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PLASMA KINETICS

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For International program “Advances and Applications in Plasma Physics”, Course “Plasma Theory”

Lections on plasma theory are part of a program “Advances and Applications in Plasma Physics”, This is an advanced course and students are supposed to be familiar with basics of plasma physics in the framework of introductory courses like “Introduction to Plasma Physics” and “Elementary Processes in Plasma”. Lectures on plasma theory are planned for two semesters. In the first semester students study kinetic theory and transport processes in plasma, while the second semester is devoted to plasma dynamics, including MHD theory, equilibrium and stability. More advanced problems like neoclassical theory, stochastization of the magnetic field, edge plasma physics are also considered. Waves in plasma are not included in this course and should be studied separately. Considered are only low frequency waves and instabilities which are closely connected with the dynamics and transport of plasma, like MHD and drift waves. Plasma kinetics is the basic part of plasma theory.

The distinctive feature of this course compared to most courses on plasma physics is that phenomena in both low and high temperature plasma are considered simultaneously so that theory of slightly ionized and fully ionized plasmas are presented. Therefore, this course might be useful for wide auditorium of students and specialists working in different areas like nuclear fusion, gas discharge physics and low temperature plasma applications, space and astrophysics etc.

Plasma Kinetics

1.1. Boltzmann Equation

Plasma state generally is described by a set of distribution functions $f_\alpha(\vec{r},\vec{V},t)$ for all the plasma components and their quantum states. The distribution function could be denoted as a particle density in a six-dimensional coordinate and velocity phase space, while the quantity $dn_\alpha(\vec{r},\vec{V},t) = f_\alpha d\vec{r}d\vec{V}$ is a number of particles in an infinitesimal element of a phase volume. Subscript α here represents different particles, neutral or ionized ones, as well as different quantum states of atoms, molecules or ions. Below considered is classical ideal nonrelativistic plasma. Variation of the number of particles in the six dimensional phase space in the absence of collisions is caused by a flow of a "phase liquid" to the neighboring regions of the phase space and change of the number of particles in time. It is controlled by a six dimensional continuity equation

$$\frac{\partial f_\alpha}{\partial t} + \sum_{i=1}^{6} \frac{\partial}{\partial x_i}(f_\alpha \dot{x}_i) = 0 \,.$$

In the phase space six "coordinates" x_i in the Cartesian coordinate system are three space coordinates r_i and three components of the velocity V_i, and "velocities" $\dot{x}_i$ correspondingly consist of three components of the velocity V_i and three components of the acceleration $\dot{V}_i$. Therefore the continuity equation in the phase space has the form

$$\frac{\partial f_\alpha}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial r_i}(f_\alpha V_i) + \sum_{i=1}^{3} \frac{\partial}{\partial V_i}(f_\alpha \dot{V}_i) = 0 \,.$$

The first term on the l.h.s. represents temporal variation of the distribution function, the second term corresponds to divergence of a flux in the real space, and the third term corresponds to the flux divergence in the velocity space. The acceleration $\dot{V}_i$ is produced by the forces applied to a particle. In the plasma

$$\dot{\vec{V}} = \frac{Z_\alpha e}{m_\alpha}\left(\vec{E} + [\vec{V}\times\vec{B}]\right) + \vec{g}\ ,$$

where Z_α is a charge number of a particle, m_α is the particle mass, $\vec{E}$ and $\vec{B}$ are electric and magnetic fields respectively, $m_\alpha \vec{g}$ is the gravitational force.

The space coordinates and velocities are the independent variables, hence $\partial V_i / \partial r_i = 0$. Also since the Lorentz force is perpendicular to the velocity of a particle, we have $\partial \dot{V}_i / \partial V_i = 0$. So the continuity equation is reduced to the form

$$\frac{\partial f_\alpha}{\partial t} + \vec{V}\frac{\partial f_\alpha}{\partial \vec{r}} + \dot{\vec{V}}\frac{\partial f_\alpha}{\partial \vec{V}} = 0\ , \qquad (1.1)$$

or

$$\frac{\partial f_\alpha}{\partial t} + \vec{V}\frac{\partial f_\alpha}{\partial \vec{r}} + \frac{Z_\alpha e}{m_\alpha}\left(\vec{E} + [\vec{V}\times\vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{V}} + \vec{g}\,\frac{\partial f_\alpha}{\partial \vec{V}} = 0\,. \qquad (1.2)$$

This equation is known as Vlasov equation.

The Vlasov equation could be rewritten in any generalized coordinates q_i and momentum p_i. The Vlasov equation in the form of Eq. (1.1) in the general case is derived from the continuity equation in the phase space using a relation $\partial \dot{q}_i / \partial q_i + \partial \dot{p}_i / \partial p_i = 0$, which follows from the Hamilton equations: $\dot{q}_i = \partial H / \partial p_i$; $\dot{p}_i = -\partial H / \partial q_i$.

The left hand side of the Vlasov equation is equal to a full derivative df_α / dt. Therefore in the stationary case according to the Liouville's theorem the distribution function is constant along the phase-space trajectories of the system. An important statement which follows from the Liouville's theorem is that the stationary distribution function in a collisionless case should be a function of integrals of motion. This important notice gives an opportunity to find the distribution function in various collisionless problems.

Account of collisions changes the Eqs. (1.1), (1.2) since the distribution function is not constant along the trajectories even in the stationary case. In the process of collisions the velocities of the particles changes as well as their quantum states (we shall not consider the change of the particle positions in the process of collisions assuming that the plasma is an ideal gas). In the presence of collisions the kinetic equation is given by

$$\frac{df_\alpha}{dt} \equiv \frac{\partial f_\alpha}{\partial t} + \vec{V}\frac{\partial f_\alpha}{\partial \vec{r}} + \frac{Z_\alpha e}{m_\alpha}\left(\vec{E} + [\vec{V}\times\vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{V}} + \vec{g}\,\frac{\partial f_\alpha}{\partial \vec{V}} = St_\alpha\,, \tag{1.3}$$

where a collision operator on the r.h.s. St_α is responsible for the change of the distribution function during collisions. The Eq. (1.3) is known as a Boltzmann equation. The collision operator is a sum over all species

$$St_\alpha = \sum_\beta St_{\alpha\beta}(f_\alpha, f_\beta)\,. \tag{1.4}$$

Each summand corresponds to collisions of the species α with all species in the plasma including particles α.

We shall consider elastic collisions when the quantum states of particles remain the same in the process of collisions and energy and momentum of the particles are conserved. We assume that before the collision two species α and β have velocities $\vec{V}_\alpha$ and $\vec{V}_\beta$, while after the collision they changes their velocities to the values $\vec{V}'_\alpha$ and $\vec{V}'_\beta$ correspondingly without changing spatial coordinate. After each collision the particle with the velocity $\vec{V}_\alpha$ escapes from the infinitesimal volume in the velocity space $d\vec{V}_\alpha$. The full number of such escapes caused by the collisions of the particles of α species with the particles of the species β with $\vec{V}_\alpha, \vec{V}_\beta \to \vec{V}'_\alpha, \vec{V}'_\beta$ per second in the infinitesimal velocity space volume $d\vec{V}_\alpha$ for a fixed value of $\vec{V}_\alpha$ is given by an expression

$$dQ^{-}_{\alpha\beta} = d\vec{r}d\vec{V}_{\alpha}\int\limits_{\vec{V}_{\beta}}\int\limits_{\Omega} f_{\alpha}(\vec{V}_{\alpha})\, f_{\beta}(\vec{V}_{\beta})(d\sigma_{\alpha\beta}/d\Omega)\left|\vec{V}_{\alpha}-\vec{V}_{\beta}\right| d\Omega d\vec{V}_{\beta}\,.$$

Here $d\sigma_{\alpha\beta}/d\Omega$ is a differential cross section of scattering to the solid angle Ω. Besides losses in the infinitesimal velocity space volume $d\vec{V}_{\alpha}$ there is also a source caused by the collisions $\vec{V}'_{\alpha},\vec{V}'_{\beta} \to \vec{V}_{\alpha},\vec{V}_{\beta}$ which transfer particles with the velocities $\vec{V}'_{\alpha},\vec{V}'_{\beta}$ to the velocity space volume $d\vec{V}_{\alpha}$:

$$dQ^{+}_{\alpha\beta} = d\vec{r}\int\limits_{\vec{V}'_{\beta}}\int\limits_{\vec{V}'_{\alpha}}\int\limits_{\Omega} f_{\alpha}(\vec{V}'_{\alpha})\, f_{\beta}(\vec{V}'_{\beta})(d\sigma_{\alpha\beta}/d\Omega)\left|\vec{V}'_{\alpha}-\vec{V}'_{\beta}\right| d\Omega d\vec{V}'_{\alpha}d\vec{V}'_{\beta}\,.$$

Velocities $\vec{V}'_{\alpha}$ and $\vec{V}'_{\beta}$ under the integral are not independent but connected by the conservation laws. Indeed, in the process of collision of the particles with the velocities $\vec{V}'_{\alpha}$, $\vec{V}'_{\beta}$ the particle of species α obtains the velocity $\vec{V}_{\alpha}$. Let us change variables in the integral and integrate over the velocities $\vec{V}_{\alpha}$, $\vec{V}_{\beta}$ using the conservation of the relative velocity $\left|\vec{V}_{\alpha}-\vec{V}_{\beta}\right| = \left|\vec{V}'_{\alpha}-\vec{V}'_{\beta}\right|$. As is known from classical mechanics, Jacobian of this transformation is equal to unity: $d\vec{V}'_{\alpha}d\vec{V}'_{\beta} = d\vec{V}_{\alpha}d\vec{V}_{\beta}$. Finally, since integration over $d\vec{V}_{\alpha}$ is carried out in the vicinity of a chosen value $\vec{V}_{\alpha}$, one obtains

$$dQ^{+}_{\alpha\beta} = d\vec{r}d\vec{V}_{\alpha}\int\limits_{\vec{V}_{\beta}}\int\limits_{\Omega} f_{\alpha}(\vec{V}'_{\alpha})\, f_{\beta}(\vec{V}'_{\beta})(d\sigma_{\alpha\beta}/d\Omega)\left|\vec{V}_{\alpha}-\vec{V}_{\beta}\right| d\Omega d\vec{V}_{\beta}\,.$$

Combining sources and sinks in the volume $d\vec{r}d\vec{V}_{\alpha}$, we find

$$dQ^{+}_{\alpha\beta} - dQ^{-}_{\alpha\beta} = St_{\alpha\beta}d\vec{r}d\vec{V}_{\alpha}\,,$$

where the collision operator is

$$St_{\alpha\beta}(f_{\alpha,} f_{\beta}) = \int\limits_{\vec{V}_\beta}\int\limits_{\Omega}(f'_\alpha f'_\beta - f_\alpha f_\beta)(d\sigma_{\alpha\beta}/d\Omega)\left|\vec{V}_\alpha - \vec{V}_\beta\right| d\Omega d\vec{V}_\beta . \qquad (1.5)$$

Here $f'_\alpha \equiv f'_\alpha(\vec{V}'_\alpha),\ f'_\beta \equiv f'_\beta(\vec{V}'_\beta)$. The Eq. (1.5) is known as Boltzmann collision operator.

Boltzmann kinetic equation (1.3) is an integro-differential equation which contains all distribution functions of the particles in the system. Therefore a system of coupled equations for all distribution functions is to be solved.

In the process of derivation of the collision operator we assumed that during the collision the particles coordinate remains unchanged. This is justified provided the potential energy is significantly smaller than their average kinetic energy which is of the order of their temperature T. For the charged particles an average potential energy of the Coulomb interaction is $Z_\alpha Z_\beta e^2 / 4\pi\varepsilon_0 \langle r_{\alpha\beta} \rangle$, and average distance between the charged particles $\langle r_{\alpha\beta} \rangle$ is of the order of $n^{-1/3}$, where n is the plasma density. Therefore, the criterion of ideal plasma used in the derivation has the form

$$Z_\alpha Z_\beta e^2 n^{1/3} / T << 1 .$$

In other words the plasma should be sufficiently hot and not very dense.

An inelastic collision operator can be constructed in a similar way. However, it is seldom used in the general case, in practice analyzed are special processes-ionization, recombination, excitation etc, so the integrals over the velocities are used which are easier to obtain directly.

Moments of the distribution function describe macroscopic plasma parameters. They are defined according to

$$M_{\alpha j,k...n} = \int V_j V_k ... V_n f_\alpha(\vec{r}, \vec{V}, t) d\vec{V} . \qquad (1.6)$$

The most important of them are the following. Particle density:

$$n_\alpha = \int f_\alpha d\vec{V} . \qquad (1.7)$$

The flux density of the particles is defined as

$$\vec{\Gamma}_\alpha = n_\alpha \vec{u}_\alpha = \int \vec{V} f_\alpha d\vec{V} , \tag{1.8}$$

where $\vec{u}_\alpha$ is the fluid velocity. The average energy of chaotic motion (for single atoms) is:

$$\frac{3}{2} n_\alpha T_\alpha = \int \frac{m_\alpha (\vec{V} - \vec{u}_\alpha)^2}{2} f_\alpha d\vec{V} . \tag{1.9}$$

A quantity T_α is called a temperature. A heat flux density is defined according to

$$\vec{q}_\alpha = \frac{m_\alpha}{2} \int (\vec{V} - \vec{u}_\alpha)^2 (\vec{V} - \vec{u}_\alpha) f_\alpha d\vec{V} . \tag{1.10}$$

Frequent collisions in the plasma tend to establish local Maxwellian distribution function, which is defined as

$$f_\alpha^M = \frac{n_\alpha}{(2\pi T_\alpha / m_\alpha)^{3/2}} \exp\left[- \frac{m_\alpha (\vec{V} - \vec{u}_\alpha)^2}{2T_\alpha} \right]. \tag{1.11}$$

In the absence of all forces in the stationary case and in the homogeneous plasma The Boltzmann equation (1.3) is reduced to the equation $St_\alpha = 0$. It is easy to see that the Maxwellian distribution functions with the common temperature and common fluid velocity turn to zero the collision operator. Indeed substituting Maxwellian functions Eq. (1.11) into the collision operator Eq. (1.5) we find that the bracket $(f'_\alpha f'_\beta - f_\alpha f_\beta)$ is zero, which is the consequence of the energy conservation during an act of collision. Therefore, the Maxwellian distribution functions of all the particles correspond to the thermodynamic equilibrium. More strong statement is known in statistical physics as Boltzmann H-theorem – in the absence of all forces a system tends to thermodynamic equilibrium and hence to establishing of the Boltzmann distribution functions.

1.2. Collision operator for Coulomb collisions

1.2.1 General expression for a flow in the velocity space caused by collisions

The Boltzmann form of the collision operator is inconvenient in the case of Coulomb collisions. This is connected with the specific character of the Coulomb collisions where small angle scattering for large impact parameters dominates and determine the Coulomb cross section. The fact that the velocity change in a single act of collision is small with respect to the particle velocity can be used to simplify the Boltzmann collision operator. Indeed during collision process ‘phase fluid’ arrives at a given phase volume from the neighboring regions, therefore almost continuous flow of ‘phase fluid’ takes place. In other words, the collision operator can be rewritten as a divergence of a flow in the velocity space

$$St_\alpha = -\nabla_{\vec{V}} \cdot \vec{\Gamma}_\alpha^{\vec{V}} . \qquad (1.12)$$

With such form of the collision operstor the kinetic equation takes the form of continuity equation in the phase space also with account of collisions. The general expression for the flow induced by the collisions has the form (summation over subscript *k* is assumed)

$$\Gamma_{j\alpha}^{\vec{V}} = \frac{F_{j\alpha}^{St}}{m_\alpha} f_\alpha - D_{jk}^\alpha \frac{\partial f_\alpha}{\partial V_k} . \qquad (1.13)$$

The higher derivatives over the velocities are neglected here and the corresponding series are truncated since the neighboring regions give the major contributions to the flow. The quantity $\vec{F}_\alpha^{St}$ is known as a dynamical force, while quantity D_{jk}^α is a diffusion tensor in the velocity space. For the collisions with the different species the flow in the velocity space is defined as

$$\Gamma_{j\alpha}^{\vec{V}} = \sum_{\beta} \Gamma_{j\alpha\beta}^{\vec{V}} \quad .$$

Now we shall demonstrate that the quantities $\vec{F}_{\alpha}^{St}$ and D_{ik}^{α} are connected with the problem of deceleration and scattering of the test particles α due to collisions with the background particles of the species β. Let us consider a cloud of test particles which had the same velocity $\vec{V} = \vec{V}_0$ at the initial moment $t = 0$. The mean velocity $\vec{u}$ which at $t = 0$ coincided with $\vec{V}_0$ is decreasing with time due to collisions. Simultaneously, due to the process of scattering diffusion of the cloud takes place. Evolution of the cloud of test particles in the velocity space is shown schematically in Fig. 1.1. In the homogeneous plasma in the absence of forces the time derivative of the cloud mean velocity can be rewritten using kinetic equation as following:

$$\frac{\partial u_j}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{n_\alpha}\int V_j f_\alpha d\vec{V}\right) = \frac{1}{n_\alpha}\int V_j \frac{\partial f_\alpha}{\partial t} d\vec{V} = -\frac{1}{n_\alpha}\int V_j \nabla_{\vec{V}} \cdot \vec{\Gamma}_{\alpha}^{\vec{V}} d\vec{V} \, .$$

After integrating by parts, taking into account that the distribution function vanishes at the infinity, so that $f_\alpha \to 0$ at $\vec{V} \to \infty$, and therefore according to Eq. (1.13), $\vec{\Gamma}_{\alpha}^{\vec{V}} \to 0$, one obtains

$$\frac{\partial u_j}{\partial t} = \frac{1}{n_\alpha}\int \Gamma_{j\alpha}^{\vec{V}} d\vec{V} = \frac{1}{n_\alpha}\int\left(\frac{F_{j\alpha}^{St} f_\alpha}{m_\alpha} - D_{jk}^{\alpha}\frac{\partial f_\alpha}{\partial V_k}\right)d\vec{V} = \frac{1}{n_\alpha}\int f_\alpha\left(\frac{F_{j\alpha}^{St}}{m_\alpha} + \frac{\partial D_{jk}^{\alpha}}{\partial V_k}\right)d\vec{V} \, . \qquad (1.14)$$

Substitution of the test particles distribution function at $t = 0$ in the form $f_\alpha\big|_{t=0} = n_\alpha \delta(\vec{V} - \vec{V}_0)$ to the Eq. (1.14), yields

$$\left.\frac{\partial u_j}{\partial t}\right|_{t=0} = \frac{F_{j\alpha}^{St}}{m_\alpha} + \frac{\partial D_{jk}^{\alpha}}{\partial V_k} \, . \qquad (1.15)$$

Diffusion of the test particles cloud is described by a tensor of dispersion $\left\langle (\vec{V} - \vec{u})_j (\vec{V} - \vec{u})_k \right\rangle$. Here averaging is defined as

$$\langle g \rangle = \int g f d\vec{V} / n \ .$$

Evaluating the expression for the dispersion in the same manner as for the time derivative of the mean velocity it is easy to find

$$\left. \frac{\partial \left\langle (\vec{V}-\vec{u})_j (\vec{V}-\vec{u})_k \right\rangle}{\partial t} \right|_{t=0} = 2 D^{\alpha}_{jk} \ . \qquad (1.16)$$

Hence we demonstrated that the dynamical force $\vec{F}^{St}_{\alpha}$, the diffusion tensor D^{α}_{ik}, and consequently the collisional flow in the velocity space and the collision operator, are connected with the parameters of the test particles cloud in the velocity space.

1.2.2 Deceleration and diffusion of test particles cloud in the velocity space

Let us calculate the quantities in the l.h.s. of the Eqs. (1.15), (1.16). The collisions between the particles of species α and β should be considered in the center of mass reference frame as a scattering of a particle with the mass $m_{\alpha\beta} = m_{\alpha} m_{\beta} / (m_{\alpha} + m_{\beta})$ and the relative velocity $\vec{u}^r = \vec{V}_{\alpha} - \vec{V}_{\beta}$ on a central potential. The change of the velocity in the laboratory frame is proportional to the change of the relative velocity:

$$\Delta \vec{V}_{\alpha} = \frac{m_{\beta}}{m_{\alpha} + m_{\beta}} \Delta \vec{u}^r \ .$$

A relative number of scattering events with a given impact parameter ρ and azimuth angle φ for a single test particle is

$$dn_{\beta}(\vec{V}') u^r dS = f_{\beta}(\vec{V}') d\vec{V}' u^r dS = f_{\beta}(\vec{V}') d\vec{V}' u^r \rho d\rho d\varphi \ .$$

Multiplying this expression by a change of the velocity $\Delta\vec{V}_\alpha$ and integrating over the ambient particles velocities we get

$$\left.\frac{\partial u_j}{\partial t}\right|_{t=0} = \int f_\beta(\vec{V}')w_j d\vec{V}' \ , \ w_j = \frac{m_\beta}{m_\alpha + m_\beta}\int \Delta u_j^r u^r dS \ . \qquad (1.17)$$

Similarly

$$\left.\frac{\partial\left\langle(\vec{V}-\vec{u})_j(\vec{V}-\vec{u})_k\right\rangle}{\partial t}\right|_{t=0} = \int f_\beta(\vec{V}')w_{jk}d\vec{V}' \ , \ w_{jk} = \left(\frac{m_\beta}{m_\alpha + m_\beta}\right)^2 \int \Delta u_j^r \Delta u_k^r u^r dS \ . \qquad (1.18)$$

Here vector w_j and tensor w_{jk} are the functions of the relative velocity, therefore due to the tensor invariance they could be expressed according to

$$w_j = \frac{u_j^r}{u^r}A, \quad w_{jk} = \delta_{jk}B + \frac{u_j^r u_k^r}{(u^r)^2}C \, , \qquad (1.19)$$

where A, B and C are scalars. Let us calculate these tensors in the reference frame with the z axis parallel to the vector of the relative velocity. In the process of scattering at a deflection angle θ in the central mass system the change of the relative velocity components is

$$\Delta u_x^r = u^r \sin\theta\cos\varphi,$$
$$\Delta u_y^r = u^r \sin\theta\sin\varphi,$$
$$\Delta u_z^r = -u^r(1-\cos\theta).$$

For the Coulomb collisions the impact parameter is expressed through the deflection angle according to

$$tg\frac{\theta}{2} = \frac{r_s}{\rho} \, ,$$

where a strong force radius r_s is defined as

$$r_s = \frac{Z_\alpha Z_\beta e^2}{4\pi\varepsilon_0 m_{\alpha\beta}(u^r)^2}\,.$$

The change of the relative velocity components as a function of the impact parameter hence is given by

$$\Delta u_x^r = 2u^r \frac{\rho r_s}{\rho^2 + r_s^2}\cos\varphi,$$
$$\Delta u_y^r = 2u^r \frac{\rho r_s}{\rho^2 + r_s^2}\sin\varphi,$$
$$\Delta u_z^r = -2u^r \frac{r_s^2}{\rho^2 + r_s^2}.$$

Let us substitute these expressions into Eqs. (1.17), (1.18). The integrals over ρ tend to infinity and should be truncated at some value $\rho = \rho_{\max}$ since in the plasma the Coulomb potential is screened at distances of the order of Debye radius. We shall choose $\rho_{\max} = r_d$, where the Debye radius is defined as

$$r_d = \sqrt{\frac{\varepsilon_0 T}{ne^2}}\,. \qquad (1.20)$$

The resulting integrals have logarithmic accuracy and are proportional to a Coulomb logarithm Λ, where Λ is a large quantity of the order of 10-15:

$$\Lambda = \ln(r_d / r_s)\,. \qquad (1.21)$$

For more details see for example [6]. In particular

$$\int_0^{\rho_{max}} \frac{\rho d\rho}{\rho^2 + r_s^2} = \Lambda,$$
$$\int_0^{\rho_{max}} \frac{\rho^3 d\rho}{(\rho^2 + r_s^2)^2} = \Lambda - 1/2 \approx \Lambda.$$

After integrating in Eqs. (1.17) and (1.18) over ρ and φ we find

$$w_j = -(1 + m_\alpha / m_\beta) L_{\alpha\beta} \frac{u_j^r}{4\pi (u^r)^3},$$
$$w_{jk} = L_{\alpha\beta} \left(\delta_{jk} - \frac{u_j^r u_k^r}{(u^r)^2} \right) \frac{1}{4\pi u^r},$$
$$L_{\alpha\beta} = \Lambda \left(\frac{Z_\alpha Z_\beta e^2}{\varepsilon_0 m_\alpha} \right)^2.$$

The time derivatives of the mean velocity and dispersion are given by

$$\left.\frac{\partial u_j}{\partial t}\right|_{t=0} = -(1 + \frac{m_\alpha}{m_\beta}) \frac{L_{\alpha\beta}}{4\pi} \int \frac{u_j^r}{(u^r)^3} f_\beta(\vec{V}') d\vec{V}',$$
$$\left.\frac{\partial \left\langle (\vec{V} - \vec{u})_j (\vec{V} - \vec{u})_k \right\rangle}{\partial t}\right|_{t=0} = \frac{L_{\alpha\beta}}{4\pi} \int \left(\frac{\delta_{jk}}{u^r} - \frac{u_j^r u_k^r}{(u^r)^3} \right) f_\beta(\vec{V}') d\vec{V}'. \quad (1.22)$$

Note that time derivative of the higher moment $\left\langle (\vec{V} - \vec{u})_i (\vec{V} - \vec{u})_k (\vec{V} - \vec{u})_j \right\rangle$ does not contain parameter Λ and hence its contribution to the flux in the velocity space cab be neglected. With account of expressions

$$\frac{\partial^2 u^r}{\partial V_j \partial V_k} = \frac{\delta_{jk}}{u^r} - \frac{u_j^r u_k^r}{(u^r)^3},$$
$$\frac{\partial}{\partial V_j} \frac{1}{u^r} = -\frac{u_j^r}{(u^r)^3},$$

Eq. (1.22) can be rewritten in the form

$$\left.\frac{\partial u_j}{\partial t}\right|_{t=0} = (1+\frac{m_\alpha}{m_\beta})L_{\alpha\beta}\frac{\partial}{\partial V_j}\left(\frac{1}{4\pi}\int\frac{f_\beta(\vec{V}')}{\left|\vec{V}-\vec{V}'\right|}d\vec{V}'\right),$$

$$\left.\frac{\partial\left\langle(\vec{V}-\vec{u})_j(\vec{V}-\vec{u})_k\right\rangle}{\partial t}\right|_{t=0} = L_{\alpha\beta}\frac{\partial^2}{\partial V_j\partial V_k}\left(\frac{1}{4\pi}\int\left|\vec{V}-\vec{V}'\right|f_\beta(\vec{V}')d\vec{V}'\right). \qquad (1.23)$$

To rewrite Eq. (1.23) in one more form it is useful to introduce 'Rosebluth potentials'

$$\varphi_\beta = -\frac{1}{4\pi}\int\frac{f_\beta(\vec{V}')}{\left|\vec{V}-\vec{V}'\right|}d\vec{V}',$$

$$\Psi_\beta = -\frac{1}{8\pi}\int\left|\vec{V}-\vec{V}'\right|f_\beta(\vec{V}')d\vec{V}'. \qquad (1.24)$$

The 'potentials' φ_β and Ψ_β are linked by the relations

$$\Delta_{\vec{V}}\Psi_\beta = \varphi_\beta,$$

$$\Delta_{\vec{V}}\varphi_\beta = \Psi_\beta, \qquad (1.25)$$

where $\Delta_{\vec{V}}$ is the Laplace operator in the velocity space. One can derive Eq. (1.25) using the following:

$$\Delta_{\vec{V}}\frac{1}{\left|\vec{V}-\vec{V}'\right|} = -4\pi\delta(\vec{V}-\vec{V}'),$$

$$\Delta_{\vec{V}}\left|\vec{V}-\vec{V}'\right| = \frac{2}{\left|\vec{V}-\vec{V}'\right|}.$$

With account of Eq. (1.24) one gets Eq.(1.23) in the form

$$\left.\frac{\partial u_j}{\partial t}\right|_{t=0} = -(1+\frac{m_\alpha}{m_\beta})L_{\alpha\beta}\frac{\partial\varphi_\beta}{\partial V_j}. \qquad (1.26)$$

$$\left. \frac{\partial \left\langle (\vec{V}-\vec{u})_j (\vec{V}-\vec{u})_k \right\rangle}{\partial t} \right|_{t=0} = -2L_{\alpha\beta} \frac{\partial^2 \Psi_\beta}{\partial V_j \partial V_k}. \tag{1.27}$$

1.2.3 Momentum an energy losses of the test particles

The derived equations are of special interest since they can be implemented to find momentum and energy losses of the test particles of the velocity $\vec{V}_\alpha$ in the background medium of β species particles. The particle momentum is decreasing according to Eq. (1.26). Let us derive the momentum loss of electrons due to collisions with ions. We assume that the electron velocity is much larger than that of background ions. Then neglecting the ion velocity in the denominator of integrand in Eq. (1.24), one obtains $\varphi_i = -n_i / 4\pi V_e$, and, neglecting the mass ratio,

$$\left. \frac{\partial p_e}{\partial t} \right|_{t=0} = -\nu_e p_e, \qquad \nu_e = \frac{\Lambda e^4 Z^2 n_i}{4\pi \varepsilon_0^2 m_e^2 V_e^3}. \tag{1.28}$$

Here the collision frequency ν_e corresponds to inverse characteristic time scale for electron deceleration at initial moment when all electrons have the same velocity. The collision frequency is inversely proportional to the cube of the electron velocity, i.e. decreases with the velocity which is typical for Coulomb collisions. Note that one cannot use this equation at later stages since electrons not only decelerate but are also deflected during electron-ion collisions. The collision frequency given by Eq. (1.28) can be called a slowing-down collision frequency. A collision frequency which corresponds to the characteristic time of their deflection can be obtained using Eq. (1.27). Choosing the z-axis in the direction of electron velocity we obtain

$$\left. \frac{\partial \left\langle (\vec{V}-\vec{u})_\perp^2 \right\rangle}{\partial t} \right|_{t=0} = \left. \frac{\partial \left\langle (\vec{V}-\vec{u})_x (\vec{V}-\vec{u})_x \right\rangle}{\partial t} \right|_{t=0} + \left. \frac{\partial \left\langle (\vec{V}-\vec{u})_y (\vec{V}-\vec{u})_y \right\rangle}{\partial t} \right|_{t=0} \tag{1.29}$$
$$= 2\nu_e V^2.$$

Hence the deflection collision frequency is twice larger than the slowing-down collision frequency.

The characteristic frequency for the decrease of the kinetic energy of electron can be calculated using the following derivation:

$$\left.\frac{\partial \varepsilon}{\partial t}\right|_{t=0} = \frac{\partial}{\partial t}\frac{m}{2}\left\langle V_j V_j \right\rangle\Big|_{t=0} = m\left(\frac{1}{2}\left.\frac{\partial\left\langle (\vec{V}-\vec{u})_j(\vec{V}-\vec{u})_j \right\rangle}{\partial t}\right|_{t=0} + \vec{u}\left.\frac{\partial \vec{u}}{\partial t}\right|_{t=0}\right).$$

Here we use identity

$$\left\langle (\vec{V}-\vec{u})_j(\vec{V}-\vec{u})_j \right\rangle = \left\langle V_j V_j \right\rangle - \vec{u}^2 .$$

After substitution of Eqs (1.26), (1.27) one finds

$$\left.\frac{\partial \varepsilon_\alpha}{\partial t}\right|_{t=0} = -m_\alpha L_{\alpha\beta}[\varphi_\beta + (1 + m_\alpha / m_\beta)\vec{V}_\alpha \nabla_{\vec{V}} \varphi_\beta]. \qquad (1.30)$$

Let us calculate for example the change of the kinetic energy for electrons in electron-ion collisions. By analogy with electrostatics where the potential and electric field outside of the cloud of distributed charge density are known to be $\varphi_i = -n_i / 4\pi V$, $\nabla_{\vec{V}}\varphi_i = \frac{\vec{V} n_i}{4\pi V^3}$, we obtain

$$\left.\frac{\partial \varepsilon_e}{\partial t}\right|_{t=0} = -\nu_e^\varepsilon \varepsilon_e, \qquad \nu_e^\varepsilon = \frac{2m_e}{m_i}\nu_e . \qquad (1.31)$$

We see that a collision frequency for energy exchange in the process of electron-ion collisions ν_e^ε is $2m_e / m_i$ times smaller than the slowing-down collision frequency ν_e. This is expected result since the kinetic energy of electron is changing rather slowly in the collisions with ions because of the small mass ratio. If the velocity of particles are comparable the corresponding expressions can be derived from Eq. (1.30).

1.2.4 Landau collision operator

Using the expressions for dynamical force and diffusion tensor in the velocity space Eq. (1.15), (1.16) and Eqs. (1.27)-(1.27), we find the flow in the velocity space Eq.(1.13):

$$\Gamma^{\vec{V}}_{j\alpha} = -L_{\alpha\beta}[\frac{m_\alpha}{m_\beta}\frac{\partial\varphi_\beta}{\partial V_j} f_\alpha - \frac{\partial^2\Psi_\beta}{\partial V_k \partial V_j}\frac{\partial f_\alpha}{\partial V_k}]. \qquad (1.32)$$

Let us use the following relations:

$$\begin{aligned} \frac{\partial\varphi_\beta}{\partial V_j} &= -\frac{1}{8\pi}\int U_{jk}\frac{\partial f'_\beta}{\partial V'_k} d\vec{V}', \\ \frac{\partial^2\Psi_\beta}{\partial V_j \partial V_k} &= -\frac{1}{8\pi}\int U_{jk} f'_\beta d\vec{V}'. \end{aligned} \qquad (1.33)$$

Here we introduced a tensor

$$U_{jk} = \frac{\delta_{jk}}{u^r} - \frac{u^r_j u^r_k}{(u^r)^3}. \qquad (1.34)$$

The first relation in Eq. (1.33) can be obtained according to

$$\frac{\partial}{\partial V_j}\int \frac{f'_\beta}{u^r} d\vec{V}' = -\int \frac{u^r_j}{(u^r)^3} f'_\beta d\vec{V}' = -\frac{1}{2}\int \frac{\partial U_{jk}}{\partial V'_k} f'_\beta d\vec{V}' = \frac{1}{2}\int \frac{\partial f'_\beta}{\partial V'_k} U_{jk} d\vec{V}',$$

with account of

$$\frac{u^r_j}{(u^r)^3} = -\frac{1}{2}\frac{\partial U_{jk}}{\partial V_k} = \frac{1}{2}\frac{\partial U_{jk}}{\partial V'_k}.$$

The second relation is obtained according to

$$\frac{\partial\Psi_\beta}{\partial V_j \partial V_k} = -\frac{1}{8\pi}\frac{\partial}{\partial V_j}\int \frac{\partial u^r}{\partial V_k} f'_\beta d\vec{V}' = -\frac{1}{8\pi}\frac{\partial}{\partial V_j}\int \frac{u^r_k}{u^r} f'_\beta d\vec{V}' = -\frac{1}{8\pi}\int U_{jk} f'_\beta d\vec{V}'.$$

Finally, substituting Eq. (1.33) into Eq. (1.32), we obtain the flow in the velocity space in the form

$$\Gamma_{j\alpha}^{\vec{V}} = \frac{Z_\alpha^2 Z_\beta^2 e^4 \Lambda}{8\pi\varepsilon_0^2 m_\alpha} \int U_{jk} \left(\frac{f_\alpha}{m_\beta} \frac{\partial f'_\beta}{\partial V'_k} - \frac{f'_\beta}{m_\alpha} \frac{\partial f_\alpha}{\partial V_k} \right) d\vec{V}' . \qquad (1.35)$$

Landau collision operator is equal to the divergence of this flow with an opposite sign:

$$St_{\alpha\beta} = -\frac{Z_\alpha^2 Z_\beta^2 e^4 \Lambda}{8\pi\varepsilon_0^2 m_\alpha} \frac{\partial}{\partial V_j} \int U_{jk} \left(\frac{f_\alpha}{m_\beta} \frac{\partial f'_\beta}{\partial V'_k} - \frac{f'_\beta}{m_\alpha} \frac{\partial f_\alpha}{\partial V_k} \right) d\vec{V}' . \qquad (1.36)$$

1.3 Collision operator in the relativistic case

In the previous Section relativistic effects were neglected. For electrons this is justified if their temperature is significantly smaller than their rest mass (~0.5MeV). In the opposite case one shall use more general expression for the collision operator. In this Section we shall restrict consideration to the practically frequent situation of the relativistic particles moving through the background of nonrelativistic particles which could be considered as practically stationary. Since the deflection angle is assumed to be small as in nonrelativistic case one can neglect radiation with respect to the collisional energy losses.

The momentum of relativistic particle α moving in the *z*-direction is

$$p = p_z = \gamma m_\alpha V , \qquad (1.37)$$

where m_α is a rest mass, and $\gamma = (1 - V^2 / c^2)^{-1/2}$ is the relativistic mass factor. The particle energy is

$$E_\alpha = \gamma m_\alpha c^2 = \sqrt{m_\alpha^2 c^4 + p^2 c^2} . \qquad (1.38)$$

The initial energy of nonrelativistic particle β is $E_\beta = m_\beta c^2$. After the collision the energies of particles α and β are given by

$$E'_\alpha = \sqrt{m_\alpha^2 c^4 + (p_z + \delta p_z)^2 c^2 + (\delta p_\perp)^2 c^2} \ . \qquad (1.39)$$

$$E'_\beta = \sqrt{m_\beta^2 c^4 + \delta p_z^2 c^2 + (\delta p_\perp)^2 c^2} \ .$$

Here $\delta p_\perp$ is the momentum change in the direction perpendicular to the *z*-axis. For the small change of momentum during the collision

$$E'_\alpha = E_\alpha \left(1 + \frac{2 p_z \delta p_z + (\delta p_\perp)^2}{2E_\alpha^2}\right) c^2 , \qquad (1.40)$$

$$E'_\beta = m_\beta c^2 \left(1 + \frac{(\delta p_z)^2 + (\delta p_\perp)^2}{2 m_\beta^2 c^2}\right) .$$

We neglect here $(\delta p_z)^2$ since from the energy conservation one obtains

$$\delta p_z = -\left(1 + \frac{E_\alpha}{m_\alpha c^2}\right) \frac{(\delta p_\perp)^2}{2 p_z} << p_z .$$

The change of the momentum in the perpendicular direction depends on the impact parameter as (see previous Section)

$$\delta p_\perp = \frac{2 m_\alpha V \rho r_s}{r_s^2 + \rho^2} .$$

The total effect of collisions with all impact parameters is described as

$$\frac{\partial \left\langle (\delta p_\perp)^2 \right\rangle}{\partial t} = \left(2 m_\alpha r_s V\right)^2 \int_0^{r_d} \frac{2\pi \rho^3 V n_\beta}{(r_s^2 + \rho^2)^2} d\rho = \frac{2 m_\alpha^2 c^3}{V} \hat{\nu}_{\alpha\beta} . \qquad (1.41)$$

$$\frac{\partial\langle(\delta p_z)\rangle}{\partial t} = -(1+\frac{E_\alpha}{m_\beta c^2})\frac{m_\alpha^2 c^3}{p_z V}\hat{\nu}_{\alpha\beta}.$$

Here it is taken into account that the number of encounters in unit time with impact parameters in the interval $(\rho, \rho + d\rho)$ is $n_\beta V 2\pi\rho d\rho$. The collision frequency at the speed of light is

$$\hat{\nu}_{\alpha\beta} = \frac{Z_\alpha^2 Z_\beta^2 e^4 \Lambda}{4\pi\varepsilon_0^2 m_\alpha^2 c^3}. \tag{1.42}$$

Analogously

$$\frac{\partial\langle\delta p_k \delta p_l\rangle}{\partial t} = \mu (m_\alpha c)^3 \hat{\nu}_{\alpha\beta} P_{kl}. \tag{1.43}$$

where

$$P_{kl} = \frac{p^2\delta_{kl} - p_k p_l}{p^3}, \quad \mu = (1+\frac{p^2}{m_\alpha^2 c^2})^{1/2.}$$

The relativistic collision operator is given by

$$St_{\alpha\beta} = m_\alpha^3 c^3 \hat{\nu}_{\alpha\beta}[\frac{m_\alpha}{m_\beta p^2}\frac{\partial}{\partial p}(\mu^2 f_\alpha) + \frac{\partial}{\partial p_k}(\frac{P_{kl}}{2}\frac{\partial(\mu f_\alpha)}{\partial p_l}]. \tag{1.44}$$

1.4. Fokker-Planck equation

Landau collision operator still remains too complicated. In some cases it could be further simplified. It particular it can be done for the test impurity of heavy ions in the ambient plasma with light ions. Let f_α be a distribution function of test heavy ions, so that $n_\alpha << n_\beta$. We assume that impact of heavy ions on the distribution function of light ions f_β is negligible, and function f_β is assumed to the Maxwellian one. Taking into account that for $f_\beta = f_\beta^M$ the relation

$$\frac{\partial f'_\beta}{\partial V'_k} = -\frac{m_\beta}{T_\beta} V'_k f'_\beta \,,$$

is valid, we can simplify Landau collision integral Eq. (1.36). The first term can be transformed as:

$$\frac{\partial}{\partial V_j}\int U_{jk}\frac{f_\alpha}{m_\beta}\frac{\partial f'_\beta}{\partial V'_k} d\vec{V}' = -\frac{\partial}{\partial V_j}\int U_{jk}\frac{f_\alpha}{T_\beta} V'_k f'_\beta d\vec{V}' = -\frac{\partial}{\partial V_j}\int U_{jk}\frac{f_\alpha}{T_\beta}(V_k - u_k^r) f'_\beta d\vec{V}' \,.$$

Here the second term under the integral is equal to zero,

$$U_{jk} u_k^r = 0. \qquad (1.45)$$

Since $\left|\vec{V}'\right| >> \left|\vec{V}\right|$, we have $\vec{u}^r \approx -\vec{V}'$, and, therefore,

$$-\frac{\partial}{\partial V_j}\int U_{jk}\frac{f_\alpha}{T_\beta} V_k f'_\beta d\vec{V}' = -\frac{\partial}{\partial V_j}\int\left[\frac{\delta_{jk}}{V'} - \frac{V'_j V'_k}{(V')^3}\right]\frac{f_\alpha}{T_\beta} V_k f'_\beta d\vec{V}' \,.$$

In the second term the contribution from components with $j \neq k$ turns to zero (in the reference frame where the mean velocity $\vec{u}^r$ is absent), therefore the remaining integral is given by

$$-\frac{\partial}{\partial V_j}\int \frac{f_\alpha f'_\beta}{T_\beta} V_j \frac{(V')^2 - (V'_j)^2}{(V')^3} d\vec{V}' = -\frac{4}{3\sqrt{2\pi}}\frac{m_\beta^{1/2}}{T_\beta^{3/2}} n_\beta \frac{\partial}{\partial \vec{V}}(f_\alpha \vec{V}) \,.$$

The second term in Landay collision integral Eq. (1.36) is reduced analogously. Finally the kinetic equation with the simplified collision operator has the form

$$\frac{df_\alpha}{dt} = \tilde{\nu}_{\alpha\beta} \frac{\partial}{\partial \vec{V}} (\vec{V} f_\alpha + \frac{T_\beta}{m_\alpha} \frac{\partial f_\alpha}{\partial \vec{V}}), \tag{1.46}$$

where a collision frequency $\tilde{\nu}_{\alpha\beta}$ is defined as

$$\tilde{\nu}_{\alpha\beta} = \frac{\sqrt{2} m_\beta^{1/2} \Lambda Z_\alpha^2 Z_\beta^2 e^4 n_\beta}{12 \pi^{3/2} \varepsilon_0^2 m_\alpha T_\beta^{3/2}} . \tag{1.47}$$

Equation (1.46) is known as Fokker-Planck equation. In the homogeneous plasmas in the absence of external forces this equation describes the process of relaxation of a distribution function f_α to the Maxwellian distribution function with the temperature T_β

$$f_\alpha^M = n_\alpha \left(\frac{m_\alpha}{2\pi T_\beta} \right)^{3/2} \exp\left(-\frac{m_\alpha V^2}{2 T_\beta} \right)$$

with the characteristic time scale $\tau = \tilde{\nu}_{\alpha\beta}^{-1}$. Indeed this Maxwellian distribution function f_α^M turns to zero the r.h.s. of Eq. (1.46).

The Fokker-Planck equation is a linear equation and can be solved analytically. Let us demonstrate how the Fokker-Planck equation can be used to calculate mobility of impurities-the coefficient which connects the applied electric field and mean (fluid) velocity of impurities. In the stationary homogeneous plasma in the electric field the Fokker-Planck equation has the form

$$\frac{Z_\alpha e \vec{E}}{m_\alpha} \frac{\partial f_\alpha}{\partial \vec{V}} = \tilde{\nu}_{\alpha\beta} \frac{\partial}{\partial \vec{V}} (\vec{V} f_\alpha + \frac{T_\beta}{m_\alpha} \frac{\partial f_\alpha}{\partial \vec{V}}) . \tag{1.48}$$

Let us multiply Eq. (1.48) by $\vec{V}$ and integrate over velocities. After integrating by part one obtains that the first term in the r.h.s. is proportional to the particle flux $n_\alpha \vec{u}_\alpha$, while the second term turns to zero. Integral in the l.h.s. is equal to $-Z_\alpha n_\alpha e\vec{E} / m_\alpha$. Finally

$$\vec{u}_\alpha = b_\alpha \vec{E}, \quad b_\alpha = \frac{Z_\alpha e}{m_\alpha \tilde{\nu}_{\alpha\beta}}. \tag{1.49}$$

Thus the mobility is given by Eq. (1.49) with the numerical coefficient equal to unity. The distribution function can also be obtained from Eq. (1.48). In relatively small electric fields when the fluid velocity of impurities is much smaller that the thermal velocity, the solution can be sought in the form of a sum of the maxwellian distribution plus small correction: $f = f^M + f^1$. In the linear approximation the distribution function in the l.h.s. of Eq. (1.48) can be taken as the maxwellian one $f_\alpha = f_\alpha^M$, since electric field is a small value, while in the r.h.s. $f_\alpha = f_\alpha^1$ can be kept since the maxwellian distribution function turns to zero the r.h.s. Then it is easy to obtain

$$f_\alpha^1 = \frac{m_\alpha}{T_\beta} (\vec{u}_\alpha \vec{V}) f_\alpha^M, \tag{1.50}$$

where the mean velocity is given by Eq. (1.49). In other words the distribution function is given by expansion of a shifted maxwellian distribution

$$
\begin{aligned}
f_\alpha &= n_\alpha \left(\frac{m_\alpha}{2\pi T_\beta} \right)^{3/2} \exp\left(-\frac{m_\alpha (\vec{V} - \vec{u}_\alpha)^2}{2T_\beta} \right) \\
&\approx n_\alpha \left(\frac{m_\alpha}{2\pi T_\beta} \right)^{3/2} \exp\left(-\frac{m_\alpha V^2}{2T_\beta} \right) (1 + \frac{m_\alpha}{T_\beta} \vec{u}_\alpha \vec{V}).
\end{aligned}
\tag{1.51}
$$

It is however worthwhile to note that in the general case the distribution function in the electric field does have such simple form.

1.5. Runaway electrons in fully ionized plasma

Let us analyze the distribution function of electrons in the homogenious plasma in the absence of magnetic field in the weak electric field

$$
E << E_D = \frac{e^3 \Lambda n}{\varepsilon_0 T_e} . \tag{1.52}
$$

Here the field E_D is known as a Dreicer field. This condition is equivalent to an assumption that the electric force is smaller than the friction force acting on the main body of electrons with the thermal velocities $V_{T_e} = \sqrt{2T_e / m_e}$:

$$
enE << nm_e \nu_e (V_{Te}) V_{Te} \sim nm_e V_{Te} \frac{ne^4 \Lambda}{\varepsilon_0 m_e^2 V_{Te}^3} ,
$$

From which one gets Eq.(1.52). In the opposite case the main part of electrons is infinitely accelerated since the friction force is unable to balance the electric force due to the inverse dependence of the friction force on the velocity.

In the weak electric field when the condition Eq. (1.52) is satisfied only small fraction of electrons is accelerated by electric field so the distribution function has a tail in the direction $-\vec{E}$, Fig. 1.2. The main bulk of electrons is approximately described by the Maxwellian distribution function while the flux to the tail in the velocity space is caused by the Coulomb collisions. Since the number of runaway electrons is small (exponentially small as shown below) the problem of runaway electrons can be treated as stationary problem. Indeed the density in the main body of distribution function is very slowly decreasing or is compensated by small source of electrons.

Let us restrict ourselves by calculation of the number of electrons per second which are accelerated and becomes runaway electrons, i.e. we shall find stationary flux in the velocity space towards larger energies. The kinetic equation in the electric field is given by

$$\frac{\partial f}{\partial t} - \frac{e\vec{E}}{m_e}\frac{\partial f}{\partial \vec{V}} + \nabla_{\vec{V}} \cdot \vec{\Gamma}^{\vec{V}} = 0 . \qquad (1.53)$$

Let us introduce spherical coordinates in the velocity space with the z axis in the $-\vec{E}$ direction. The second term in the l.h.s. of Eq. (1.45) can be rewritten in the form (dependence on azimuth angle is absent due the symmetry of the problem)

$$-\frac{e\vec{E}}{m_e}\frac{\partial f}{\partial \vec{V}} = \frac{eE}{m_e}(\cos\theta\frac{\partial f}{\partial V} - \frac{\sin\theta}{V}\frac{\partial f}{\partial \theta})$$
$$= \frac{eE}{m_e}\left[\frac{\cos\theta}{V^2}\frac{\partial}{\partial V}(V^2 f) - \frac{1}{V\sin\theta}\frac{\partial}{\partial\theta}(f\sin^2\theta)\right].$$

Divergence of the collisional flux in the velocity space (the third term in the l.h.s. of Eq. (1.53) in the spherical coordinates has the form

$$\nabla_{\vec{V}} \cdot \vec{\Gamma}^{\vec{V}} = \frac{1}{V^2}\frac{\partial}{\partial V}(V^2\Gamma_V^{\vec{V}}) + \frac{1}{V\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\Gamma_\theta^{\vec{V}}) .$$

We are interested in the collisional flux averaged over the angles in the velocity space

$$\overline{\Gamma}_V^{\vec{V}} = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \Gamma_V^{\vec{V}} .$$

Now one can also average kinetic equation for fast electrons over the angles assuming $\cos\theta \approx 1$, since the distribution function of fast electrons is strongly elongated in the $-\vec{E}$ direction. From Eq. (1.53) one obtains

$$\frac{\partial \bar{f}}{\partial t}+\frac{eE}{m_e V^2}\frac{\partial}{\partial V}(V^2\bar{f})+\frac{1}{V^2}\frac{\partial}{\partial V}(V^2\overline{\Gamma}_V^{\vec{V}})=0\,. \tag{1.54}$$

The average distribution function $\bar{f}$ is sufficient to calculate the net flux in the velocity space. In the collision operator it is sufficient to take into account collisions of the fast electrons with the slow Maxwellian background. Collisions of fast electrons with ions gives small contribution to the flux over energies due the small mass ratio, and collisions between fast electrons are negligible.

According to Eq. (1.35) a component of a flux of the fast electrons in the velocity space is

$$\begin{aligned}\Gamma_j^{\vec{V}} &= \frac{\Lambda e^4}{8\pi\varepsilon_0^2 m_e}\int U_{jk}\left(\frac{f}{m_e}\frac{\partial f^M(\vec{V}')}{\partial V_k'}-\frac{f^M(\vec{V}')}{m_e}\frac{\partial f}{\partial V_k}\right)d\vec{V}' \\ &= -\frac{\Lambda e^4}{8\pi\varepsilon_0^2 m_e T_e}\int U_{jk}\left(V_k' f f^M(\vec{V}')+\frac{T_e}{m_e}f^M(\vec{V}')\frac{\partial f}{\partial V_k}\right)d\vec{V}'.\end{aligned} \tag{1.55}$$

Here f^M is unshifted Maxwellian distribution function. Let us evaluate the first term in the integrand. In the zero approximation the relative velocity coincides with the fast electron velocity $\vec{u}^r=\vec{V}$, and tensor U_{jk} is independent on V' so that the integral turns to zero. In the first approximation using the expansion $u^r=V(1-V_jV_j'/V^2)$ and expanding the tensor U_{jk}, one obtains

$$\begin{aligned}\int U_{jk}V_k' f f^M(\vec{V}')d\vec{V}' = \int\left(\frac{\delta_{jk}}{V}\frac{V_iV_i'V_k'}{V^2}+\frac{V_j'V_kV_k'}{V^3}+\frac{V_jV_kV_k'}{V^3}-\frac{V_jV_k}{V^3}\frac{3V_iV_i'}{V^2}V_k'\right). \\ \times f f^M(\vec{V}')d\vec{V}'\end{aligned} \tag{1.56}$$

The last and the previous terms cancel during integration, since

$$3\int\frac{V_kV_k'V_iV_i'}{V^2}f^M(\vec{V}')d\vec{V}' = 3\frac{V_k^2}{V^2}\int(V_k')^2 f^M(\vec{V}')d\vec{V}' = \frac{V_k^2}{V^2}\int(V')^2 f^M(\vec{V}')d\vec{V}'\,.$$

The first two terms in Eq. (1.56) are identical. Hence

$$\int U_{jk} V'_k f f^M (\vec{V}') d\vec{V}' = \frac{2T_e}{m_e V^3} V_j f \ .$$

The second term in Eq. (1.53) can be evaluated analogously. One has to take into account the inequality $V_\perp << V$, and expansion of the tensor U_{jk} up to the second order terms proportional to $(V')^2$ is required. Finally flux in the $\vec{V}$ direction is

$$\Gamma_V^{\vec{V}} = -\nu_e(V)(Vf + \frac{T_e}{m_e}\frac{\partial f}{\partial V}) \, ,$$

where

$$\nu_e(V) = \frac{ne^4 \Lambda}{4\pi\varepsilon_0^2 m_e^2 V^3}$$

is the collision frequency of fast electrons. After averaging over the angles in the velocity space

$$\overline{\Gamma}_V^{\vec{V}} = -\nu_e(V)(V\bar{f} + \frac{T_e}{m_e}\frac{\partial \bar{f}}{\partial V}) \, . \qquad (1.57)$$

The net flux in the velocity space as follows from Eq. (1.54) is given by

$$S_V = \overline{\Gamma}_V^{\vec{V}} + \frac{eE}{m_e}\bar{f} \ . \qquad (1.58)$$

After neglecting time derivative in Eq. (1.54) one obtains the conservation of the net flux in the velocity space

$$4\pi V^2 S_V = const = \dot{n}_r \, . \qquad (1.59)$$

Here a constant $\dot{n}_r$ is a number of fast electrons per second escaping in the velocity space, which is to be found. This equation should be considered as equation for the averaged distribution function of fast electrons. In the dimensionless form

$$\alpha = \frac{E}{E_D}, \qquad u = \frac{V m_e^{1/2} \alpha^{1/2}}{T_e^{1/2}} .$$

the Eq. (1.59) can be rewritten as

$$\frac{d\bar{f}}{du} + \frac{u - u^3}{\alpha} \bar{f} = -Cu, \qquad C = \frac{\dot{n}_r m_e^2}{16\pi^2 n e^2 \Lambda \alpha^{1/2}} . \tag{1.60}$$

The solution of Eq. (1.60) is given by

$$\bar{f} = F - CF \int_0^u \frac{u'}{F(u')} du', \qquad F = A \exp\left[\frac{1}{2\alpha} \left(\frac{u^4}{2} - u^2 \right) \right]. \tag{1.61}$$

When $\vec{u} \to 0$ the distribution function $\bar{f} \to f^{(M)}$. This condition determines the constant A:

$$A = n \left(\frac{m_e}{2\pi T_e} \right)^{3/2} .$$

For $u \to \infty$ the function $F \to \infty$, while the function $\bar{f}$ remains finite, so the ratio $\bar{f} / F \to 0$. This condition determines the constant C

$$C^{-1} = \int_0^\infty \frac{u}{F} du = \frac{1}{n} \left(\frac{2\pi T_e}{m_e} \right)^{3/2} \int_0^\infty \exp\left[-\frac{1}{2\alpha} \left(\frac{u^4}{2} - u^2 \right) \right] u du .$$

This integral can be calculated using Laplace method. Finally the number of runaway electrons per time unit is proportional to

$$\dot{n}_r = \beta(\alpha) n \nu_e(V_{Te}) \exp(-E_D / 4E) . \qquad (1.62)$$

A function $\beta(\alpha)$ is some dimensionless power function of the parameter α, which cannot be found in the framework of approximation considered. The reason is connected with the fact that the Dreicer electric field Eq. (1.52) is proportional to the Coulomb logarithm Λ, so that the latter is under the exponent in Eq. (1.62). Since the Coulomb logarithm Λ is specified with logarithmic accuracy the quantity β could be correctly calculated only using the next approximation. So the Eq. (1.62) determines the number of runaway electrons with logarithmic accuracy.

For the electric field larger than the Dreicer electric field, the main mechanism of the runaway electron generation is associated with so-called avalanche effect-multiplication of the fast electrons due to low impact parameter collisions of the fast electrons with the thermal ones. During such collisions electrons with the velocities of the order of the speed of light c produce new fast electrons, and the generation speed is proportional to the existing number of the fast electrons. The collision frequency for the electrons moving with the speed of light in accordance with Eq. (1.28) is

$$\nu_e(c) = \frac{\Lambda e^4 n}{4\pi\varepsilon_0^2 m_e^2 c^3} . \qquad (1.63)$$

If the electric field is larger than the critical electric field $(E_c = E_D(c))$

$$E_c = \frac{\Lambda e^3 n}{\varepsilon_0 m_e c^2} , \qquad (1.64)$$

Then the fast electron can transfer part of their energy to the slow electrons. For $E >> E_c$ accelerated are electrons with the velocities $V > V_0 = c(E_c / E)^{1/2}$ since their friction force is smaller than the electric field. Since the Coulomb cross section is inversely proportional to the cube of the scattering angle and therefore is inversely proportional to cube of the transferred velocity, such electrons are born as a result of scattering of fast electrons with the frequency $\nu_e^* = \dfrac{e^4 n}{4\pi\varepsilon_0^2 m_e^2 V^3}$. The number of such electrons born per second by order of magnitude can be estimated as (f_r is a 1D distribution function of fast electrons which is independent of the velocity)

$$\int_{V_0}^{c} \nu_e^* f_r dV = \frac{f_r}{8\pi\varepsilon_0^2 c^2}(\frac{c^2}{V_0^2} - 1)\frac{e^4 n}{m_e^2} .$$

Assuming that the distribution function of fast electrons is constant, so that their density is $n_r = cf_r$, one obtains an estimate for the number of fast electrons, which are born per time unit

$$\dot{n}_r = n_r \nu_e(c)(E / E_c - 1)/(2\Lambda) . \qquad (1.65)$$

Here the factor $(E / E_c - 1)$ demonstrates that the avalanche effect has threshold character.

1.6. Distribution function of electrons in slightly ionized plasma

As in the previous section we shall consider the electrons being the test particles. Then the collision operator could be linearized and kinetic equation could be simplifies significantly. Slightly ionized plasma is defined as the plasma where the following inequality is fulfilled

$$\nu_{ee} << \nu_{eN} , \qquad (1.66)$$

where ν_{ee} is electron-electron collision frequency столкновений (for clarity let us consider ν_{ee} to be identical to $\nu_e(V)$ of the previous section), and ν_{eN} is electron-neutral collision frequency. However, the condition given by Eq. (1.66) is not sufficient to treat electrons as test particles. There is another frequency responsible for the change of electron energy in the electron-neutral collisions:

$$\nu_{eN}^{\varepsilon} = \delta_{eN}\nu_{eN} = \frac{2m_e}{m_N}\nu_{eN} . \qquad (1.67)$$

This frequency contains electron to neutral mass ratio, while in electron-electron collisions energy of electron is changed practically during one collision at a time scale $\sim \nu_{ee}^{-1}$. Hence the Coulomb collisions could be neglected only if more strong condition is satisfied:

$$\nu_{ee} << \nu_{eN}^{\varepsilon} . \tag{1.68}$$

The intermediate case

$$\nu_{eN}^{\varepsilon} << \nu_{ee} << \nu_{eN} \tag{1.69}$$

Also correspond to slightly ionized plasma, and due to frequent electron-electron collisions the distribution function of electrons is close to the Maxwellian one.

1.6.1 Approximation $f_0, \vec{f}_1$

Let us assume the condition Eq. (1.66) to be fulfilled. Due to the small energy loss of electron in the electron-neutral collisions the mean velocity of electron is small with respect to the chaotic velocity. As a result the anisotropic part of the distribution function should be also small with respect to the isotropic one. Hence it is reasonable to seek the distribution function as an expansion over spherical harmonics decaying with number:

$$f(\vec{r},\vec{V},t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m(\vec{r},V,t) Y_l^m(\theta,\varphi) . \tag{1.70}$$

The coefficients f_l^m depend on the velocity absolute value. Spherical functions could be expressed through the associated Legendre polynomials

$$Y_l^m = P_l^m(\cos\theta)\exp(im\varphi).$$

The first spherical functions are

$$Y_0^0 = 1, \quad Y_1^0 = \cos\theta, \quad Y_1^1 = \sin\theta(\cos\varphi + i\sin\varphi).$$

Let us truncate expansion Eq. (1.70) and keep the first two terms. Then Eq. (1.70) can be rewritten in the form

$$f(\vec{r},\vec{V},t) = f_0(\vec{r},V,t) + \frac{\vec{f}_1(\vec{r},V,t)\vec{V}}{V}. \qquad (1.71)$$

This distribution function should be inserted into the kinetic equation for electrons. Below is the result of evaluation of the different terms of the kinetic equation. The first term:

$$\frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial t} + \frac{\vec{V}}{V}\frac{\partial \vec{f}_1}{\partial t}. \qquad (1.72)$$

The term with the spatial derivative has the form

$$\vec{V}\nabla_{\vec{r}} f = \vec{V}\nabla_{\vec{r}} f_0 + \vec{V}\nabla_{\vec{r}}(\vec{f}_1\vec{V}/V).$$

Using the identity known from vector algebra

$$\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\nabla)\vec{G} + (\vec{G}\nabla)\vec{F} + [\vec{F}\times[\nabla\times\vec{G}]] + [\vec{G}\times[\nabla\times\vec{F}]],$$

one obtains

$$\vec{V}\nabla_{\vec{r}} f = \vec{V}\nabla_{\vec{r}} f_0 + \frac{\vec{V}}{V}(\vec{V}\nabla_{\vec{r}})\vec{f}_1. \qquad (1.73)$$

Other terms are evaluated analogously:

$$
\begin{aligned}
&-\frac{e}{m_e}\left(\vec{E}+[\vec{V}\times\vec{B}]\right)\nabla_{\vec{V}} f \\
&= -\frac{e\vec{E}}{m_e}\frac{\vec{V}}{V}\frac{\partial f_0}{\partial V} - \frac{e\vec{E}}{m_e}(\vec{V}\nabla_{\vec{V}})\frac{\vec{f}_1}{V} - \frac{e\vec{E}\vec{f}_1}{m_e V} - \frac{e}{m_e}[\vec{V}\times\vec{B}]\frac{\vec{f}_1}{V}.
\end{aligned}
\tag{1.74}
$$

The collision integral is also separated onto two components (for homogeneous neutral gas at rest)

$$
St_{eN} = St_{eN}^0 + St_{eN}^1 ,
$$

which depends on f_0 and $\vec{f}_1$ correspondingly. Now multiplying kinetic equation by $(1/4\pi)d\Omega$, where $d\Omega$ is the solid angle in the velocity space, for simplicity axis could be directed along $\vec{f}_1$, after integrating over solid angle we obtain the averaged kinetic equation:

$$
\frac{\partial f_0}{\partial t} + \frac{V}{3}\nabla_{\vec{r}}\cdot\vec{f}_1 - \frac{e\vec{E}}{m_e}\frac{V}{3}\frac{\partial(\vec{f}_1/V)}{\partial V} - \frac{e\vec{E}}{m_e V}\vec{f}_1 = St_{eN}^0 . \tag{1.75}
$$

Multiplying kinetic equation by quantities V_j/V and $(3/4\pi)d\Omega$ and integrating over solid angle $d\Omega$, one obtains three equations. After summation we have one equation in the vector form

$$
\frac{\partial\vec{f}_1}{\partial t} + V\nabla_{\vec{r}} f_0 - \frac{e\vec{E}}{m_e}\frac{\partial f_0}{\partial V} + \frac{e}{m_e}[\vec{B}\times\vec{f}_1] = \left\langle St_{eN}^1 \right\rangle . \tag{1.76}
$$

Collision integral for the electron-neutral collisions could be obtained from the general expression Eq. (1.5). It is also possible to derive such expression directly assuming the neutral particles to be cold and staying at rest. Indeed, since the relative velocity coincides with the velocity of electron, the number of particles coming and leaving the differential volume in the velocity space is (for brevity the differential cross section is denoted as σ)

$$St_{eN} d\vec{V} = \int_{\Omega} n_N \sigma(V', \theta) V' f' d\Omega d\vec{V}' - \int_{\Omega} n_N \sigma(V, \theta) V f d\Omega d\vec{V} .$$

Electron velocities before and after collision are connected by the relation

$$V' = V[1 + \frac{2m_e}{m_N}(1 - \cos\theta)]^{1/2} .$$

It then follows that $dV'/V' = dV/V$. Hence

$$d\vec{V}' = V'^2 dV' d\Omega = \frac{V'^3}{V} dV d\Omega = \frac{V'^3}{V^3} d\vec{V} .$$

Substituting this expression to the collision integral one obtains

$$St_{eN} = \frac{n_N}{V^3} \int_{\Omega} [V'^4 f' \sigma(V', \theta) - V^4 f \sigma(V, \theta)] d\Omega . \qquad (1.77)$$

Let us first calculate collision integral Eq.(1.69) in the first approximation with respect to the ratio m_e / m_N, i.e. the mass ratio is put to zero. In this approximation $V' = V$ and therefore $f_0 = f_0'$, so according to Eq. (1.77) $St_{eN}^0 = 0$. Inserting the correction $\vec{f}_1 \vec{V} / V$ into Eq. (1.77) we find

$$St_{eN}^1 = n_N V |f_1| \int_{\Omega} [\sigma(V, \tilde{\theta}) \cos\theta' - \sigma(V, \tilde{\theta}) \cos\theta] d\Omega . \qquad (1.78)$$

In this equation scattering angle between vectors $\vec{V}$ and $\vec{V}'$ is denoted as $\tilde{\theta}$, while θ, φ and θ', φ' are the angles related to the spherical coordinates for vectors $\vec{V}$ and $\vec{V}'$ correspondingly. Let us now turn from the integration over scattering angle to the equivalent integration over the angles θ', φ'. Let us evaluate the first term under the integral. Expansion of the cross section over Legendre polynomials is given by

$$\sigma(V,\tilde{\theta}) = \sum_k \sigma_k P_k(\cos\tilde{\theta}) .$$

The cosine of the angle between two vectors is given by the expression

$$\cos\tilde{\theta} = \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\varphi-\varphi') .$$

According to addition theorem for spherical functions

$$P_k(\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\varphi-\varphi')) = P_k(\cos\theta)P_k(\cos\theta')$$
$$+2\sum_{m=1}^{k}\frac{(k-m)!}{(k+m)!}P_k^m(\cos\theta)P_k^m(\cos\theta')\cos(m(\varphi-\varphi')).$$

Integrating the first term in Eq. (1.70) with account of orthogonal properties of Legendre polynomials, one obtains

$$\int_\Omega \sigma(V,\tilde{\theta})\cos\theta' d\Omega = \int_\Omega \sigma_1\cos^2\theta' d\Omega = (4\pi/3)\sigma_1 .$$

It is convenient to rewrite this expression again with account of orthogonality in the form

$$\int_\Omega \sigma(V,\tilde{\theta})\cos\theta' d\Omega = \int_\Omega \sigma(V,\tilde{\theta})\cos\tilde{\theta}\, d\Omega .$$

Combining two terms in Eq. (1.70), we find

$$St_{eN}^1 = -\nu_{eN}(V)\vec{f}_1\vec{V}/V , \qquad (1.79)$$

where

$$\nu_{eN}(V) = n_N V\int_\Omega (1-\cos\theta)\sigma(V,\theta)d\Omega \qquad (1.80)$$

is known as transport collision frequency for electron-neutral collisions. After averaging with the weight $3V_j/(4\pi V)$, we have

$$\left\langle St^1_{eN} \right\rangle = -\nu_{eN}(V)\vec{f}_1 . \tag{1.81}$$

To obtain the nonzero value of St^0_{eN} it is necessary to keep the terms of the order of m_e/m_N in the collision integral Eq. (1.77). Inserting the value f_0 into Eq. (1.77) and expanding the integrand in series, one obtains

$$St^0_{eN} = \frac{n_N}{V^3}\int_\Omega [V'^2 - V^2]\frac{\partial}{\partial V^2}(V^4 f_0 \sigma)d\Omega . \tag{1.82}$$

Finally, inserting the value V' into Eq. (1.74) we have

$$St^0_{eN} = \frac{2m_e}{m_N}\frac{1}{V}\frac{\partial}{\partial V^2}\left[\nu_{eN}(V)V^3 f_0\right]. \tag{1.83}$$

Collison integrals given by Eqs. (1.81) and (1.83) enters the right hand sides of equations for f_0 and $\vec{f}_1$, and therefore close the system for isotropic and anisotropic parts of the distribution function.

1.6.2 Distribution function in the electric field

Let us find a distribution function of electrons in the homogeneous plasma in the presence of stationary homogeneous electric field, and in the absence of magnetic field. In this case the Eqs. (1.81), (1.83) has the form

$$\begin{aligned} &\frac{\partial f_0}{\partial t} - \frac{e\vec{E}}{m_e}\frac{V}{3}\frac{\partial(\vec{f}_1/V)}{\partial V} - \frac{e\vec{E}}{m_e V}\vec{f}_1 = \frac{2m_e}{m_N}\frac{1}{V}\frac{\partial}{\partial V^2}\left[\nu_{eN}(V)V^3 f_0\right] , \\ &\frac{\partial \vec{f}_1}{\partial t} - \frac{e\vec{E}}{m_e}\frac{\partial f_0}{\partial V} = -\nu_{eN}(V)\vec{f}_1 . \end{aligned} \tag{1.84}$$

Since there is only one featured direction along the electric field, the distribution function $\vec{f}_1 \| \vec{E}$. As follows from the second equation the characteristic time scale for establishing stationary distribution function is given by ν_{eN}^{-1}, i.e. occurs practically at a single collision and corresponds to the relaxation time scale for the momentum. In the stationary case

$$\vec{f}_1 = \frac{e\vec{E}}{m_e \nu_{eN}(V)} \frac{\partial f_0}{\partial V}. \qquad (1.85)$$

The equation for f_0 can be rewritten in the form

$$\frac{\partial f_0}{\partial t} - \frac{1}{3V^2} \frac{e\vec{E}}{m_e} \frac{\partial(\vec{f}_1 V^2)}{\partial V} = \frac{m_e}{m_N} \frac{1}{V^2} \frac{\partial}{\partial V} \left[\nu_{eN}(V) V^3 f_0 \right]. \qquad (1.86)$$

After substituting $\vec{f}_1$ into Eq. (1.86) we have

$$\frac{\partial f_0}{\partial t} - \frac{1}{3V^2} \frac{e^2 E^2}{m_e^2} \frac{\partial}{\partial V} \left[\frac{V^2}{\nu_{eN}(V)} \frac{\partial f_0}{\partial V} \right] = \frac{m_e}{m_N} \frac{1}{V^2} \frac{\partial}{\partial V} \left[\nu_{eN}(V) V^3 f_0 \right]. \qquad (1.87)$$

The process of f_0 relaxation has a diffusive character as can be seen from Eq. (1.87), the stationary distribution function is established at a $\nu_{eN}^{-1} m_N / m_e$ time scale. The stationary distribution function can be easily found by integrating Eq. (1.87) for $\partial f_0 / \partial t = 0$:

$$f_0 = A \exp\left[-\int_0^V \frac{3 m_e^3 V' \nu_{eN}^2(V')}{e^2 E^2 m_N} dV' \right]. \qquad (1.88)$$

This distribution function of electrons is determined by the velocity dependence of the electron-neutral collision frequency.

If the collision frequency is velocity independent (such situation is typical for *He* in the wide range of velocities) then according to Eq. (1.88)

$$f_0 = A\exp\left[-\frac{m_e V^2}{2T_{eff}}\right], \tag{1.89}$$

where

$$T_{eff} = \frac{e^2 E^2 m_N}{3m_e^2 \nu_{eN}^2}. \tag{1.90}$$

For the constant collision frequency distribution function is a Maxwellian one with effective temperature given by Eq. (1.90). Since in accordance with Eq. (1.87) the process of gaining energy has a diffusive character with the diffusion coefficient $D_V \sim e^2 E^2 /(m_e^2 \nu_{eN})$, then during time scale $\nu_{eN}^{-1} m_N / m_e$ the average square of the velocity reaches $V^2 \sim D_V \nu_{eN}^{-1} m_N / m_e$, and average energy is of the order of that given by Eq. (1.90).

As a second case let us consider constant mean free path λ_{eN}. The collision frequency is then proportional to velocity: $\nu_{eN}(V) = V/\lambda_{eN} \sim V$. According to Eq. (1.80)

$$f_0 = A\exp\left[-\frac{3m_e^3 V^4}{4e^2 E^2 m_N \lambda_{eN}^2}\right]. \tag{1.91}$$

This distribution function is known as Druyvesteyn distribution function. Here the velocity dependence is much stronger than for the case of Maxwellian distribution function.

1.6.3 Impact of electron-electron collisions

When the condition Eq. (1.69) is satisfied electron-electron collisions have strong impact on the distribution function of electrons. In the r.h.s. of equations for f_0 and $\vec{f}_1$ (1.75)-(1.76) now it is necessary to take into account electron-electron collision integral St_{ee}. This additional collision integral in the equation for f_0 is of the order of

$\nu_{ee} f_0$, while electron-neutral collision term $St_{eN}^0 \sim (m_e / m_N)\nu_{eN} f_0$. Hence, when condition Eq. (1.69) is fulfilled ,the main term in the r.h.s. of the equation for f_0 is $St_{ee}(f_0, f_0)$. For weak electric fields and gradual density and temperature gradients terms in the l.h.s. of Eq. (1.75) are small with respect to $\nu_{ee} f_0$, and the solution of the equation for f_0 is a function which turn to zero the r.h.s. $St_{ee}(f_0, f_0) = 0$, i.e. the Maxwellian distribution function f^M. On the other hand in the r.h.s. of the equation for $\vec{f}_1$ Eq. (1.76) the additional term from electron-electron collisions $St_{ee} \sim \nu_{ee}\left|\vec{f}_1\right|$ remains small with respect to $\left\langle St_{eN}^1 \right\rangle$. Therefore the equation for $\vec{f}_1$ remains the same as without electron-electron collisions, and $\vec{f}_1$ is expressed through f_0 according to Eq. (1.76), where f_0 is equal to the Maxwellian distribution function.

In the inhomogeneous plasma this approach is valid if the characteristic spatial scale of the inhomogeneity L exceeds the relaxation length of the distribution function λ_f. Relaxation of f_0 to f^M takes place at a time scale ν_{ee}^{-1}, hence the length λ_f corresponds to the random walk shift during ν_{ee}^{-1}. Along magnetic field the step of radom walk is mean free path λ_{eN}, and across the magnetic field the step is electron gyroradius ρ_{ce}. Hence

$$\lambda_{f\|} = \lambda_{eN}(\nu_{eN} / \nu_{ee})^{1/2} \;\; , \lambda_{f\perp} = \rho_{ce}(\nu_{eN} / \nu_{ee})^{1/2} .$$

1.6.4 General expression for $\vec{f}_1$

General solution of the Eq. (1.76) for $\vec{f}_1$ in the case of slow processes when time derivative can be neglected has the form

$$\vec{f}_1 = -\hat{M}\left(V \frac{\partial f_0}{\partial \vec{r}} - \frac{e\vec{E}}{m_e} \frac{\partial f_0}{\partial V}\right), \qquad (1.92)$$

where

$$\hat{M} = \begin{pmatrix} \frac{\nu_{eN}}{\omega_{ce}^2 + \nu_{eN}^2} & -\frac{\omega_{ce}}{\omega_{ce}^2 + \nu_{eN}^2} & 0 \\ \frac{\omega_{ce}}{\omega_{ce}^2 + \nu_{eN}^2} & \frac{\nu_{eN}}{\omega_{ce}^2 + \nu_{eN}^2} & 0 \\ 0 & 0 & 1/\nu_{eN} \end{pmatrix}. \tag{1.93}$$

Here $\omega_{ce} = eB/m_e$. The function f_0 is close to the Maxwellian one provided the condition Eq. (1.69) is satisfied.

1.7. Transport coefficients for electrons in slightly ionized plasma

Particle and heat fluxes in slightly ionized plasma with the distribution function close to the Maxwellian one could be obtained by direct integrating of the distribution function. Particle flux of electrons is

$$\vec{\Gamma}_e = n\vec{u}_e = \int \vec{V}(f_0 + \frac{\vec{f}_1\vec{V}}{V})d\vec{V} = \frac{4\pi}{3}\int_0^\infty \vec{f}_1 V^3 dV \ . \tag{1.94}$$

Since the function $\vec{f}_1$ is linear with respect to density, temperature and potential gradients, the particle flux can be written in the form

$$\vec{\Gamma}_e = -\hat{D}_e \nabla n - \hat{b}_e \vec{E} n - \hat{D}_e^T n \nabla T_e / T_e , \tag{1.95}$$

where $\hat{D}_e$ is a diffusion tensor, $\hat{D}_e^T$ is a tensor of thermal diffusion, $\hat{b}_e$ is a mobility tensor. Inserting $\vec{f}_1$ as a function of f_0 Eq. (1.92) into Eq. (1.92), for f_0 equal to the Maxwellian distribution function, one obtains

$$\hat{b}_e = \frac{4\pi e}{3nT_e}\int_0^\infty V^4 \hat{M}(V) f_0 dV,$$
$$\hat{D}_e = \frac{4\pi}{3n}\int_0^\infty V^4 \hat{M}(V) f_0 dV, \tag{1.96}$$
$$\hat{D}_e^T = \frac{4\pi}{3n}\int_0^\infty V^4 \hat{M}(V)(\frac{m_e V^2}{2T_e} - \frac{3}{2}) f_0 dV = T_e \frac{d\hat{D}_e}{dT_e}.$$

These tensors can be calculated provided the velocity dependence of the electron-neutral collison frequency $\nu_{eN}(V)$ is known. According to Eq. (1.96) the mobility and diffusion tensors are linked by Einstein relation

$$\hat{D}_e = \frac{T_e}{e}\hat{b}_e \, . \tag{1.97}$$

The heat flux is defined as

$$\vec{q}_e = \frac{m_e}{2}\int (\vec{V} - \vec{u}_e)^2 (\vec{V} - \vec{u}_e)(f_0 + \frac{\vec{f}_1 \vec{V}}{V}) d\vec{V} \, . \tag{1.98}$$

Assuming the directed velocity to be small with respect to the chaotic one, we have

$$\vec{q}_e = \frac{2\pi m_e}{3}\int_0^\infty V^5 \vec{f}_1 dV - \frac{5}{2} n T_e \vec{u}_e \, . \tag{1.99}$$

The electron heat flux can be written in the form

$$\vec{q}_e = -\hat{\kappa}_e \nabla T_e + \hat{c}^T n T_e \vec{u}_e \, , \tag{1.100}$$

where $\hat{\kappa}_e$ is the thermal conductivity tensor, and $\hat{c}^T$ is a dimensionless tensor. Here the first term corresponds to the heat flux caused by the temperature gradient, while the second term represents the heat flux caused by the directed velocity. Also a heat conductivity tensor $\hat{\chi}_e$ is often used which is connected with $\hat{\kappa}_e$ by the relation

$$\hat{\kappa}_e = \frac{3}{2} n \hat{\chi}_e \,. \qquad (1.101)$$

Using Eq. (1.92) tensors $\hat{\kappa}_e$ ($\hat{\chi}_e$) and $\hat{c}^T$ could be calculated for known dependence $\nu_{eN}(V)$. They are linked to tensors of diffusion and thermal diffusion by the relation

$$\begin{aligned} \hat{\kappa}_e &= \frac{3}{2} n \hat{\chi}_e = n\left(\frac{3}{2}\hat{D}_e + \hat{D}_e^T - T_e \frac{d\hat{D}_e^T}{dT_e} + \hat{D}_e \hat{D}_e^{-1} \hat{D}_e^T\right), \\ \hat{c}^T &= \hat{D}_e^T \hat{D}_e^{-1} - \hat{I} \,, \end{aligned} \qquad (1.102)$$

where $\hat{I}$ is the unity tensor. Diffusion and thermal diffusion coefficients are the same order quantities.

When the collision frequency ν_{eN} is the velocity independent quantity, expressions for diffusion, thermal diffusion and mobility coefficients have a simple form. From (1.96), (1.02) it follows

$$\hat{D}_e = \hat{D}_e^T = \frac{T_e}{e}\hat{b}_e = \frac{3}{5}\hat{\chi}_e$$
$$= \frac{T_e}{m_e \nu_{eN}} \begin{pmatrix} \frac{1}{\omega_{ce}^2/\nu_{eN}^2 + 1} & -\frac{\omega_{ce}/\nu_{eN}}{\omega_{ce}^2/\nu_{eN}^2 + 1} & 0 \\ \frac{\omega_{ce}/\nu_{eN}}{\omega_{ce}^2/\nu_{eN}^2 + 1} & \frac{1}{\omega_{ce}^2/\nu_{eN}^2 + 1} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (1.103)$$

For the weak dependence $\nu_{eN}(V)$ or when the detailed description of the transport coefficients is not required a so-called elementary theory is used (elementary theory corresponds to the quasihydrodynamic approximation, see the next Chapter). In this approximation Eq. (1.95) is used with the average transport collision frequency $\nu_{eN}(T) = \langle \nu_{eN}(V) \rangle = \int \nu_{eN}(V) f_0 d\vec{V}/n.$ For the strong magnetic field $\omega_{ce} >> \nu_{eN}(T)$ Eq. (1.95) remains exact for arbitrary dependence остается $\nu_{eN}(V)$, which could easily seen from Eq. (1.96) and definition of $\nu_{eN}(T)$. More accurate approximations for arbitrary dependence $\nu_{eN}(V)$ without magnetic can be found in [7].

For ions simplified expressions of the elementary theory are normally used.

$$\hat{D}_i = \hat{D}_i^T = \frac{T_e}{e}\hat{b}_i = \frac{3}{5}\hat{\chi}_i$$
$$= \frac{T_i}{m_i \nu_{iN}} \begin{pmatrix} \dfrac{1}{\omega_{ci}^2/\nu_{iN}^2+1} & \dfrac{\omega_{ci}/\nu_{iN}}{\omega_{ci}^2/\nu_{iN}^2+1} & 0 \\ -\dfrac{\omega_{ci}/\nu_{iN}}{\omega_{ci}^2/\nu_{iN}^2+1} & \dfrac{1}{\omega_{ci}^2/\nu_{iN}^2+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1.104)$$

For the important case of the ions in their own gas the charge exchange collision frequency is

$$\nu_{iN} = \frac{32\sigma_{ex}}{3\pi^{1/2}}\left(\frac{T_i}{m_i}\right)^{1/2} n_N. \quad (1.105)$$

1.8. Drift kinetic equation in a stationary electric and magnetic fields

For fully ionized plasma immersed in a strong magnetic field, it is often useful to switch to the description of the plasma by following the guiding centers of the Larmor orbits. It can be done when ion Larmor radius is small with respect to the typical scale of the problem. In the guiding center approximation of the particle motion, averaging on the cyclotron motion is performed, and only the guiding center motion is considered. In this approximation along magnetic field the guiding center moves with the parallel velocity $V_{||}$, and across the magnetic field – with the drift velocity. Expression for the drift velocity is obtained by averaging of the particle motion equation over the small Larmor circle with the velocity $V_{\perp}$. Result of such averaging [8] is

$$\dot{\vec{R}} = V_{\|}\vec{h} + \frac{1}{B^2}[\vec{E}\times\vec{B}] + \frac{mV_{\perp}^2}{2ZeB}[\vec{h}\times\frac{\nabla B}{B}] + \frac{mV_{\|}^2}{ZeB}[\vec{h}\times(\vec{h}\nabla\vec{h})] \quad , \tag{1.106}$$

where $\vec{h} = \vec{B}/B$. The second term in the r.h.s. is $\vec{E}\times\vec{B}$ drift, the third term is known as gradB drift and the last term is the curvature drift. The expression for the $\vec{E}\times\vec{B}$ drift is valid only for the weak electric field so that drift velocity is smaller than the perpendicular velocity: $E/B << V_{\perp}$.

The variation of the parallel and perpendicular velocity are given by [8]:

$$\begin{gathered}
\dot{V}_{\|} = \frac{e\vec{E}\vec{h}}{m} + \frac{V_{\perp}^2}{2}div\vec{h} + \frac{cV_{\|}}{B}(\vec{E}[\vec{h}\times(\vec{h}\nabla)\vec{h}]) - \frac{cmV_{\perp}^2V_{\|}}{2ZeB}(\frac{\nabla B}{B}[\vec{h}\times(\vec{h}\nabla)\vec{h}]) \\
-\frac{cmV_{\perp}^2V_{\|}}{2ZeB}\vec{h}rot(\vec{h}\nabla)\vec{h} \quad , \\
\dot{V}_{\perp} = -\frac{V_{\|}V_{\perp}}{2}div\vec{h} - \frac{cV_{\perp}}{2B}(\vec{h}rot\vec{E}) + \frac{cV_{\perp}}{2B}(\vec{E}[\vec{h}\times\frac{\nabla B}{B}]) + \frac{cV_{\perp}}{2B}(\vec{E}\vec{h})(\vec{h}rot\vec{h}) \\
+\frac{cmV_{\|}^2V_{\perp}}{2ZeB}(\frac{\nabla B}{B}[\vec{h}\times(\vec{h}\nabla)\vec{h}]) + \frac{cmV_{\|}^2V_{\perp}}{2ZeB}\vec{h}rot(\vec{h}\nabla)\vec{h} \quad .
\end{gathered} \tag{1.107}$$

The terms with electric field represents the work of electric field during the particle motion along the field and work of inductive electric field for time-dependent magnetic field at a particle rotating over its Larmor radius. The terms connected with magnetic field inhomogeneity represents the change of the parallel velocity while the magnetic moment and the full energy remain constant.

It is convenient to introduce five variables: $\vec{R}$, μ, $V_{\|}$, where

$$\mu = \frac{mV_{\perp}^2}{2B} \tag{1.108}$$

is a magnetic moment of a rotating particle and is an adiabatic invariant of motion. Then the equation for the parallel velocity can be reduced to

$$\dot{V}_{\|} = -(\frac{\mu}{m}\nabla B + \frac{Ze}{m}\nabla\varphi)[\vec{b} + \frac{V_{\|}}{\omega_c}\nabla\times\vec{b}]. \qquad (1.109)$$

This expression can be derived by using $\mu B + mV_{\|}^2/2 + Ze\varphi = E = const$:

$$\dot{V}_{\|} = \frac{d}{dt}[\frac{2}{m}\sqrt{E-\mu B - Ze\varphi}] = -(\frac{\mu}{m}\frac{dB}{dt} + Ze\frac{d\varphi}{dt})(E--\mu B - e\varphi)^{-1/2}. \qquad (1.110)$$

Here $\frac{dB}{dt} = \dot{\vec{R}}\nabla B$ and $\frac{d\varphi}{dt} = \dot{\vec{R}}\nabla\varphi$. With account of

$$\left[\frac{1}{B^2}[\vec{E}\times\vec{B}] + \frac{mV_{\perp}^2}{2ZeB}[\vec{h}\times\frac{\nabla B}{B}] + \frac{mV_{\|}^2}{ZeB}[\vec{h}\times(\vec{h}\nabla\vec{h})]\right](\mu\nabla B - e\nabla\varphi) = 0,$$

after substituting Eq.(1.106) for $\dot{\vec{R}}$ into Eq. (1.110) we obtain Eq. (1.109).

The drift kinetic equation could be derived by averaging the Boltzmann kinetic equation over gyromotion, see below. Alternatively it is possible to introduce a distribution function of the guiding centers in the corresponding 5 dimensional space. This distribution function F depends on $\vec{R}, V_{\|}$ and μ. The phase space volume element is defined to give number of particles guiding centers in space unit

$$dV = \frac{2\pi}{m}Bd\vec{R}dV_{\|}d\mu, \quad dN = FdV. \qquad (1.111)$$

In the absence of collisions the continuity equation for the guiding center fluid in the phase space leads to the Vlasov kinetic equation for guiding centers distribution function

$$\frac{\partial F}{\partial t} + \dot{\vec{R}}\frac{\partial F}{\partial \vec{R}} + \dot{V}_{\|}\frac{\partial F}{\partial V_{\|}} = 0. \qquad (1.112)$$

This is collisionless drift kinetic equation. It could be extended also to the collisional case by using collision integral when it is possible to neglect the difference between the particle position and guiding center coordinate. Note also that calculation of self-consistent magnetic field requires knowledge of real plasma currents including e.g. diamagnetic currents which differs from guiding center currents, so guiding center kinetic equation cannot be used for this purpose.

1.9. Gyrokinetic equation

Gyrokinetic equation is more general and can treat time dependent cases in a strong magnetic field for low frequences. It can be used even for $k_{\perp}\rho_{ci} \sim 1$, where $k_{\perp}$ is a wave vector across a magnetic field, so waves and turbulence with the scales of the order of ion Larmor radius can be considered. It is also assumed that

$$\begin{aligned} &\frac{\omega}{\omega_{ci}} \sim \frac{\rho_{ci}}{L} \sim k_{\|}\rho_{ci} \sim \frac{e\varphi}{T_e} \sim \frac{\delta B}{B} \sim O(\varepsilon), \\ &k_{\perp}\rho_{ci} \sim O(1). \end{aligned} \tag{1.113}$$

Let us start with Vlasov kinetic equation (1.2)

$$\frac{\partial f_\alpha}{\partial t} + \vec{V}\frac{\partial f_\alpha}{\partial \vec{r}} + \frac{Z_\alpha e}{m_\alpha}\left(\vec{E} + [\vec{V}\times\vec{B}]\right)\frac{\partial f_\alpha}{\partial \vec{V}} = 0\,. \tag{1.114}$$

Now we introduce gyrokinetic variables

$$f_\alpha(\vec{r},\vec{V},t) \to f_\alpha(\vec{R},V_{\|},\mu,\theta,t)\,, \tag{1.115}$$

where relations between initial and gyrokinetic variables are

$$\begin{aligned} &\vec{r} = \vec{R} + \vec{\rho},\ \vec{\rho} = [\vec{h}\times\vec{V}_{\perp}/\omega_{c\alpha}]\,,\quad \vec{V}_{\perp} = V_{\perp}(\cos\theta\vec{e}_1 + \sin\theta\vec{e}_2), \\ &\vec{V}_{\perp} = V_{\|}\vec{h} + \vec{V}_{\perp}. \end{aligned} \tag{1.116}$$

Here θ is the gyrophase angle. Electric field is assumed to be electrostatic $\vec{E} = -\nabla\varphi$.

Let us express derivatives $\frac{\partial}{\partial \vec{r}}$ and $\frac{\partial}{\partial \vec{V}}$ through the new gyrokinetic variables.

$$\frac{\partial}{\partial \vec{r}} = \frac{\partial \vec{R}}{\partial \vec{r}} \frac{\partial}{\partial \vec{R}} + \frac{\partial \mu}{\partial \vec{r}} \frac{\partial}{\partial \mu} + \frac{\partial V_{\|}}{\partial \vec{r}} \frac{\partial}{\partial V_{\|}} + \frac{\partial \theta}{\partial \vec{r}} \frac{\partial}{\partial \theta},$$

$$\frac{\partial}{\partial \vec{V}} = \frac{\partial \vec{R}}{\partial \vec{V}} \frac{\partial}{\partial \vec{R}} + \frac{\partial \mu}{\partial \vec{V}} \frac{\partial}{\partial \mu} + \frac{\partial V_{\|}}{\partial \vec{V}} \frac{\partial}{\partial V_{\|}} + \frac{\partial \theta}{\partial \vec{V}} \frac{\partial}{\partial \theta}. \tag{1.117}$$

For simplicity let us consider uniform magnetic field. Since derivatives $\frac{\partial}{\partial \vec{r}}$ are taken at $\vec{V} = const$ only the first term on the r.h.s. is nonzero and since $\vec{r} = \vec{R} + \vec{\rho}$, and $\frac{\partial \vec{\rho}}{\partial \vec{R}} = 0$, we have $\frac{\partial}{\partial \vec{r}} = \frac{\partial}{\partial \vec{R}}$. For derivatives over velocity

$$\frac{\partial V_{\|}}{\partial \vec{V}} = \frac{\partial (\vec{V}\vec{h})}{\partial \vec{V}} = \vec{h}, \quad \frac{\partial \mu}{\partial \vec{V}} = \frac{\partial (\vec{V}^2 - V_{\|}^2) m_\alpha / 2B}{\partial \vec{V}} = \frac{m_\alpha \vec{V}_\perp}{B},$$

$$\frac{\partial \vec{R}}{\partial \vec{V}} = \frac{\partial (\vec{r} - [\vec{h} \times \vec{V} / \omega_{c\alpha}])}{\partial \vec{V}} = -\frac{\partial [\vec{h} \times \vec{V} / \omega_{c\alpha}]}{\partial \vec{V}}.$$

Hence

$$\frac{\partial}{\partial \vec{V}} = \vec{h} \frac{\partial}{\partial V_{\|}} + \frac{m_\alpha \vec{V}_\perp}{B} \frac{\partial}{\partial \mu} - \frac{1}{V_\perp} [\vec{h} \times \vec{e}_\perp] \frac{\partial}{\partial \theta} - \frac{\partial [\vec{h} \times \vec{V} / \omega_{c\alpha}]}{\partial \vec{V}} \frac{\partial}{\partial \vec{R}}$$

Different terms in the kinetic equation are

$$\vec{V} \frac{\partial}{\partial \vec{r}} = V_{\|} \vec{h} \frac{\partial}{\partial \vec{R}} + \vec{V}_\perp \frac{\partial}{\partial \vec{R}},$$

$$\frac{Z_\alpha e}{m_\alpha} \vec{E} \frac{\partial}{\partial \vec{V}} = \frac{Z_\alpha e}{m_\alpha} (E_{\|} \frac{\partial}{\partial V_{\|}} + \frac{\vec{E}\vec{V}_\perp m_\alpha}{B} \frac{\partial}{\partial \mu} - \frac{\vec{E}[\vec{h} \times \vec{V}_\perp]}{V_\perp^2} \frac{\partial}{\partial \theta}) + \frac{\vec{E} \times \vec{B}}{B^2} \frac{\partial}{\partial \vec{R}},$$

$$\frac{Z_\alpha e}{m_\alpha} [\vec{V} \times \vec{B}] \frac{\partial}{\partial \vec{V}} = -\omega_{c\alpha} \frac{[\vec{V} \times \vec{B}][\vec{B} \times \vec{V}]}{B^2 V_\perp^2} \frac{\partial}{\partial \theta} + [[\vec{V} \times \vec{B}] \times \vec{B}] \frac{\partial}{\partial \vec{R}}$$

$$= \omega_{c\alpha} \frac{\partial}{\partial \theta} - \vec{V}_\perp \frac{\partial}{\partial \vec{R}}.$$

Electrostatic potential and electric field should also be expressed in the new variables. Since $\varphi = \varphi(\vec{R} + \vec{\rho})$, we have

$$\frac{\partial \varphi}{\partial \theta} = \frac{\partial \varphi}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial \theta} = \frac{\partial \varphi}{\partial \vec{r}} \frac{\partial \vec{\rho}}{\partial \theta} = \frac{\vec{V}_\perp}{\omega_{c\alpha}} \frac{\partial \varphi}{\partial \vec{r}} = -\frac{\vec{E}\vec{V}_\perp}{\omega_{c\alpha}},$$

and

$$Z_\alpha e \frac{\vec{E}\vec{V}_\perp}{B} \frac{\partial}{\partial \mu} = -\left(\frac{Z_\alpha e}{m_\alpha}\right)^2 m_\alpha \frac{\partial \varphi}{\partial \theta} \frac{\partial}{\partial \mu}.$$

Now we can rewrite Vlasov equation in the new variables

$$\begin{aligned} &\frac{\partial f_\alpha}{\partial t} + (V_\| \vec{h} + \frac{\vec{E}\times\vec{B}}{B^2}) \frac{\partial f_\alpha}{\partial \vec{R}} + \frac{Z_\alpha e}{m_\alpha} E_\| \frac{\partial f_\alpha}{\partial V_\|} + \omega_{c\alpha} \frac{\partial f_\alpha}{\partial \theta} - \frac{Z_\alpha e \omega_{c\alpha}}{B} \frac{\partial \varphi}{\partial \theta} \frac{\partial f_\alpha}{\partial \mu} \\ &- \omega_{c\alpha} \frac{\vec{E}\times\vec{B}}{B^2} \frac{\vec{V}_\perp}{V_\perp^2} \frac{\partial f_\alpha}{\partial \theta} = 0. \end{aligned} \quad (1.118)$$

The distribution function can be sought in the form $f_\alpha = f_\alpha^0 + f_\alpha^1$ with expansion parameter ε Eq. (1.113). Keeping the largest term in the zero order approximation one obtains $\omega_{c\alpha} \frac{\partial f_\alpha^0}{\partial \theta} = 0$, i.e. f_α^0 is independent on gyrophase angle. Therefore $f_\alpha = \langle f_\alpha \rangle + f_\alpha^1$, where $\langle f_\alpha \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_\alpha d\theta$.

In the first order approximation

$$\frac{\partial f_\alpha^0}{\partial t} + (V_\| \vec{h} + \frac{\vec{E}\times\vec{B}}{B^2}) \frac{\partial f_\alpha^0}{\partial \vec{R}} + \frac{Z_\alpha e}{m_\alpha} E_\| \frac{\partial f_\alpha^0}{\partial V_\|} + \omega_{c\alpha} \frac{\partial f_\alpha^1}{\partial \theta} - \frac{Z_\alpha e \omega_{c\alpha}}{B} \frac{\partial \varphi}{\partial \theta} \frac{\partial f_\alpha^1}{\partial \mu} = 0. \quad (1.119)$$

The last small in Eq. (1.118) is neglected. After averaging of Eq. (1.119) over gyrophase angle, we have

$$\frac{\partial\langle f_\alpha\rangle}{\partial t}+(V_\parallel\vec{h}+\frac{\langle\vec{E}\rangle\times\vec{B}}{B^2})\frac{\partial\langle f_\alpha\rangle}{\partial\vec{R}}+\frac{Z_\alpha e}{m_\alpha}\langle E_\parallel\rangle\frac{\partial\langle f_\alpha\rangle}{\partial V_\parallel}=0. \qquad (1.200)$$

This is gyrokinetic equation in a uniform magnetic field in the electrostatic approximation. In this equation electric field is averaged over gyrophase angle $\langle\vec{E}\rangle=\frac{1}{2\pi}\int_0^{2\pi}\vec{E}(\vec{r})d\theta$. In the general case of nonuniform magnetic field drift kinetic equation can be derived in a similar way and is given by

$$\frac{\partial\langle f_\alpha\rangle}{\partial t}+\dot{\vec{R}}\frac{\partial\langle f_\alpha\rangle}{\partial\vec{R}}+\dot{V}_\parallel\frac{\partial\langle f_\alpha\rangle}{\partial V_\parallel}=0\,, \qquad (1.201)$$

where

$$\dot{\vec{R}}=V_\parallel\vec{h}+\frac{1}{B^2}[\langle\vec{E}\rangle\times\vec{B}]+\frac{m_\alpha\mu}{Z_\alpha e}[\vec{h}\times\frac{\nabla B}{B}]+\frac{m_\alpha V_\parallel^2}{Z_\alpha eB}[\vec{h}\times(\vec{h}\nabla\vec{h})]\quad , \qquad (1.202)$$

$$\dot{V}_\parallel=(\frac{Z_\alpha e}{m_\alpha}\langle\vec{E}\rangle-\frac{\mu}{m_\alpha}\nabla B)(\vec{b}+\frac{V_\parallel}{\omega_{c\alpha}}[\nabla\times\vec{b}])\,. \qquad (1.203)$$

Now we have to calculate gyrophase average of the potential and electric field.

$$\varphi(\vec{r})=\varphi(\vec{R}+\vec{\rho})=\sum_{\vec{k}}\varphi_{\vec{k}}\exp(i\vec{k}\vec{R}+i\vec{k}\vec{\rho})=\sum_{\vec{k}}\varphi_{\vec{k}}\exp(i\vec{k}\vec{R}+ik_\perp\rho\sin\theta)\,. \quad (1.204)$$

We use Fourier-Bessel expansion

$$\exp(ik_\perp\rho\sin\theta)=\sum_n J_n(k_\perp\rho)\exp(in\theta)\,. \qquad (1.205)$$

The averaged value is

$$\left\langle \exp(ik_\perp \rho \sin\theta) \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \sum_n J_n(k_\perp \rho) \exp(in\theta) d\theta = J_0(k_\perp \rho) . \qquad (1.206)$$

Finally

$$\left\langle \varphi \right\rangle = \sum_{\vec{k}} \varphi_{\vec{k}} J_0(k_\perp \rho) \exp(i\vec{k}\vec{R}) . \qquad (1.207)$$

With this expression one can solve gyrokinetic equation (1.201) with account of finite Larmour radius effects.

Additional problem in gyrokinetic approach arises from the fact that particle density and current calculated using gyroaveraged distribution function is different from the real particle density and current due to guiding center approach. For example, to calculate the real density, which one can use in the Poisson's equation, one has to calculate correction f_α^1 to the averaged distribution function. This can be done using Eq. (119). The sum of the first four terms on the l.h.s. of Eq.(1.119) is small, with the average fields it is zero according to Eq. (1.200), so that we can neglect it. Integrating the remaining terms we obtain

$$f_\alpha^1 = \frac{Z_\alpha e}{B} (\varphi - \left\langle \varphi \right\rangle) \frac{\partial \left\langle f_\alpha \right\rangle}{\partial \mu} . \qquad (1.208)$$

The real density is given by

$$n_\alpha = 2\pi \int d\vec{R} dV_\parallel d\mu \frac{B}{m_\alpha} \left\langle f_\alpha \right\rangle \delta(\vec{R} + \vec{\rho} - \vec{r}) + \int d\vec{R} dV_\parallel d\mu d\theta \frac{B}{m_\alpha} f_\alpha^1 (\vec{R} + \vec{\rho} - \vec{r}) . (1.209)$$

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