 shows the distribution of the transport current density and magnetic field in the cross section of the wire. With an increase of the current the magnetic field increases. In 1916 F. Silsbee suggested that the critical current density is achieved when at the surface of the sample the magnetic field reaches a critical value. This assumption was brilliantly confirmed by experiment. With Silsbee's hypothesis one can also find critical currents of superconductors in an external magnetic field. To do this, we have to sum the external field and the field of the transport current. The critical value of the current corresponds to the moment, when at some point of the sample, the magnetic field is equal to the critical value.

The critical current density can be very high (~10^7 A / cm^2), but due to the thinness of the surface layer the total current is not high.

Consider a wire with radius a, through which a current J flows. The field at the surface of the wire is $h(a) = J / 2\pi a$ (from the theorem of circulation of magnetic strength). If $h(a) < H_C$ the whole wire can be in a superconducting state. This condition defines a critical current value $J_C = 2\pi a H_C$. If $J > J_C$ we have $h(a) > H_C$ and the wire near the surface must go to the normal state. However, the entire wire can not be in a normal state, because in this case the current would be distributed uniformly over the cross section of the wire and the field near the axis would be less than critical. To verify this, we find the field inside the wire at a distance r from the axis when the current is flowing with a current density j in all points of the cross section. Using Stokes theorem on the circulation of the magnetic strength ($\oint \vec{H}d\vec{l} = J$) we obtain:

$$h(r) = \frac{J_r}{2\pi r} = \frac{j\pi r^2}{2\pi r} = \frac{jr}{2} \qquad (4.1)$$

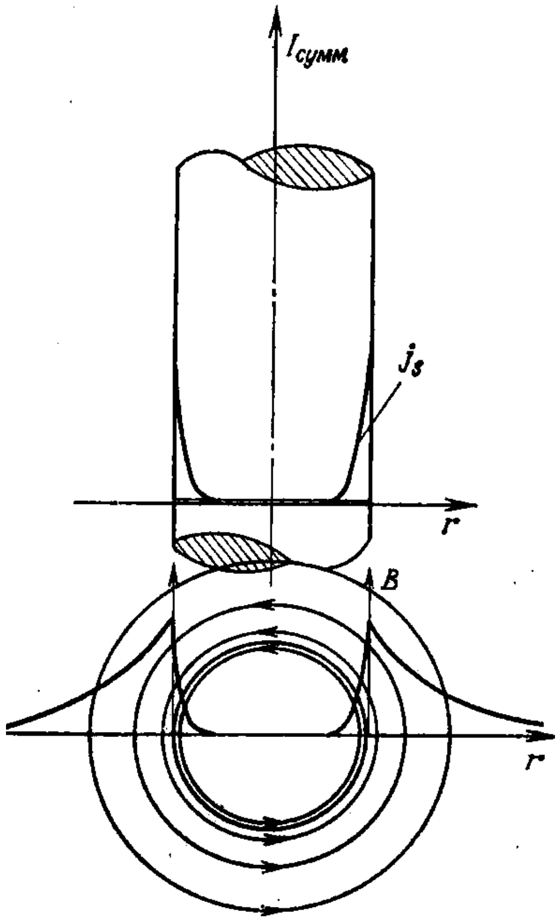


Figure 4.1. Distributions of current density and magnetic field in a superconducting wire with a transport current.

From (4.1) it follows that near the axis ($r \approx 0$) the field is weak. Therefore, the transition of these areas to the normal state is energetically unfavorable. Hence, the outer region of the wire ($R < r < a$) is normal and the inner part ($0 < r < R$) should be either in a superconducting, or in an intermediate state. The radius of interface R corresponds to the condition $h(R) = H_C$. Thus, the current flowing in the inner part is

$$J_1 = 2\pi RH_C = \frac{R}{a}J_C < J_C \qquad (4.2)$$

The rest of the current ($J - J_1$) flows in the outer part of the section. Since it is normal, then the current flowing through it requires a voltage along it. Consequently, the inner part can not be completely superconducting since it would short-circuit the poles of the generator. Thus, the inner part of the wire is in the intermediate state. But the option of alternating flat layers, as in Figure 3.5, does not solve the problem, since at the interfaces of superconducting and normal areas the magnetic field must be equal to the critical value, and in a flat version it is not fulfilled.

The detailed calculation gives the structure shown in Figure 4.2. Once the current reaches a critical value, the wire jumps to the state in which the superconducting cells reach the surface. With further increase of the current something like a normal phase shell arises covering a core in the intermediate state; with increasing current, the thickness of the shell increases, and the superconducting core region decreases. At all points on the N-S interfaces $h = H_C$, i.e. the closer to the axis the higher is the current density (see. 4.1), which is obtained by increasing the size of the superconducting phase.

From above, in particular, it follows that the current of $J > J_C$ can not exist in a superconducting ring without a power supply.

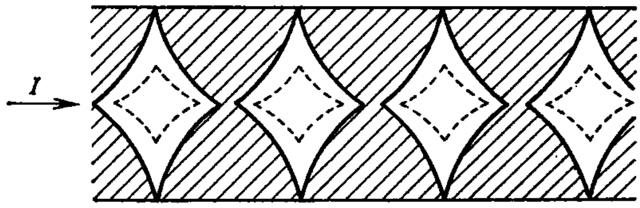


Figure 4.2. The structure of the intermediate state of the round cross-section wire with a transport current. The normal area is shaded.

§4.2. Critical currents in type II superconductors.

In weak magnetic fields and at low transport currents, they behave the same way as type I superconductors, i.e. push the magnetic field and current into a shallow surface layer. If the magnetic field on the sample surface is higher than H_{C1}, the sample is in a mixed state, i.e. it is penetrated by the filaments of magnetic flux. It turns out that in this state at any, even very small, transport currents the sample has a finite resistance.

To understand the cause of this phenomenon, let us consider a rectangular plate along the surface of which the electric current flows, and due to the perpendicular external magnetic field the plate itself is in the mixed state (Figure 4.3).

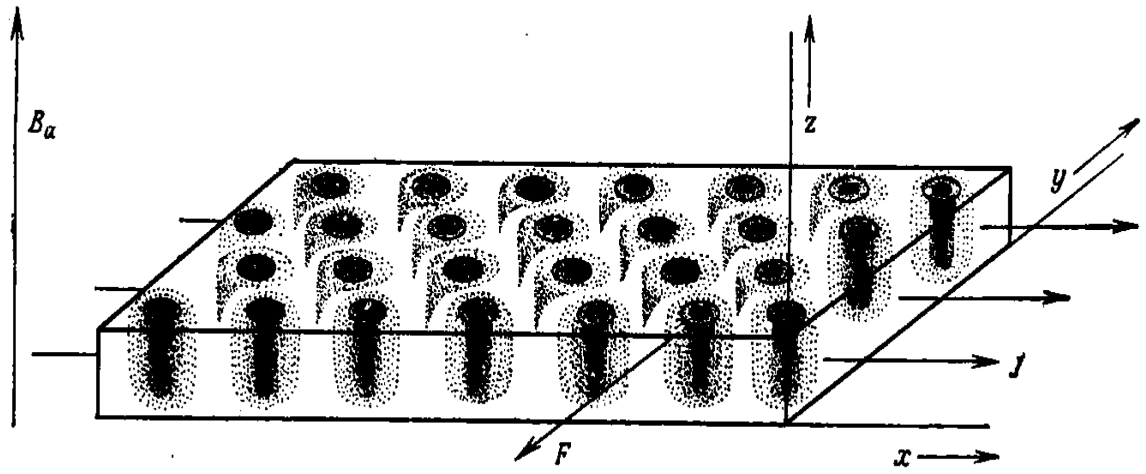


Figure 4.3. The mixed state in the presence of a transport current

An important conclusion from this consideration is the fact that in these conditions the current is distributed evenly over the cross section of the plate and is not limited to a thin layer near the surface. With the penetration of magnetic flux into the sample the transport current can also penetrate into the interior of the superconductor.

This creates an extremely important interaction between the transport current and the threads of magnetic flux. The Ampere force (3.37),

$$\vec{f} = \vec{j}(\vec{r}) \times \vec{\Phi}_0, \qquad (4.3)$$

directed perpendicular to the field and the current, acts on a thread.

From the expression (4.3) it follows that the vortex can be in equilibrium only if the sum of the superconducting velocities from all other sources is equal to zero in each point. This condition can occur if each of the vortices is surrounded by a symmetrical vortex lattice, for example, square or triangular. However, the square lattice corresponds to unstable equilibrium, so that small displacements of the vortices will increase. A triangular lattice is stable, since it has the lowest energy.

In addition, the expression (4.3) shows that the vortices will be subjects to a force from any transport current, with the result that the vortices will move and there will be loss of energy. This energy is drawn from the energy transport current, thus there is a voltage on the sample that corresponds to the appearance of resistance. Energy losses are defined by two basic mechanisms.

The losses are caused by two main mechanisms.

1. When a vortex moves the magnetic field at every point changes, an alternating electric field arises which accelerates the unpaired electrons, which later give energy to the lattice.

2. At the movement of threads there is a continuous process of disintegration and formation of Cooper pairs. If the thread moves so slowly that the distribution of pairs remains in equilibrium, the energy spent for a rupture of pairs on the forward front of a vortex is released again behind it at formation of pairs with the result that there is no energy loss. But at rather fast movement of a vortex the equilibrium density of pairs doesn't have enough time to be reestablished and the energy is dissipated.

Let us now consider the question of the critical current in the absence of an external magnetic field. Consider again the wire of radius a along which the current J flows. The field on the surface of the wire equals $h(a) = J/2\pi a$. If $h(a) < H_{C1}$, the whole wire can be in the superconducting state. This condition defines a critical current value $J_C = 2\pi a H_{C1}$. When $J > J_C$ the field at the surface exceeds H_{C1}, and t region at the surface of the wire must go to the Shubnikov phase. Since the magnetic field lines of the transport current are the concentric circles, the vortex filaments are also formed in the shape of closed circles (toroidal). At the beginning they have a radius a, but then for reasons of minimizing the energy, i.e. length, they are compressed to the wire axis, and finally disappear. The formation of vortices, their compression and disappearance occurs continuously, so there is a constant conversion of energy into heat. Since $H_{C1} < H_C$ the critical currents in type II superconductors are lower than in similar samples of the type I.

It should be said that in the intermediate state of type I superconductors under the influence of sufficiently strong transport currents the movement of areas can also occur, which leads to the emergence of resistance.

§4.3. Type III superconductors.

In paragraph 4.2 the important result is formulated: if a type II superconductor is in the mixed state (a Shubnikov phase), arbitrarily small transport currents lead to the movement of vortices. In other words, the critical current of a superconductor in the Shubnikov state is equal to zero. Nonzero critical currents can be obtained, only if to carry out the fixing (pinning) of vortex threads on certain sites of substance obstructing their traffic. Type II superconductors containing such centers of fixing of vortices are called rigid type II superconductors or type III superconductors.

Various impurities, violations of structure, defects can serve as points energetically preferable for vortices. As it often happens, samples, ideal from the point of view of the theory, aren't the best in the relation of their practical application.

Curves of magnetization of an alloy of Nb and Ta, typical for type III superconductors are given in Figure 4.4. Very careful annealing of a sample allows for it to receive very uniform solid solutions which possess an almost reversible curve of magnetization, characteristic for type II superconductors (a curve 1). If this alloy is subjected to deformation (for example, by drawing during the process of wire production), a set of defects in the lattice is formed and can be centers of for the pinning of vortices. Thus the curve of magnetization takes a completely different form (a curve 2).

It is possible to note the following features:

a) significantly increased values of magnetization,

b) total absence of reversibility,

c) after the removal of the external magnetic field the flux remains "frozen" in a sample,

d) the top critical field H_{C2} remains invariable.

These facts can be easily explained qualitatively. Up to field H_{C1} we don't observe anything new: the sample is in the Meissner phase, almost insensitive to presence of defects. Once at field H_{C1} vortices escape from the sample surface into its volume. However, the pinning doesn't allow them to penetrate the entire sample evenly all at once, as it would be in a uniform material. Therefore, vortices settle in a near-surface layer. In the area of their placement the shielding currents can flow. The expansion of the area in which thevortices are situated, leads to increasing of the shielding current in comparison with a Meissner phase that leads to an increase in the value of magnetization, M.

In other words, the penetration of vortex threads into a near-surface layer increases the effective thickness of the shielding layer and therefore the total shielding current.

For type II superconductors, at a decrease in the external field, in the absence of a pinning, a number of vortices would leave the sample, and the density of distribution of vortices in the section of a sample would decrease to equilibrium value. When a pinning exists, vortices are more or less strongly fixed on defects of the lattice. Therefore, at decreasing field they leave the sample with difficulty, which accounts for lack of reversibility.

Even at zero external field, the sample contains a number of vortices that provide the "frozen" magnetic flux directed along the external field.

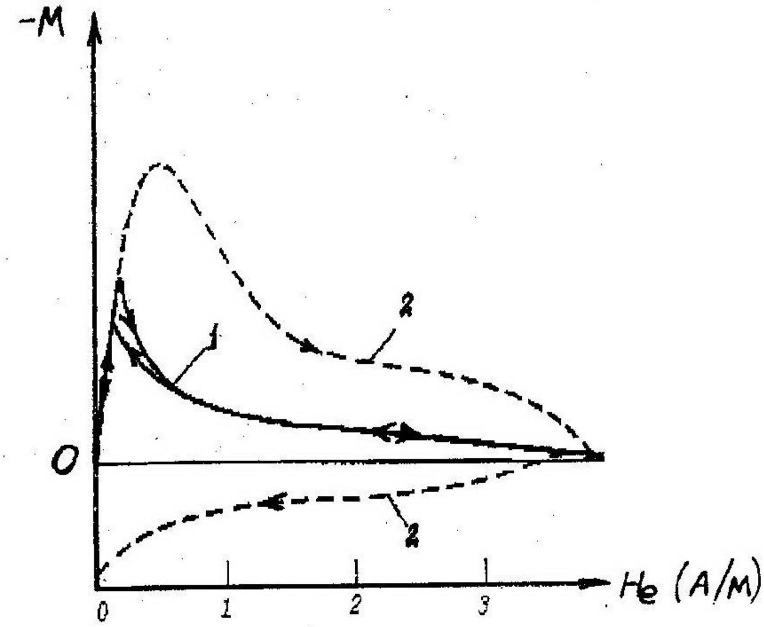


Fig. 4.4. Magnetization curve of an alloy $Nb_{0.55}Ta_{0.45}$*:*

1 - a well-annealed sample, 2 – a sample with a large amount of defects.

Figure 4.5 shows the entire hysteresis loop for the same alloy.

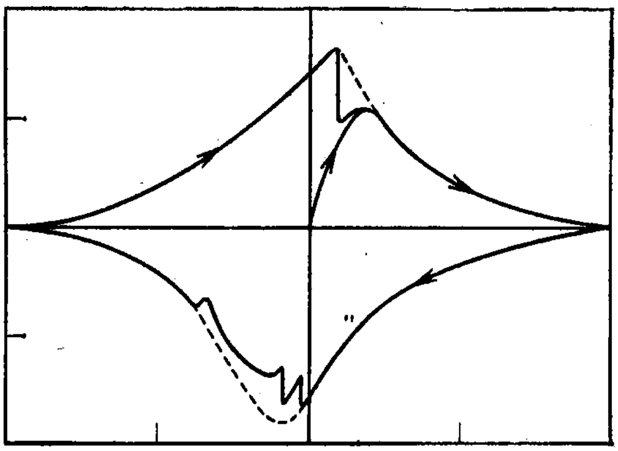


Fig.4.5. The entire cycle of magnetization.

It is clear that there must be some critical pinning separating the two possible states. For small values of the pinning, where it can be neglected, we have a type II superconductor, in which the vortices fill the entire cross section of the sample. At a high pinning, vortices are located near the surface. How and when is there a transition from one mode to another with a gradual change of the pinning parameter?

Calculations show that there exists a critical pinning separating the two possible modes of penetration of an external magnetic field into the medium. If it is exceeded, at any value of the external field there is a “near the surface” current configuration of finite length, fully compensating the external field in the depth of the sample, i.e. the situation is similar to type III superconductor. If pinning is less than critical, such a situation is realized only up to a certain value of the external field. For higher values of the field, it penetrates into the medium at infinite depth, which resembles the situation in type II superconductors.

CHAPTER 5. THE PHASE COHERENCE - JOSEPHSON EFFECTS.

§5.1. Stationary and non-stationary Josephson effects

In the chapter "Basic facts" we talked about the Josephson effects, caused by tunneling of Cooper pairs through an insulating layer. B. Josephson was the first who considered this effect in 1962. For these works in 1973 he was awarded a Nobel Prize. He showed that the tunneling of Cooper pairs becomes essential at a barrier thickness of 10-20 angstrom. Additionally, he predicted some unusual and interesting phenomena taking place when electrons tunnel in pairs. Later all his predictions were excellently confirmed by experiment. Besides their basic importance for understanding of superconductivity the Josephson effects (this name is accepted to call this complex of the phenomena) provide opportunities for carrying out the most exact measurements. We will emphasize that they play an especially important role in the processes occurring in the high temperature ceramic superconductors (HTSC) because, in them, Josephson contacts already exist naturally (contacts between granules). For this reason these substances are sometimes called Josephson media.

The stationary effect of Josephson is a percolation of not fading superconducting current through a thin isolating layer at zero voltage on contact. The magnitude of this current is determined by the phase values φ_1 and φ_2 on different sides of the contact and can not exceed a certain critical value J_C:

$$J = J_C \sin(\varphi_1 - \varphi_2) \quad (5.1)$$

The non-stationary Josephson effect appears at a nonzero voltage, U_S, on the contact. In this case a high-frequency alternating current percolates through the contact with frequency ν, which is proportional to the voltage:

$$\nu = \frac{2eU_S}{h} \quad (5.2)$$

To understand a practical situation, we will consider the circuit represented in Figure 1.14. When a constant superconducting current flows through the contact (stationary effect of Josephson) the voltage at the contact is equal to zero, i.e. all U_e falls on the resistor R. This state can exist if the current (equal to U_e/R) does not exceed the critical value I_C. Thus, the stationary effect of Josephson takes place if $U_e < I_C R$. If $U_e > I_C R$, the generation of high-frequency current begins. Then the mathematical description of the circuit becomes very difficult.

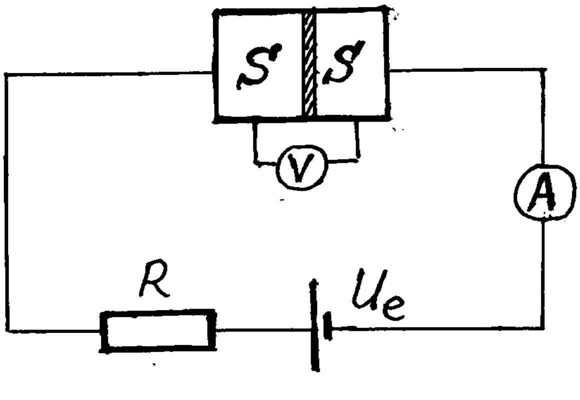


Fig. 5.1. The scheme for demonstration of Josephson effects.

From the point of view of quantum mechanics, all Cooper pairs are in the same state. This macroscopic filling of one state is the cause of the Josephson effects. Because all the pairs are in the same state, they must coincide for all parameters, in particular, the phases. This strong correlation by phase applies to very large (virtually unlimited) distance.

Josephson equations (5.1) and (5.2) follow from the basic equations for the weakly coupled quantum systems. Let the systems be described by wave functions ψ_1 and ψ_2. If the systems are completely isolated, the change in the wave functions is described by equations

$$\frac{\partial \psi_1}{\partial t} = -\frac{i}{\hbar} E_1 \psi_1 \qquad \frac{\partial \psi_2}{\partial t} = -\frac{i}{\hbar} E_2 \psi_2 \tag{5.3}$$

If the systems are weakly linked, the time dependence of ψ_1 affects ψ_2 and vice versa. This effect is taken into account by the following independent equations

$$\frac{\partial \psi_1}{\partial t} = -\frac{i}{\hbar}\left(E_1 \psi_1 + K \psi_2\right) \tag{5.4}$$

$$\frac{\partial \psi_2}{\partial t} = -\frac{i}{\hbar}\left(E_2 \psi_2 + K \psi_1\right) \tag{5.5}$$

The existence of a link means the ability to exchange Cooper pairs between the superconductors 1 and 2. The intensity of the exchange is defined by the constant *K*.

Functions ψ_1 and ψ_2 describe the states with macroscopic filling. Then the square of the amplitude can be regarded as the concentration of the Cooper pairs. In this case, we can write

$$\psi_1 = \sqrt{n_{C1}} \cdot e^{i\varphi_1}; \qquad \psi_2 = \sqrt{n_{C2}} \cdot e^{i\varphi_2} \tag{5.6}$$

Substituting these wave functions in (5.4) and (5.5), we obtain

$$\frac{\dot{n}_{C1}}{2\sqrt{n_{C1}}} e^{i\varphi_1} + i\sqrt{n_{C1}} e^{i\varphi_1} \dot{\varphi}_1 = -\frac{i}{\hbar}\left(E_1 \sqrt{n_{C1}} e^{i\varphi_1} + K\sqrt{n_{C2}} e^{i\varphi_2}\right) \tag{5.7}$$

$$\frac{\dot{n}_{C2}}{2\sqrt{n_{C2}}} e^{i\varphi_2} + i\sqrt{n_{C2}} e^{i\varphi_2} \dot{\varphi}_2 = -\frac{i}{\hbar}\left(E_2 \sqrt{n_{C2}} e^{i\varphi_2} + K\sqrt{n_{C1}} e^{i\varphi_1}\right) \tag{5.8}$$

Splitting into real and imaginary parts gives

$$\frac{\dot{n}_{C1}}{2\sqrt{n_{C1}}} = \frac{K}{\hbar}\sqrt{n_{C2}} \sin(\varphi_2 - \varphi_1)\ ; \tag{5.9}$$

$$\frac{\dot{n}_{C2}}{2\sqrt{n_{C2}}} = \frac{K}{\hbar}\sqrt{n_{C1}} \sin(\varphi_1 - \varphi_2) \tag{5.9 '}$$

$$\sqrt{n_{C1}}\dot{\varphi}_1 = -\frac{1}{\hbar}\left(E_1\sqrt{n_{C1}} + K\sqrt{n_{C2}} \cos(\varphi_2 - \varphi_1)\right) \tag{5.10}$$

$$\sqrt{n_{C2}}\dot{\varphi}_2 = -\frac{1}{\hbar}\left(E_2\sqrt{n_{C2}} + K\sqrt{n_{C1}} \cos(\varphi_1 - \varphi_2)\right) \tag{5.10'}$$

At an exchange of Cooper pairs between the systems 1 and 2 the condition $\dot{n}_{C1} = -\dot{n}_{C2}$ takes place. When superconductors are the same, $n_{C1} = n_{C2}$. Then from (5.9) and (5.9 '), we obtain

$$\dot{n}_{C1} = \frac{2K}{\hbar} n_{C1} \sin(\varphi_2 - \varphi_1) = -\dot{n}_{C2} \tag{5.11}$$

The time variation of particle concentration in the superconductor 1, multiplied by its volume V, gives the change of the number of particles, i.e. the particle flow through the junction. An electric current is obtained by multiplying the flow of particles by the charge of the particle, i.e. 2*e*. Then we get the Josephson equation (5.1)

$$J = J_C \sin(\varphi_1 - \varphi_2),$$

where $J_C = \frac{4Ke}{\hbar} V n_C$.

From (5.10) and (5.10'), we obtain the differential equation

$$\frac{d}{dt}(\varphi_2 - \varphi_1) = \frac{1}{\hbar}(E_1 - E_2) \quad (5.12)$$

When $E_1 = E_2$ the phase difference is constant over time. If a voltage, U, is applied between the superconductors, then $E_1 - E_2 = 2eU$ and the phase difference increases linearly with time

$$\varphi_2 - \varphi_1 = \frac{2eU}{\hbar} t + \varphi_0 \quad (5.13)$$

This means that alternating current flows through the contact

$$J = J_C \sin(\frac{2eU}{\hbar} t + \varphi_0), \quad (5.14)$$

the frequency of which is equal to $\nu = \frac{2eU}{h}$. When the voltage U on the contact equals 1 mV we have $\nu = 4{,}85 \cdot 10^{11}$ Hz.

Josephson junctions are also called “weak links”, and events associated with them are called "weak superconductivity". The weak link can be obtained also by a decrease in the contact area, for example, by pressing the sharp tip of a superconductor to a superconducting surface.

§5.2. Interference of stationary superconducting currents.

Let us analyze some experimental results of stationary Josephson effect.

5.2.1. Superconducting interferometer

We consider a closed contour, Γ, passing inside a superconducting ring, containing two tunneling contacts, and crossing them at points 1 and 2 (Fig. 5.2)

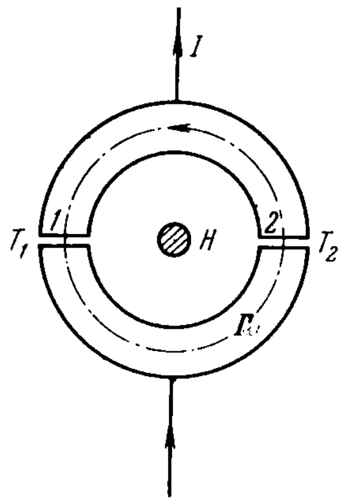


Fig. 5.2. Scheme of the superconducting interferometer

A phase change upon completion of a passing along the contour is defined by the relation

$$\Delta\varphi = \oint_{\Gamma} \vec{\nabla}\varphi \cdot d\vec{l} + \varphi_1 - \varphi_2 \qquad (5.15)$$

where φ_1 and φ_2 - jumps of a phase on contacts T_1 and T_2, i.e. differences of values of a phase on the different sides of a contact. The current flowing through the interferometer is equal to

$$J = J_{C1}\sin\varphi_1 + J_{C2}\sin\varphi_2 \qquad (5.16)$$

One of Ginzburg-Landau equations (3.16) has a form

$$\vec{j} = \frac{e}{m} n_S (\hbar\vec{\nabla}\varphi - 2e\vec{A}) \qquad (5.17)$$

where e and m - the charge and the mass of an electron, n_S - concentration of superconducting electrons. Finding from (5.17) $\vec{\nabla}\varphi$ and taking the integration contour in the superconductor depth where $j = 0$, we obtain

$$\oint_{\Gamma} \vec{\nabla}\varphi \cdot d\vec{l} = \frac{2e}{\hbar}\oint_{\Gamma} \vec{A}\cdot d\vec{l} = \frac{2e}{\hbar}\int \vec{B}\cdot d\vec{S} = 2\pi\frac{\Phi}{\Phi_0} \qquad (5.18)$$

where Φ is the magnetic flux penetrating the section of a ring, $\Phi_0 = \frac{\pi\hbar}{e}$ - the magnetic flux quantum.

At one revolution of a ring the wave functions have to be unambiguous, i.e. the change of a phase has to be a multiple 2π: $\Delta\varphi = 2\pi k$. From (5.15) we obtain a condition of quantization of a fluxoid

$$\underbrace{\varphi_1 - \varphi_2 + 2\pi\frac{\Phi}{\Phi_0}}_{Fluxoid} = 2\pi k \qquad (5.19)$$

Having entered a new variable $\psi = \varphi_1 + \frac{\pi\Phi}{\Phi_0}$, we get from (5.16)

$$J = J_{C1}\sin(\psi - \frac{\pi\Phi}{\Phi_0}) + J_{C2}\sin(\psi + \frac{\pi\Phi}{\Phi_0}) = J_m \sin(\psi - \alpha) \qquad (5.20)$$

where

$$J_m = \sqrt{(J_{C1} - J_{C2})^2 + 4J_{C1}J_{C2}\cos^2(\pi\Phi/\Phi_0)}, \qquad (5.21)$$

$$\alpha = arctg\left(\frac{J_{C1} - J_{C2}}{J_{C1} + J_{C2}} tg\frac{\pi\Phi}{\Phi_0}\right) \qquad (5.22)$$

From (5.20) we can see that the greatest possible current via the interferometer is equal to J_m. Changing at the fixed value of a magnetic field the value of the current J through the interferometer, for example, by means of the scheme of Fig. 5.1, and finding the value J at which there is a transition to non-stationary effect of Josephson, we can find J_m. Changing a magnetic field, it is possible to draw the dependence of J_m on a magnetic flux through the opening of the interferometer. As it is clear from (5.21), it has to be a periodic function with the period being equal to Φ_0. Since Φ_0 is very small, such a device can be used for the registration of very small magnetic fields.

If the interferometer is made up of identical contacts ($J_{C1} = J_{C2} = J_C$), the expression for J_m takes the form

$$J_m = 2J_C \left| \cos \frac{\pi\Phi}{\Phi_0} \right| \qquad (5.23)$$

from where it is clear that the critical current of the interferometer J_m becomes zero each time the flux Φ is equal to a half-integer number of flux quantum Φ_0 (Fig. 5.3a). If $J_{C1} \neq J_{C2}$, J_m doesn't equal zero at any values of a flux Φ and oscillates between the minimum and maximum values $|J_{C1} - J_{C2}|$ and $J_{C1} + J_{C2}$ (Fig. 5.3b).

We will note an important circumstance. The nature of behavior of the current doesn't depend on how the field in a ring is distributed. Only the value of a magnetic flux Φ is important. In particular, the magnetic field can be entirely concentrated within some area, smaller than the ring openings, and equal to zero in the points where the superconductor is located.

We can realize such a situation, for example, having installed into the interferometer the long solenoid, outside of which the field is absent. In this case the impact on current is carried out solely by vector potential $\vec{A}$. Thus, in quantum mechanics the vector potential plays an especially essential role and observed characteristics are defined not only by the magnetic field induction $\vec{B}$ or strength $\vec{H}$, but also by the vector potential $\vec{A}$. This is a purely quantum effect and it is difficult to offer an evident explanation for it within classical physics.

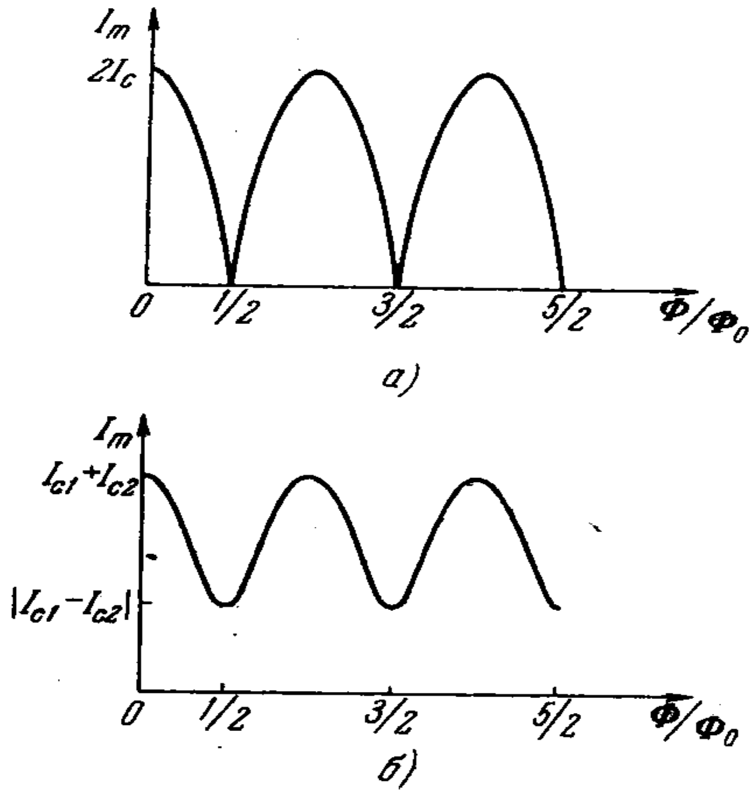


Fig. 5.3. Dependence of critical current of the interferometer on a magnetic field

5.2.2. A superconducting ring with a weak link

We consider features of behavior of a single tunnel junction in the closed superconducting chain (Fig. 5.4). If the critical current of the contact is rather high, the magnetic field in the ring opening can't be considered equal to the external.

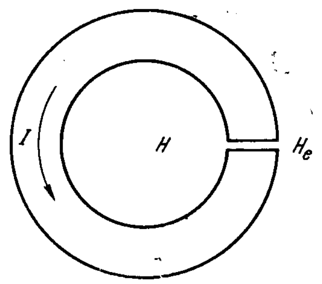


Fig. 5.4. A superconducting ring with a weak link.

The equation of a quantum interference (5.19) takes the form

$$2\pi \frac{\Phi}{\Phi_0} + \varphi = 2\pi k \qquad (5.24)$$

where φ is a difference of phases on a barrier, k is an integer.

The current through the contact is connected with a phase jump on it

$$J = J_C \sin\varphi \qquad (5.25)$$

The expression for the magnetic flux Φ through the ring is added to these two equations

$$\Phi = \Phi_e + LJ\,, \qquad (5.26)$$

where $\Phi_e = B_e S$ is a flux of an external field B_e through the ring, L - the inductance of the ring, S - its area. Condition (5.26) corresponds to the fact that the field in the opening of the ring consists of the external field and the field created by the current J flowing in the ring.

From the equations (5.24) - (5.26) we obtain

$$\Phi + LJ_C \sin\frac{2\pi\Phi}{\Phi_0} = \Phi_e \qquad (5.27)$$

The dependences of Φ and J on Φ_e are given in Figure 5.5. It is convenient to build the graph $\Phi_e(\Phi)$, and then rotate it by 90 degrees.

At small values of J_C the dependence $\Phi(\Phi_e)$ does not differ much from a straight line, and the dependence $J(\Phi_e)$ - from a sinusoid. When J_C increases the character of these curves changes. For example, if J_C exceeds a certain critical value $J_C^{\kappa p}$, these dependences become ambiguous so that for each value of Φ_e there can correspond several values of Φ and J (Fig. 5.5b). Critical value $J_C^{\kappa p}$ can be found from a condition $\partial\Phi/\partial\Phi_e = \infty$:

$$J_C^{\kappa p} = \frac{\Phi_0}{2\pi L} \qquad (5.28)$$

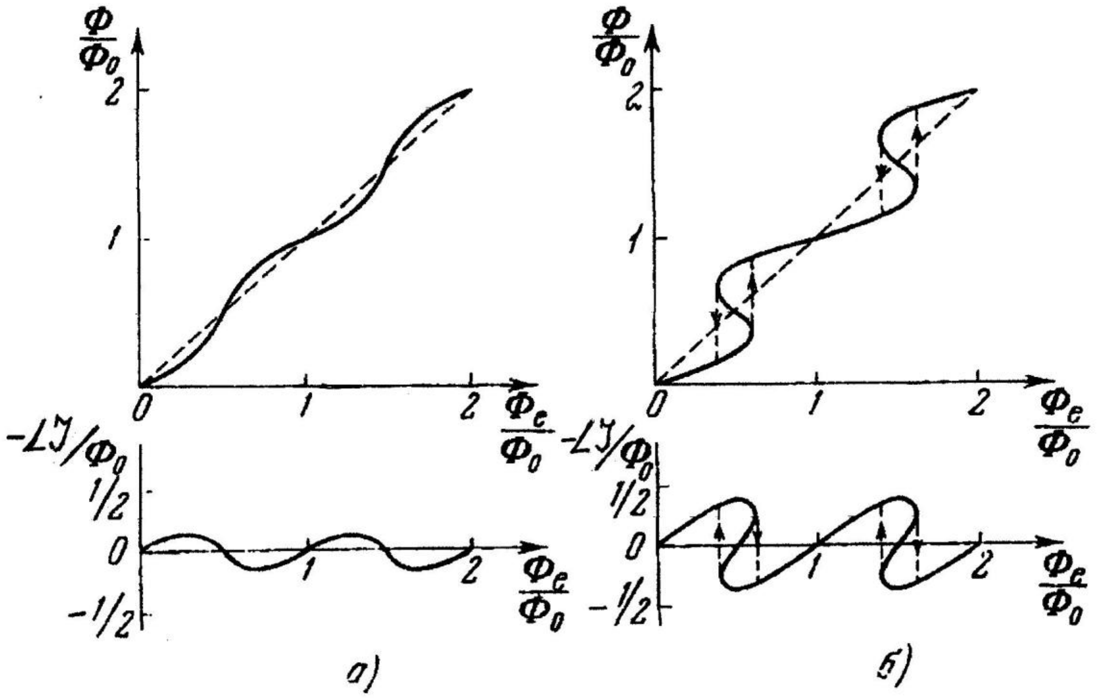


Fig. 5.5. Dependence of the flux Φ *through the ring and the circulating current* J *on an external flux* Φ_e *: a)* $J_C < J_C^{кр}$ *, b)* $J_C > J_C^{кр}$ *.*

At $J_C > J_C^{кр}$ the dependences $\Phi(\Phi_e)$ and $J(\Phi_e)$ have jumps at some values of Φ_e. At increase and decrease of the external flux Φ_e these dependences have various forms, i.e. there is a hysteresis. The direction of transition at a hysteresis is shown by arrows in Figure 5.5b.

A similar effect has to exist in superconducting interferometers with two tunnel junctions if the value of the critical current J_C is comparable with the parameter Φ_0 / L, where L is the inductance of a loop of the interferometer. The consideration of operation of such interferometers, carried out in the previous paragraph, concerned only the case $J_C << \Phi_0 / L$.

§5.3. Interaction of an alternating Josephson current with an external electromagnetic radiation - Shapiro's steps.

Discovery of non-stationary Josephson effect gave the chance of creation of a new type of generators of electromagnetic radiation with very high frequency which value is regulated by the operating voltage.

Historically, however, some indirect experimental confirmations of the existence of this effect were originally obtained. In these experiments the features on volt-ampere characteristics arising due to te interaction of an alternating superconducting current with external electromagnetic microwave radiation were studied.

In the experiments of Shapiro with employees (1963), a tunnel contact of Al-Sn was located in the microwave resonator in which microwave oscillations of frequency ν were created. The constant voltage U_0 was applied to the contact. Under the influence of a microwave field on a volt-ampere characteristic some horizontal sites - steps - were observed. Their positions correspond to a relation $2eU_0 = nh\nu$, where n are arbitrary integers (Fig. 5.6).

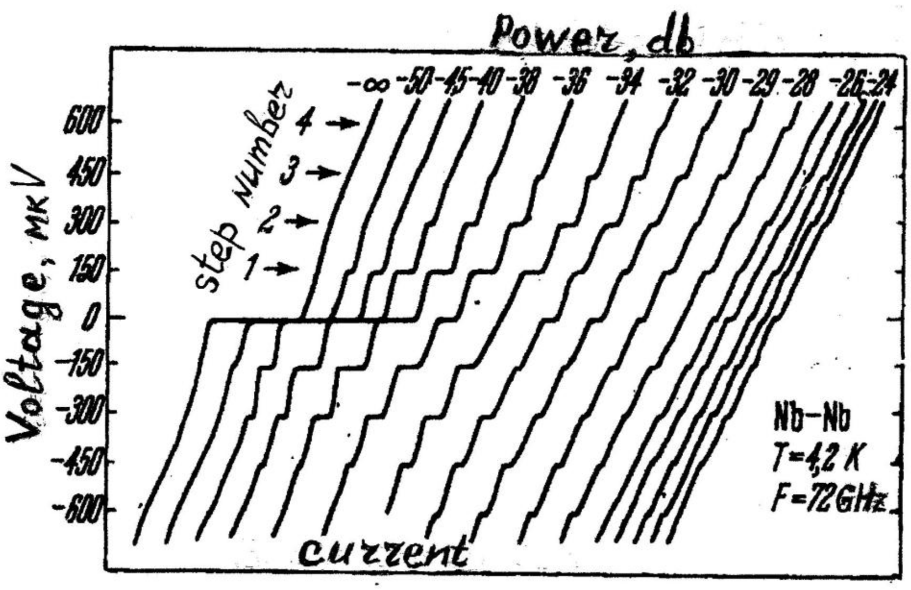


Fig. 5.6. Volt-ampere characteristics of Josephson contact of Nb-Nb in the presence of a microwave radiation of various power with a frequency of 72 GHz.

For clarification of the origin of this effect we will consider simply that the microwave field causes the appearance on the contact of the additional voltage oscillating with a frequency ν, which leads to the modulation of frequency of Josephson current ν_J in accordance with (5.2)

$$\nu_J = \frac{2e}{h}\left[U_0 + u\cos(2\pi\nu t + \theta)\right] \qquad (5.29)$$

where u is an amplitude of an oscillating additive proportional to the intensity of a microwave field.

The total current through the contact includes a Josephson current and a current of uncoupled electrons, equal to U/R, where $U = U_0 + u\cos(2\pi\nu t + \theta)$ is a full voltage on the contact, R – the contact resistance. Since the voltage on the contact depends on time, instead of (5.13) we will come to $\varphi_2 - \varphi_1 = \int \frac{2eU}{\hbar} dt + \varphi_0$.

Thus, the current is equal to

$$J = J_C \sin\left(\int_0^t 2\pi\nu_J\, dt + \varphi_0\right) + \frac{U}{R} = J_C \sin\left[\frac{2e}{\hbar}U_0 t + \frac{2eu}{h\nu}\sin(2\pi\nu t + \theta) + \varphi_0\right] + \frac{U}{R} \qquad (5.30)$$

where φ_0 - the initial value of a phase jump on the contact.

After a number of transformations with use of series

$$\sin(z\sin x) = 2\sum_{k=0}^{\infty} J_{2k+1}(z)\sin(2k+1)x$$

$$\cos(z\sin x) = J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos 2kx$$

the expression (5.30) takes a form

$$J = J_C\left\{J_0(\frac{2eU}{h\nu})\sin(\frac{2e}{\hbar}U_0t+\varphi_0)+\sum_{n=1}^{\infty}J_n\left(\frac{2eu}{h\nu}\right)\left(\sin\left[\frac{2e}{\hbar}U_0t+\varphi_0+n(2\pi\nu t+\theta)\right]\right.\right.$$

$$\left.\left.+(-1)^n\sin\left[\frac{2e}{\hbar}U_0t+\varphi_0-n(2\pi\nu t+\theta)\right]\right)\right\}+\frac{U_0+u\cos(2\pi\nu t+\theta)}{R} \qquad (5.31)$$

where $J_n(x)$ - the Bessel function of n-th order.

From (5.31) we can see that the total current oscillates in time with very high frequency ν or with a Josephson frequency $2eU_0/h$. The measured value of the current is a time average from the expression (5.31). It is easy to see that this value is equal to $\bar{J}=\frac{U_0}{R}$ at all values of U_0, except those at which $2eU_0/\hbar=2\pi n\nu$, i.e. when the Josephson frequency is multiple of the microwave field frequency. At these values of U_0 one of the terms of the sum stops oscillating and makes a constant contribution to the value of the current. Thus the current becomes equal to

$$\bar{J}=\frac{U_0}{R}+(-1)^nJ_n(\frac{2eu}{\hbar\nu})\sin(\varphi_0-n\theta). \qquad (5.32)$$

Since $\varphi_0-n\theta$ can assume arbitrary values, at the values of U_0, corresponding to the condition $2eU_0=nh\nu$, the current can take values of a certain range. It means that at these voltages on the current-voltage characteristic there exist horizontal steps. It is confirmed by the experimental curves of Figure 5.6. This figure shows clearly not only the coinciding of the steps with the theoretical predictions, but satisfying quantitative agreement between the periodic change of the length of the steps with increasing microwave power and the dependence of the Bessel function on u.

CHAPTER 6. HIGH-TEMPERATURE SUPERCONDUCTORS

In chapter 1 it was already told about the history of discovery of high-temperature superconductivity (HTSC). We remind you that till 1986 a maximum critical temperature equal to 23,2 K, was observed at an alloy of Nb_3Ge (1973). Despite considerable efforts of theorists and experimenters, scientists didn't manage to rise T_C above this value up to that moment when Bednorts and Müller found that the ceramics of $La-Ba-Cu-O$ showed signs of transition to a superconducting state when cooling lower than 35 K. The studied samples were a mix of different phases. In January, 1987 it was established that the phase $La_{2-x}Ba_xCuO_4$ is responsible for superconductivity. Critical temperature depends on structure and has maximum (35 K) at x=0.2.

A new direction of researches was opened. Physicists of the whole world began a search for superconductors close in structure. Replacement of La by other close elements from the 1st and 2nd groups of the table of Mendeleyev, and also the variation of structure created a large number of superconducting ceramics. Within 2 months the critical temperature 92 K was reached in ceramics $YBa_2Cu_3O_{7-x}$. Thus the "nitric" barrier was broken, i.e. there was an opportunity to obtain superconductors not by means of expensive and inconvenient liquid helium with a temperature of boiling of 4,2 K, but with use of the cheap and simple in using liquid nitrogen boiling at 77 K. From that moment a large number of matters with critical temperature above a boiling point of liquid nitrogen are created. The record belongs to the ceramic $HgBa_2Ca_2Cu_3O_{8+x}$ opened in 2003, the critical temperature of which is equal 135 K.

The properties of HTSC are in many respects similar to usual superconductors, but at the same time there are both essential quantitative and qualitative differences.

1) As well as in the case of usual superconductors, their resistance becomes zero when cooling below critical temperature. The values of T_C are essentially higher and reach 100 K and above (nowadays to 135 K).

2) At zero external magnetic field there is no release or absorption of heat, but a jump of a thermal capacity is observed, i.e. a phase transition of the 2nd type takes place.

3) Meissner effect takes place, but the penetration depth λ, equal to $10^2 - 10^3$ angstrom, is much more than in usual superconductors.

4) Experiments show the existence of an energy gap, the order of value of which is coordinated with the theory of BCS ($2\Delta \approx 3{,}5kT_C$). However some experiments allow to find two different values of a gap. There are reasons to believe that a higher value is connected with the planes in a crystal, and a smaller - with linear chains.

5) A dependence of T_C on the mass of atoms (isotopic effect) has been found to testify to a role of lattice fluctuations.

6) In a magnetic field HTSC behave as type II superconductors. It is due to both a high penetration depth λ, and a very small length of coherence ξ - from 0.5 to the 30 angstrom (in usual superconductors - thousands angstrom). Thus, the condition $\lambda > \xi$ characterizing type II superconductors is carried out for certain.

7) The stationary effect of Josephson takes place and, from experiments on dependence of oscillation of the maximum Josephson current on a magnetic field (see §5.2.1), it follows that the magnetic flux quantum Φ_0 is equal to $h/2e$ which points to the transfer of current by Cooper pairs with charge $2e$.

8) In the case of applying to Josephson contact direct and alternating voltages simultaneously the Shapiro steps (see §5.3) testifying to non-stationary effect of Josephson can be observed. Thus the period between steps is equal to $h\nu/2e$ which also speaks about Cooper pairs with charge $2e$.

9) The quantization of a magnetic flux takes place, i.e. a magnetic flux through an opening in a superconductor is equal to an integer of quanta of magnetic flux $\Phi_0 = h/2e$.

10) The lattice of vortex threads was observed, and it was established that each thread contains the same quantum of a magnetic flux $\Phi_0 = h/2e$.

The above-stated facts (points 7-10) give strong reasons to believe that current is transferred by Cooper pairs. However, it appeared that in the majority of HTSC, Cooper pairs are formed not by electrons, but by holes.

Apparently from all aforesaid, the majority of the phenomena, considered in previous paragraphs for usual superconductors, take place as well in HTSC. But there are also essential features, general for all HTSC, distinguishing them from usual superconductors.

1) Unlike the usual superconductors which are in a normal state metals or metal alloys, HTSC represent oxides of metals and in a normal state have considerably high resistance. Though it should

be noted that the linear growth of specific resistance with a temperature testifies, nevertheless, to the metallic nature of their conductivity.

2) It is difficult to produce these metal-oxide compounds in the form of a monocrystal. The existing technology (agglomeration of previously mixed mixture of ingredients) allows to obtain ceramics made up of crystals ("granules") of the sizes from units to hundreds of microns, the space between which is filled by dielectric.

3) In places of contacts of granules with each other, Josephson contacts - on which there can be processes connected with Josephson effects - are formed. As the sizes of granules are small, the number of such contacts is very great. Therefore superconducting ceramics sometimes are called Josephson media. Josephson effects are described by nonlinear equations. Therefore when such samples are placed into external constant and alternating electromagnetic fields different complicated processes in them can happen which weren't observed in other substances earlier.

4) The transition to a state with zero resistance in HTSC happens in a wider temperature range, than in usual superconductors. So, for example, in Bednorts and Müller's first article it was reported that a sharp falling of resistance of oxide $La_{2-x}Ba_xCuO_4$ with $x = 0.2$ began at 35 K and the resistance reached zero at $T \approx 25$ K. Big blurring of the transition is explained by the existence in ceramics of various phases with different values of the T_C .

5) Monocrystals of HTSC are prepared using special technologies. They have small sizes (up to several millimeters), layered structure and associated with it strong anisotropy of the majority of properties.

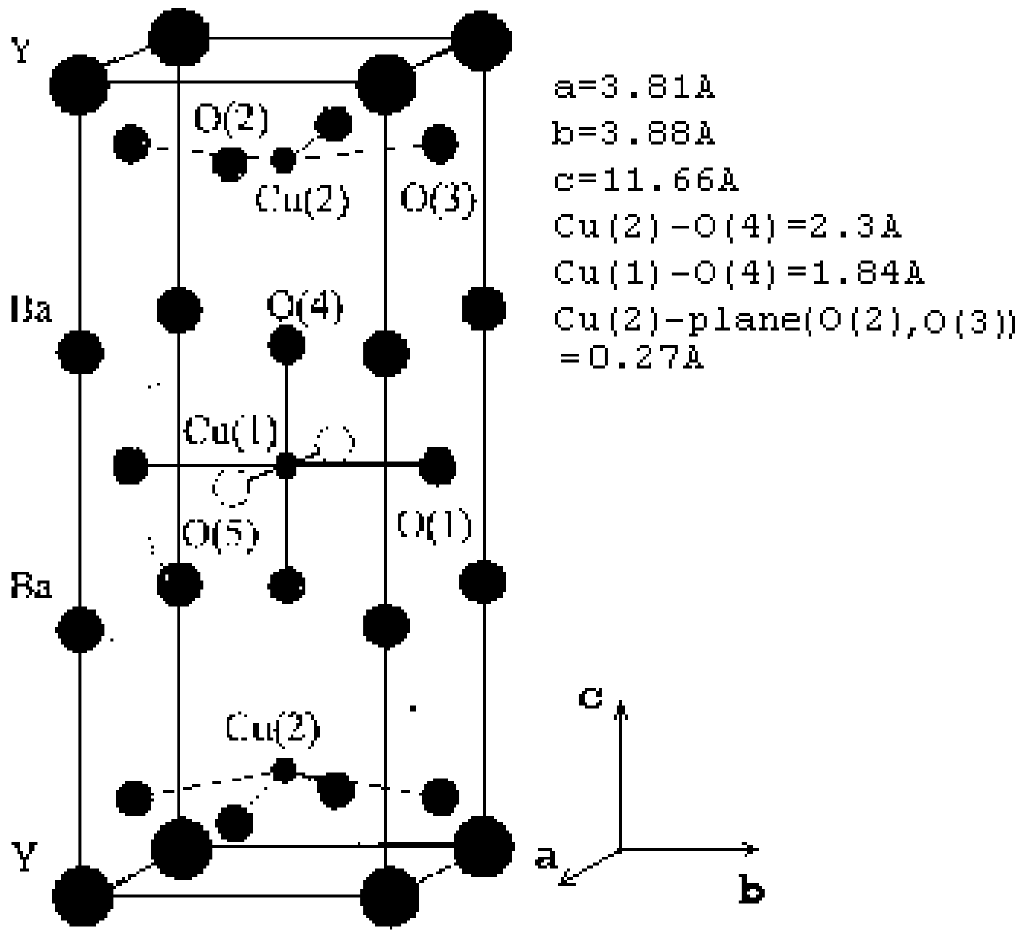


Fig. 6.1. An elementary cell of a crystal $YBa_2Cu_3O_{7-x}$.

In Figure 6.1 the elementary cell of a crystal $YBa_2Cu_3O_{7-x}$ is represented. Its distinctive feature is the existence of two nonequivalent positions of atoms of copper: Cu(1) and Cu(2). Atoms of Cu(2) are put into a pyramid with the square base formed by four atoms of oxygen - O(2) and O(3) - and are almost in the base plane. These layers located perpendicularly to the axis c are called planes of CuO_2. Unlike atoms of Cu(2) atoms of Cu(1) in the plane, perpendicular to the axis c, adjoin only to two atoms of oxygen O(1), forming so-called chains of CuO. Thus, in the structure of the $YBa_2Cu_3O_{7-x}$ there are two various elements - planes of CuO_2 and chains of CuO - which weakly interact with each other through bridging oxygens O(4). The positions of O(5) remain vacant.

The planes of CuO_2 are available in all ceramics, while in some of them there are CuO chains. There are bases to believe that both planes and chains play very important roles in high-temperature superconductivity.

The theory of HTSC is not created yet. There are essential bases to believe that for its construction it is enough to modify the theory of BCS, having found new, other than phonon, mechanism of an attraction of electrons (or holes) leading to their association in Cooper pairs. The phonon mechanism doesn't allow obtaining such high values of critical temperature. It is necessary to find some other, stronger type of interaction. The exchange of some particles can be the cause of such attraction. As the elementary cell of a crystal of HTSC is very complicated (Fig. 6.1), in a sample there can be a large number of different types of particles - phonons, excitons, polarons, bipolarons, magnons, etc. It is possible also that in different substances various particles are responsible for pairing.

The answer to a question of the nature of this interaction isn't found so far as well as there is no explanation for some other facts, such as the existence of two gaps in one sample, anomalies in dependence of a thermal capacity on temperature, etc.

In 2008, the new class of superconducting substances with high values of critical temperature - layered compounds on the basis of iron and elements of the 5th group (pniktogen) or Se - were discovered. These substances are called ferropniktids (or selenids of iron). The superconducting state for substances containing magnetic atoms (Fe) was revealed for the first time, because usually magnetic field suppresses superconductivity. The crystal structure of all ferriferous superconductors (6 families are already known) has a form of alternating layers in which atoms of iron are surrounded with a tetrahedron from atoms of As or Se which suppress magnetic properties of atoms of Fe. At the moment the champion on critical temperature is a compound of $GdOFeAs$ (Gd-1111), doped by F (fluorine) which replaces oxygen. Its T_C reaches 55 K.

In 2001 MgB_2 alloy (magnesium diborid) with $T_C = 40$ K - record for intermetalloid (chemical compounds of two or more metals) – was discovered.

Use of very high pressure allows to increase critical temperatures. For example, the critical temperature of the ceramic $HgBa_2Ca_2Cu_3O_{8+x}$ mentioned above (with record-breaking high $T_C = 135$ K) at a pressure of 40 GPa increases to 165 K.

Mentioned before selenid of iron loses superconducting properties at 10 GPa, but at 11.5 GPa gets them again, and zero resistance remains to record for iron selenid critical temperature 48 K.

At high pressures superconducting properties can arise even in substances which couldn't be suspected of such abilities. For example, in 2014 it was revealed that hydrogen sulfide (H_2S) at a pressure of 180 GPa and a temperature of 190 K suddenly sharply reduces the resistance that suggests an idea of superconductivity. However, this interpretation still needs to be tested.

The reasons for all these phenomena are still unclear.

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