Министерство науки и высшего образования Российской Федерации

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ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ ПЕТРА ВЕЛИКОГО

А.В. Костарев Т.А. Костарева

ТЕОРЕТИЧЕСКАЯ МЕХАНИКА

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Abstract

The derivation of Conditions of equilibrium is based not on unnecessary axioms of statics, but on Newton's axiomatics and the theorem on the equivalence of systems of forces applied to a rigid body.

The conditions for constructing statically determinate constraints are formulated.

A set of interactive excel problems has been created that allow the student to independently prepare for the test, and the teacher to conduct tests, the results of which are determined automatically by the program.

The concept of force and coefficient of resistance of a soft road to the movement of the center of mass of a hard wheel is introduced. The phenomenon of slipping of such a wheel is explained.

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Vector algebra of forces

Equilibrium conditions for a rigid body

Vector algebra of forces

Subject and models of mechanics.

Classical or Newtonian mechanics is a part of physics that explores the fundamental laws of mechanical interaction and movement of solids.

History of mechanics development totals a Millenium. Practically people became interested in mechanics and intuitively used it's laws when trying to accurately throw a stone at the hunt. Since then, the mechanic has made a long way. Such thinkers of antiquity, as Archimedes (3 century b.c.), Leonardo da Vinci (15 c), Galileo and Descartes (16B) were able to generalize the experience of the first explorers, laying the foundations of classical mechanics. Mechanics has acquired a modern form thanks to the geniuses of Huygens and Newton (17th century), Euler and Lagrange (18V).

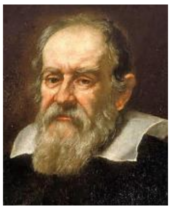
Archimedes (287 BC-212 BC) was a Greek mathematician, physicist, and engineer from Syracuse. Famous for many mechanical designs. Lever was known before Archimedes, but only Archimedes gave his complete theory and successfully applied it in practice. Invented by him Archimedes ' screw (Auger) for the water lift is still used in Egypt. He said: "give me a fulcrum and I will move the Earth!" Video

Archimedes-screw.gif

Archimedes.flv

Leonardo da Vinci (1452-1519) was interested in the problems of flight. List of inventions, both real and ascribed to him: parachute-1483, bicycle, Tank, Light portable bridges for the army, spotlight, Catapult, robot, telescope. He said: "experience is the true teacher. Video

Leonardo Da Vinci.flv



Galileo Galilei (1564-1642) was the first who used the telescope to observe celestial bodies. Galileo is a founder of experimental physics. His experiments he convincingly refuted Aristotle's metaphysics and laid the foundations of classical mechanics. Was known as an active supporter of the heliocentric system of the world, led Galileo to a serious conflict with the Catholic Church. He said, "and yet it moves!" Video

René Dekárt (1596-1650) was a French mathematician, philosopher, physicist, founder of analytical geometry and modern algebraic symbols, the author of the method of radical doubt in philosophy, mehanicizm in physics. He said: «I think, therefore I am. " Video

Sir Isaac Newton (1642-1727) was an English physicist, mathematician, and astronomer, one of the founders of classical physics. Author of fundamental work philosophiæ Naturalis, in which he set out the law of universal gravitation and the three laws of mechanics, which became the basis of classical mechanics. He has developed differential and integral calculus, color theory and many other mathematical and physical theories. He said: "if I have seen further than others, it is by standing on the shoulders of giants". Video

Leonárd Éjler (1707-1783) was a Swiss, a German and a Russian mathematician who made significant contributions to the development of mathematics, physics, astronomy and applied sciences. He suggested a modern approach to the dynamics of a rigid body. He said: "all that we now know from physics, it was first the guesswork, and if never would be allowed guesses, even erroneous, then we would not have any truth." Video

Joseph Louis Lagrange (1736-1813), french mathematician and engineer of Italian origin. The author of the classic treatise of analytical mechanics, which established the fundamental principle of motion "and concluded the mathematical mechanics. He has made great contributions to the development of analysis, number theory, probability theory and numerical methods, developed variations calculus.

Course of mechanics can be divided into three main parts: STATICS, KINEMATICS AND DYNAMICS. In STATIC we explore the conditions of a rigid body rest, KINEMATICS is the language of description of motion of the body, and DYNAMICS, being the mechanics itself, explore the laws of motion of a rigid body under the action of forces. Since the rest is a special case of motion, it would be easier to get statics equations from the laws of motion of the body. However, since statics equations are necessary for you now to explore other mechanical disciplines, we start with the static.

Models in mechanics

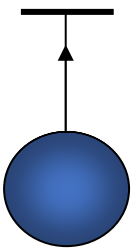
As any exact science mechanics deals not with real, endlessly complex physical objects, but with their models, reflecting only their main properties under given conditions.

The object of classical mechanics is a system of interacting material (with weight) particle, known as a mechanical system.

A special case of mechanical system is a solid model of a real deformable body, the distance between any two points of which does not change with time. Since the strains of the most structures are negligible, the rigid body model is quite justified. Moreover, it simplifies significantly the study of movement and rest of the body, and its results are applicable to the deformable body.

Force. Module, projection and component of force.

All the bodies are in interaction. For example, a hanging ball interacts with the thread. The action of the thread on the ball has a point of application, a line of action (vertical), direction (up) and value (modulus).



Values, characterized by a line of action, direction and a module are known in mathematics as vectors. Therefore, the measure of the interaction is vector **F** which is called ***force***.

When solving a problem the numbers rather than vectors are used. That's why we use scalar representation of the vector by, for example, its three projections on the Cartesian axis x, y, z. They form a vector-column

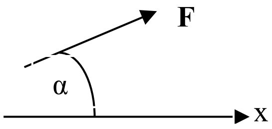


Рис. 1

Remind that the projection of vector onto the x-axis is a scalar value, which is equal to

$$F_x = FCos\alpha$$

The projection sign is that of the cosine of the angle between the directions of the forces and the axis. If the angle is acute, the projection is positive if it's dumb- the projection is negative. To put it simply, the projection is positive if the direction of the force is the same with the direction of the axis with the accuracy of $\pi/2$.

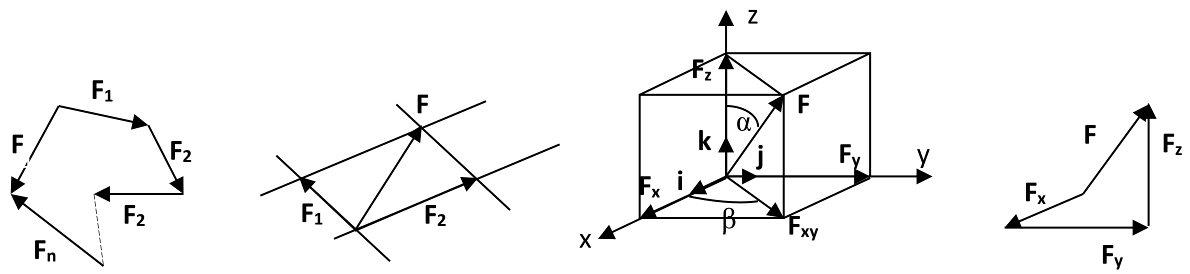
It is important to remember that the projection of force **perpendicular** to the axis IS EQUAL TO ZERO.

The force vector, arbitrarily directed in space, is characterized by three projections on the Cartesian coordinate axes, forming a vector column.

$$F = \begin{pmatrix} Fx \\ Fy \\ Fz \end{pmatrix}$$

In writing vector $\vec{F}$ is over lined, in the books it is printed in bold **F**. Vector module is designed by the same letter, but without the line over the letter in writing or not bold in print: F. Force module is measured in newtons (SI system) or kilograms kg (technical units). Formal mathematical operations on vectors are presented in the annex. Next, let's look at the most important for us operations with vectors. It is known that in math vectors sum up according to the rule of the parallelogram (Figure 2 b).

Procedures of vectors addition are shown on the slide



a) b) Рис.1 c) d)

From the rule of parallelogram follows the rule of decomposition of a vector into components along two directions. For this purpose we need to draw by the ends of the vector F the lines parallel to the given directions (Pic1 (b)). ***Component*** of a vector is any of the components of the vector sum

$$\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2} + \cdots + \boldsymbol{F_n}$$

On the Figure 2 a components of the vector **F** form a vector polygon in which the beginning of the follow-up force coincides with the end of the previous vector. The vector **F** encloses the vector polygon.

Represent the force **F** with its projections on the Cartesian axis with the unit vectors **i, j, k** (Figure 1 c)

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} \quad (3)$$

Let's call the components in this expression ***components-projections*** of the vector F (fig. 1 c, d).

$$\boldsymbol{F} = \boldsymbol{F_{xy}} + \boldsymbol{F_z}; \quad \boldsymbol{F_{xy}} = \boldsymbol{F_x} + \boldsymbol{F_y}; \quad \boldsymbol{F} = \boldsymbol{F_x} + \boldsymbol{F_y} + \boldsymbol{F_z}$$

$$\boldsymbol{F_x} = F_x\boldsymbol{i}; \qquad \boldsymbol{F_y} = F_y\boldsymbol{j}; \qquad \boldsymbol{F_z} = F_z\boldsymbol{k};$$

$$F_{xy} = FSin\alpha; \quad F_x = F_{xy}Cos\beta = FSin\alpha Cos\beta; \quad F_y = F_{xy}Sin\beta = FSin\alpha Sin\beta; \quad F_z = FCos\alpha$$

The projections of the vector determine its module by the Pythagorean theorem:

$$F^2 = F_x^2 + F_y^2 + F_z^2$$

Thus, we will distinguish the following values for a vector:

vector F in print, $\overline{F}$ or $\vec{F}$ in writing

component projection along the x axis: $\boldsymbol{F_x}$, in print, $\overrightarrow{F_x}$, or $\overline{F_x}$ in writing

component projection onto the xy plane: $\boldsymbol{F_{xy}}$, in print, $\overrightarrow{F_{xy}}$, or $\overline{F_{xy}}$ in writing ***module*** of the vector: F

projection of the vector: F_x

In the written works the vector designation with letter F without arrow is unacceptable!

System of forces.

Main vector of System of forces.

The system of forces $\{\boldsymbol{F}\} = \{\boldsymbol{F_1}\ \boldsymbol{F_2} \ldots \boldsymbol{F_n}\ \}$ is the set of forces applied to the points of the mechanical system. The ***main vector*** is the vector sum of all forces of the system:

$$\boldsymbol{V} = \sum_{k=1}^{n} \boldsymbol{F_k} \qquad (6)$$

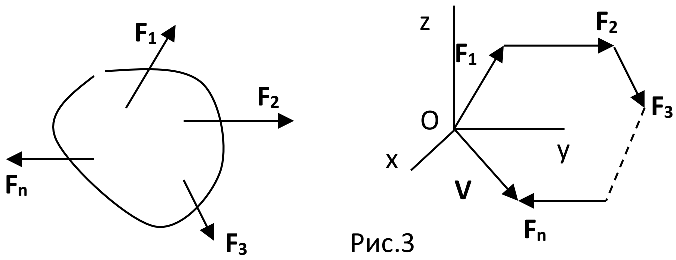


Рис.3

Main vector **V** can be built in any centre O, with the help of the vector polygon (Figure 3).

For a spatial system of forces it is practically difficult to build the polygon. It's easier to find the main vector analytically. Projecting formula (6) on the coordinate axes, we define the projections of the main vector, its module and guide cosines:

$$V_x = \sum_{k=1}^{n} F_{kx}; \qquad V_y = \sum_{k=1}^{n} F_{ky}; \quad V_z = \sum_{k=1}^{n} F_{kz} \qquad (7)$$

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2}; \quad Cos(\boldsymbol{V};x) = \frac{V_x}{V}; \; Cos(\boldsymbol{V};y) = \frac{V_y}{V}; \; Cos(\boldsymbol{V};z) = \frac{V_z}{V}$$

Moment of force about a point.

Theorems on the moment

The concept of moment arises when the solid is considered. Experience shows that if we fix a point of the body, the force **F** applied at another point can rotate body around the fixed point. The ability of the force to rotate the body is characterized by moment. Let us denote by **r** the radius-vector of the point of application of force relative to the center O. **Moment of force F about the center O** is the vector **m₀ (F),** equal to the vector product of the radius-vector of the point of application of force to the force vector (fig. 4)

$$\boldsymbol{m_o}(\boldsymbol{F}) = \boldsymbol{r} \times \boldsymbol{F} \qquad (8)$$

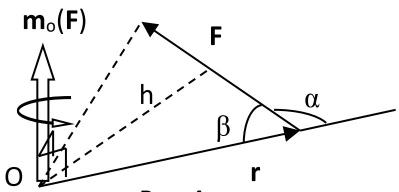
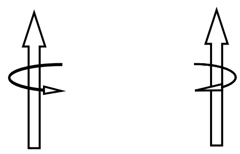


Рис.4

Direction of the vector product is conventionally, and depends on the orientation of space. The orientation of space- is the rule of left or right screw, which we accept to correspond the direction of vector perpendicular to the plane of the arc of physical sense of rotation (fig. 5). Vector which direction depends on the orientation of space is called ***axial***.


Правый винт Левый винт
Рис.5

We will consider only the right oriented space, i.e. the direction of the cross product will be determined by the right screw rule: from the end of the moment $\mathbf{m_o}$ it seems that the force intend to turn the body counterclockwise.

The moment module is equal to the product of the force module F on the arm h- the length of the perpendicular omitted from the center O on the line of action of the force

$$m_o(F) = FrSin\alpha = FrSin\beta = Fh \quad (9)$$

We see that moment of force is less the less is its shoulder h, and it is zero about any center lying on the line of action of the force. This is the expected result because experience shows that such force cannot rotate the body.

Theorem 1. ***The dependence of the moment on Center.***

Let's find the connection between the moments of force **F** about the centers A and B. Figure 6 shows that

$$\boldsymbol{r_A} = \boldsymbol{AB} + \boldsymbol{r_B}; \quad \boldsymbol{m_A}(\boldsymbol{F}) = \boldsymbol{r_A} \times \boldsymbol{F} = (\boldsymbol{AB} + \boldsymbol{r_B}) \times \boldsymbol{F} = \boldsymbol{r_B} \times \boldsymbol{F} + \boldsymbol{AB} \times \boldsymbol{F}$$

Thus

$$\boldsymbol{m_A}(\boldsymbol{F}) = \boldsymbol{m_B}(\boldsymbol{F}) + \boldsymbol{AB} \times \boldsymbol{F} \quad (10)$$

Formula (10) shows that:

a) in general case, the moment of force depends on the center

б) moving the center along the line parallel to the force does not change the moment (in this case, the second term in (10) becomes zero).

Theorem 2. On projections of the moments.

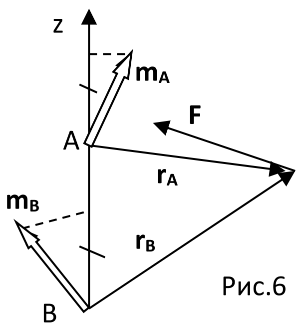


Рис.6

Projecting (10) on the z axis, passing through A and B, we find

$$\text{пр}_z \boldsymbol{m_A}(\boldsymbol{F}) = \text{пр}_z \boldsymbol{m_B}(\boldsymbol{F}) = m_z(\boldsymbol{F}) \quad (11)$$

Since the product **AB x F** is perpendicular to **AB** its projection on z is zero. Thus, we come to the theorem:

Projections of the moments of force about all points of an axis on this axis are equal.

Therefore, we can conclude that the m_z (F) characterizes the action of force with respect to the z axis, and will call it a ***moment of force about an axis*** (see below)

Matrix calculation of cross product.
Attached matrix.
Analytical expression of the moment.

It is known that the product

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$

of two vectors

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \quad \text{и} \quad \boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$$

can be calculated as the determinant of the matrix

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \left| \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} \right| = (a_y b_z - a_z b_y)\boldsymbol{i} + (a_z b_x - a_x b_z)\boldsymbol{j} + (a_x b_y - a_y b_x)\boldsymbol{k} \qquad (12)$$

Or

$$c_x = a_y b_z - a_z b_y; \quad c_y = a_z b_x - a_x b_z; \quad c_z = a_x b_y - a_y b_x$$

Vectors **a, b** and **c** can be presented by the columns of their projections.

$$a = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \qquad b = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \qquad c = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} \qquad (13)$$

It is easy to verify that the column matrix c can be calculated by multiplying an skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \qquad (14)$$

composed of projections of vector **a** by the column matrix b (13). Matrix A is called ***attached matrix of vector a.***

We conclude that to any vector product

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$

corresponds a matrix formula

$$c = Ab \qquad (16)$$

The reverse is also true. To any expression of (16) type where A is a skew-symmetric matrix corresponds the vector product $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$.

Analytical expression of the moment

Let the radius vector **r** and force **F** be defined analytically, i.e. by their projections on the axis x, y, z. Then to the vector product of the moment $\boldsymbol{m_0}(\boldsymbol{F}) = \boldsymbol{r} \times \boldsymbol{F}$ corresponds a matrix formula:

$$m_0(F) = Rf$$

where R- is an attached matrix of the radius vector **r** (x, y, z)

$$R = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

Thus, we get the analytical expressions of axis projections of the moment

$$m_x(F) = yF_z - zF_y; \qquad m_y(F) = zF_x - xF_z; \quad m_z(F) = xF_y - yF_x;$$

which let us find the module and the direction of the vector $\boldsymbol{m_o}(\boldsymbol{F})$

Moment of force about an axis

Theorem on projections (see above) allows us to introduce a new characteristic of the force with respect to the axis. ***Moment of force F about the z axis*** is an algebraic value equal to the projection on the axis of the moment of force about an arbitrary point on the specified axis.

$$m_z(\boldsymbol{F}) = \text{пр}_z \boldsymbol{m_A}(\boldsymbol{F}) \qquad (A \in z) \qquad (1)$$

Let us consider the method of calculation and the properties of the moment about the axis. Using the arbitrariness of the center choice on the axis, we select the projection point O on the z axis of the point of force application. Denoting as **k** the ort of the z axis, and applying a circular permutation in the mixed product, we write

$$m_z(\boldsymbol{F}) = \boldsymbol{k} \cdot (\boldsymbol{OA} \times \boldsymbol{F}) = (\boldsymbol{k} \times \boldsymbol{OA}) \cdot \boldsymbol{F} = hFCos\alpha \qquad (2)$$

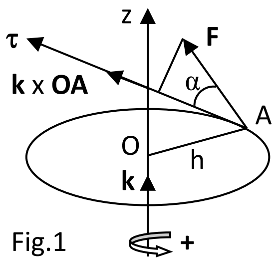


Fig.1

It is considered that the product $\boldsymbol{k} \times \boldsymbol{OA}$ is directed along τ according to the right screw rule. Its module equals the shortest distance OA = h from the point of application of the force to the axis.

Equation (2) shows that:

1. Moment of force about an axis depends only on the component of the force, tangential to a circle of radius h.
2. The sign of the moment is determined by the sign of the Cosα. Fig.1 gives us the following rule of the sign:

Moment of force about an axis is positive if we see from the end of the axis, that the force tends to turn the body counterclockwise.

It follows from equation (2) that moment of force about an axis is zero if the force and axis lie in one plane (α =π/2). This occurs when

1. the force is parallel to the axis
2. line of action of the force crosses the axis

You feel it when rising a bucket from the well, and therefore try to apply the force so that to create a bigger force arm.

Algebraic moment of force about the center for a coplanar force system.

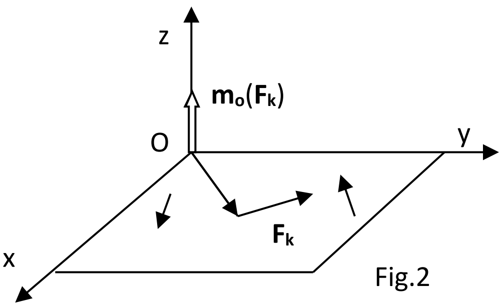


Fig.2

System of forces in one plane is called a ***coplanar system***. Let us combine the force system with the xy-plane.

In this case, all the forces moment vectors are along the z axis. Combining force plane with the plane of the sheet, the reader sees the axis z as the point O and calls the moment about the axis z an ***algebraic moment of force about the point O.***

$$mo(F) \equiv mz(F) = \pm Fh \qquad (3)$$

Rule of signs:

The moment is positive, if you see that moment tends to rotate the body counterclockwise.

Main moment of force system.
Dependence of main moment on the Center.

Main moment of the force system {**F**} about the center A is a vector $\mathbf{M_A}$ equal to the vector sum of the moments of all the forces of the system about this center.

$$\boldsymbol{M_A} = \sum_{k=1}^{n} \boldsymbol{m_A}(\boldsymbol{F_k}) \qquad (4)$$

Analytically the main moment is defined by its projections on the Cartesian axis

$$M_x = \sum_{k=1}^{n} m_x(F_k); \qquad M_y = \sum_{k=1}^{n} m_y(F_k); \qquad M_z = \sum_{k=1}^{n} m_z(F_k)$$

that are called ***the main moments of the forces about the axes x, y, z*** . Knowing them it's easily to find the module and the direction of the main moment:

$$M_A = \sqrt{M_x^2 + M_y^2 + M_z^2}; \quad Cos(\boldsymbol{M_A}; x) = \frac{M_x}{M_A}; \; Cos(\boldsymbol{M_A}; y) = \frac{M_y}{M_A}; \; Cos(\boldsymbol{M_A}; z) = \frac{M_z}{M_A} \quad (5)$$

Let us find the dependency of the main moment on two points A and B. Summarizing the previously received dependency for a single force for all forces of the system, we get:

$$\sum \boldsymbol{m_A}(\boldsymbol{F}) = \sum \boldsymbol{m_B}(\boldsymbol{F}) + \boldsymbol{AB} \times \sum \boldsymbol{F} \quad (4)$$

$$\boldsymbol{M_A} = \boldsymbol{M_B} + \boldsymbol{AB} \times \boldsymbol{V}$$

where the definition of the main vector is considered

$$\boldsymbol{V} = \sum_{k=1}^{n} \boldsymbol{F_k}$$

We see that unlike the one force, the main moment of the force system may not depend on the center if the main vector of the system is zero.

$$\boldsymbol{M_A} = \boldsymbol{M_B} = \boldsymbol{M} \quad \text{if} \quad \boldsymbol{V} = \boldsymbol{0} \qquad (5)$$

Rotational system of forces.
Pair of forces.

We will call the force system with zero main vector the ***rotational system.*** The name can be explained by the fact that such a system gives the rotation to a free body originally in rest, leaving its center of gravity in rest. Formula (4) shows that the main moment of the rotational system is independent from the center.

The simplest rotational system is the ***pair of forces***: the ***system of two equal by module, oppositely directed forces, not lying on the same straight line.*** Distance h between the lines of the forces of the pair is called the ***couple arm***. The main vector of a pair is equal to zero, so the main moment does not depend on the center and is called the ***pair moment m.*** It can be found as the moment of one of the forces about the application point of the second force.

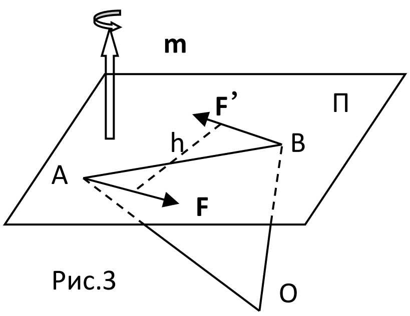


Рис.3

$$\boldsymbol{M_O}\{\boldsymbol{F}, \boldsymbol{F'}\} = \boldsymbol{m}_A(\boldsymbol{F}) = \boldsymbol{m}_B(\boldsymbol{F'}) = \boldsymbol{m} \qquad (6)$$

$$m = Fh$$

The pair moment is perpendicular to the plane of the couple and is directed so that from the end of the moment you can see that the couple is tending to turn the body counterclockwise.

Necessary conditions of the rest for any mechanical system.

We will call ***mechanical system*** a system of interacting material points with the masses $\{m_1 m_2 \ldots . m_k \ldots . m_n\}$. The movement of the system is considered relatively to a certain frame reference. ***Frame reference*** is a space, in which the observer is able to measure the distances and time.

The ***rest*** is when velocities of all points relative to the reference system are equal to zero. In statics, we seek the conditions to maintain the rest of any mechanical system, in other words the lack of the point's acceleration. It is logical to examine first the conditions of the rest for one of the system points. They may be derived from the principles (axioms) of Mechanics.

Principles (axioms) of Mechanics.
Conditions of the rest of a point.

As all the sciences, Mechanics is based on unprovable principles derived from experience and called ***axioms***. Being the fruit of the reflection of many generations of researchers, the axioms were finalized by ***Isaac Newton*** in the 17 century and therefore bear his name.

1. Galileo's principle of inertia

There is a reference system called "inertial", in which an isolated point saves the state of rest (or rectilinear uniform motion).

An isolated point is an abstract point, not interacting with the World.

Before Galileo, it was believed that for a uniform motion (of a cart) a force is needed.

All the laws of Mechanics are formulated and valid only in inertial reference system.

2. The basic principle *(Newton's second law)*

Acceleration of any material point is proportional to the applied force

and inversely proportional to the point mass

$$\boldsymbol{W} = \frac{1}{m}\boldsymbol{F} \qquad (7)$$

Consequence 1: *In inertial reference frame the rest of the point can be broken only by the action of a force* ***F*** *.*

Consequence 2 *: It is impossible to set both force and acceleration.*

3. ***Principle of internal additivity*** *(Newton's third law).* ***Properties of internal forces.***

The impact of environment on material system is equal to the vector sum of its impacts on all the parts of the system.

For simplicity, consider the material system of just two points m_1 and m_2. Let us denote as $\mathbf{F_1}$ the external impact on the point m_1 and as $\mathbf{F_2}$ on m_2. In addition to external impact, the points interact. We denote the impact of the point m_1 on m_2 as $\mathbf{F_i}$ and of the point m_2 on m_1 as $\mathbf{F_i}$**'.**

External impact on the system is $\boldsymbol{F_1} + \boldsymbol{F_2}$**.** Impact on the m_1 is $\boldsymbol{F_1} + \boldsymbol{F_i}'$**.** Impact on point m_2 is $\boldsymbol{F_2} + \boldsymbol{F_i}$ In accordance with the principle

$$\boldsymbol{F_1} + \boldsymbol{F_i}' + \boldsymbol{F_2} + \boldsymbol{F_i} = \boldsymbol{F_1} + \boldsymbol{F_2}$$

Hence $\boldsymbol{F_i}' + \boldsymbol{F_i} = \mathbf{0}$, or

$$\boldsymbol{F_i}' = -\boldsymbol{F_i} \qquad (10)$$

- known as "***the principle of equality of action and reaction*** »:
  Forces of interaction between two points are equal by modulo, opposite by direction and lie on a straight line passing through the points.

The forces of interaction are called ***internal system of forces*** (index **i**).

Consequence- Properties of internal forces

Internal forces are pairwise, so their main vector and main moment are zero.

$$\boldsymbol{V^i} = \mathbf{0}, \qquad \boldsymbol{M_O^i} = \mathbf{0} \qquad (11)$$

4. ***Principle of external additivity*** *(rule of forces summation).* ***A necessary and sufficient condition of the point rest***

Impact of environment on the point is the sum of the impacts of all the parts of the environment.

Consider a system consisting of a single point. Let the environment consisting of n points impact this point with the forces $\mathbf{F_k}$ (k= 1,2 ...n).

The principle states that all the forces of impact can be replaced by one force **F** equal to the sum of the forces with which the environment act on a material point.

Systems of force that cause the same acceleration are called ***equivalent*** systems. One force equivalent to the system of forces is called the ***resultant.***

It means that any system of forces $\{\mathbf{F_k}\}$**,** acting on the point, has a resultant

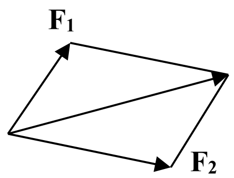


Fig.4

$$\boldsymbol{F} \sim \{\boldsymbol{F_k}\}, \qquad \boldsymbol{F} = \sum \boldsymbol{F_k} \qquad (\mathbf{8})$$

For the two forces principle gives the ***rule of the parallelogram*** (Fig. 4):

$$\{\mathbf{F_1}\ \mathbf{F_2}\} \sim \mathbf{F} = \mathbf{F_1} + \mathbf{F_2}$$

Consequence 1: Newton's second law can be generalized to the case of several forces

$$m\boldsymbol{W} = \sum \boldsymbol{F_k}$$

Consequence 2: necessary and sufficient condition of rest of a point is the balance of forces applied to the point:

$$\sum \boldsymbol{F_k} = \boldsymbol{0} \qquad (9)$$

Thus, not only an isolated point remains at rest, but a point under the action of balanced forces. If the polygon of forces is closed.

Necessity of condition (9) means that if the point is at rest, the condition (9) is met. In this case there will be unknown forces, which can be found from equation (9). For example, the forces acting on the chandelier, satisfy the condition (9) because the chandelier is at rest. We know only the force of gravity of the two forces acting on the chandelier. From the equation (9) we find that the tension of the cable is equal by modulo and opposite by direction to the force of chandeliers gravity.

Sufficiency means that if all the forces are given, then using the (9) you can check whether the point will remain at rest. For example, we know that the point is under two forces. Making their sum (9), we understand that the balance is possible only when the forces are pairwise.

Consequence 3: force may occur not only as a consequence of inertia of the point, but also as a reaction to another force. So the tension of the rope, on which hangs a chandelier, is defined only by the force of chandeliers gravity.

Necessary conditions for the equilibrium of external forces

Consider a discrete system of **n** material points. The rest requirement means that all points should stay at rest. So, the forces acting on each point, should in balance.

Denoting for the point number k: $\mathbf{F_k}^{(e)}$ as external forces resultant and $\mathbf{F_k}^{\mathbf{i}}$- as internal forces resultant, we receive the ***necessary and sufficient conditions*** of rest for arbitrary discrete mechanical system:

$$\boldsymbol{F_k^e} + \boldsymbol{F_k^i} = \boldsymbol{0} \qquad (\boldsymbol{k} = \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{n}) \qquad (\boldsymbol{12})$$

Any part or combination of conditions (12) is not sufficient, but necessary for the system rest. For example, one of the conditions (12) for the point number $k = 2$ is necessary and sufficient for this point, but only necessary for the entire system.

Properties of internal forces provide us with the ***necessary conditions of equilibrium of external forces***

Summing up (12) and taking in consideration that the main vector of internal forces equal to zero, we get

$$\boldsymbol{V^e} = \boldsymbol{0}$$

Multiplying (12) from the left by the radius vector $\mathbf{r_k}$ of the point, after the summation we will get the second condition

$$\boldsymbol{M_O^e} = \boldsymbol{0}$$

Conditions for external forces

$$\boldsymbol{V^e} = \boldsymbol{0}, \qquad \boldsymbol{M_O^e} = \boldsymbol{0} \qquad (13)$$

are necessarily met for any system at rest, including deformable body. System of external forces that meet conditions (13), is called ***a balanced system.***

The fact of rest of the system turns the expression (13) in the equation from which we can determine the unknown forces. In Statics of Rigid body these are usually the support reactions.

Furthermore, we will see that conditions (13) are also ***sufficient*** for the rest of only a ***solid body.***

Sufficient conditions of equilibrium of external forces to keep the rest of a rigid body

Load and reactions of statically determinate constraints.
Direct Statics problem

Consider a body T that is at rest under the action of a remote body T_0 (remote interaction), and of the bodies T_1, T_2, T_3, being in contact with T, and called ***constraints***.

Forces of remote interaction are determined by the laws of physics, so are generally considered known. All known forces acting on the body, are called the ***load***. Unknown forces of interaction with the fixed supports are called ***reactions.*** Let the body be supplied with ***sufficient supports*** which provide the rest under an arbitrary load.

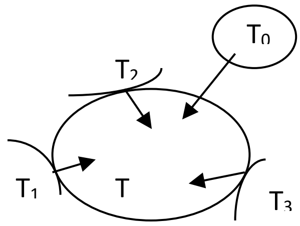


Рис.1

Direct problem of Statics is definition of reactions caused by the load. Since the body with sufficient supports remains at rest under any load, then the equilibrium conditions of the external forces are necessarily met:

$$\boldsymbol{V^e} = \boldsymbol{V^R} + \boldsymbol{V^a} = \boldsymbol{0}; \qquad \boldsymbol{M_o^e} = \boldsymbol{M_o^R} + \boldsymbol{M_o^a} = \boldsymbol{0}$$

from where

$$\boldsymbol{V^R} = -\boldsymbol{V^a}; \qquad \boldsymbol{M_o^R} = -\boldsymbol{M_o^a} \qquad (1)$$

Here index R identifies the reaction and index a - load.

Projections of two vector conditions (1) on the axis x,y,z give six algebraic equations for reactions that can be represented in matrix form

$$Ax = y \qquad (2)$$

Here A is the system matrix, depending only on the support configuration, x - column of unknown reactions, y - load projections column. As it is known, the algebraic system has a unique solution if and only if A is a square matrix (6 x 6), and the determinant of A is not zero.

$$|A| \neq 0 \qquad (3)$$

Supports with such matrix are called ***statically determinate*** (or short ***determinate***) because only for such supports the reactions can be determined from the equations of statics (2).

Please note that condition (3) provides a homogeneous system

$$Ax = 0 \qquad (4)$$

with zero solution. This means that the reactions of determinate supports disappear when lifting the load. In other words, if

$$\boldsymbol{V^R} = \boldsymbol{0} \quad \boldsymbol{M_o^R} = \boldsymbol{0} \qquad (5)$$

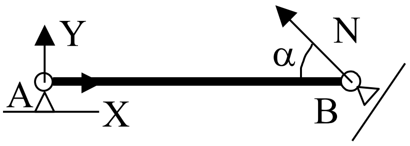
then all reactions of determinate supports are equal to zero.

Condition (3) means that the matrix A must not have linearly dependent or zero rows or columns. The rows are independent because the coordinate axes are orthogonal and projections are independent of the moments. A zero row in the matrix would mean that supports do not “stand” the corresponding load i.e. are not sufficient in this direction. We agree to deal only with sufficient supports.

Columns cannot be zero, because no force can be with no projection or moment. Dependent columns may exist only if two reactions may appear on the same line. Hence the ***rule of building determinate suppots***

Putting a new support, please be sure that its reaction would not appear on the same line with the reaction of the existing support

If condition (3) is false supports are regarded as ***redundant in this directiom***. The existence of redundant supports can be identified by two imaginary experiments: heating up the body or slightly shifting the support. If the reactions vary, the supports are redundant.



In case if the number of unknowns is equal to the number of equations, and supports are redundant in one direction, then they are not sufficient in another direction. Thus, supports of the rod (Fig.) have the "right" number of unknowns (3), but become redundant in direction AB, and not sufficient with respect to rotation around A when α = 0.

Sufficient conditions of equilibrium of external forces to keep the rest of a rigid body.
Inverse Statics problem

Absolutely solid body is a model, in which the distance between any two points is unchangeable with time. This model greatly simplifies the study of the body rest and movement and it is important because the deformations of most of machine parts are small compared with the size of the parts.

Theorem:

Conditions $\boldsymbol{V^e = 0}$; $\boldsymbol{M_o^e = 0}$ ***are sufficient to keep the solid at rest.***

Consider a free rigid body, the movement of which is not restricted. Let us apply to the body a load **{F}**, satisfying the conditions

$$\boldsymbol{V\{F\} = 0}; \qquad \boldsymbol{Mo\{F\} = 0} \qquad (6)$$

We will show that the body will remain at rest.

Assume the opposite i.e. that after the load application, the body will start to move. To stop the movement, we put the determinate supports. Thus the rest will be provided by the support reactions **{R}.** It means that the unified force system of **{F}** and **{R}** will be balanced and will fulfill the conditions:

$$\boldsymbol{V\{F\} + V^R = 0},$$
$$\boldsymbol{Mo\{F\} + M_O^R = 0}.$$

According to (6) the main vector and moment of reactions are equal to zero:

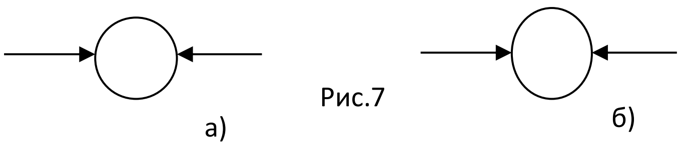
$$\boldsymbol{V^R = 0}; \qquad \boldsymbol{M_o^R = 0}$$

Since the supports are statically determinable, it follows that all reactions are equal to zero. Thus the supports are not needed, and the body remains at rest after application of **{F}.**

It means that conditions. (6) are sufficient to keep the solid at rest.

The theorem is proved.

In case of deformable body conditions (6) are not sufficient, but they are necessary. This means that apart of conditions (6), you need to have some others. However, if the rest is a fact, the equations (6) are met.



Example: If we apply to the ball at rest two forces $F = -F'$ (Fig. 7 a), the ball will not remain at rest. It will start to deform, though conditions (6) are satisfied. It means that conditions (6) are not sufficient to keep the ball at rest. But deformed ball (Fig. 7 a) keep the rest.

Conditions (6) allow us to solve the ***inverse problem of Statics***: check the balance of forces **{F},** applied to the solid.

Scalar conditions of equilibrium of various systems of forces

Although all mechanical notions have vector nature, the calculations are usually carried out in a scalar form. To pass to the scalar form of equations we can project (6) on the Cartesian axis.

a) Arbitrary spatial force system

Vector equilibrium conditions **V = 0, Mo = 0** in projections on the Cartesian coordinate axes give six scalar conditions:

$$V_x = \sum F_{kx} = 0; \qquad M_x = \sum m_x(F_k) = 0;$$
$$V_y = \sum F_{ky} = 0; \qquad M_y = \sum m_y(F_k) = 0; \qquad (7)$$
$$V_z = \sum F_{kz} = 0; \qquad M_z = \sum m_z(F_k) = 0;$$

b) Spatial system of intersecting forces.

It refers to a system of forces whose lines of action intersect at a single point (Fig.8). The main moment of such system about the point O of the forces intersection is equal to zero **Mₒ = 0**. Therefore, the equations of moments 4, 5, 6 in (7) are satisfied and leave only three conditions in projections:

$$V_x = 0; \quad V_y = 0; \quad V_z = 0; \qquad (8)$$

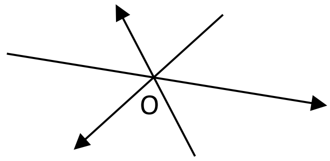


Fig.8

Theorem of 3 forces: if the body is at rest under the action of the three forces, and two of them intersect, all three intersect at this point.

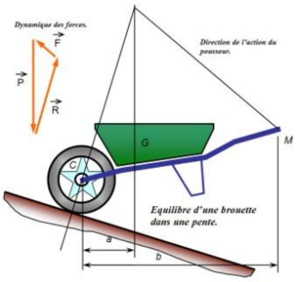


Fig.9

Indeed, the main moment of the system about the point of intersection of the two forces is equal to the moment of the third force and is equal to zero. Thus, the third force passes through the specified point. The theorem allows solving graphically, for example, the problem (Fig. 9) and finding the triangle of forces applied to the wheelbarrow.

c) Spatial system of parallel forces

If we direct the z axis parallel to the forces, then the main vector **V** will be parallel to z, and the main moment **Mo** will belong to the plane x y. i.e. **V** ⊥ **Mₒ** Three equations in (7) are satisfied and we come to the following conditions of equilibrium:

$$V_z = \sum F_{kz} = 0; \quad M_x = \sum m_x(F_k) = 0; \quad M_y = \sum m_y(F_k) = 0 \qquad (9)$$

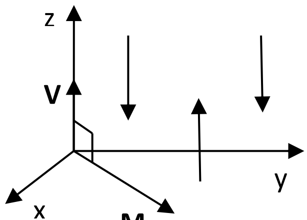


Fig.10

d) coplanar forces system.

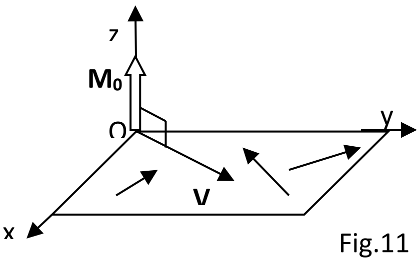


Fig.11

Construct a system of coordinates x,y,z at an arbitrary point O of the force plane (fig. 11). The main vector **V** lies in the plane xy. The main moment **Mₒ** is perpendicular to **V**. Hence, we come to three equations:

I) $V_x = 0; \quad V_y = 0; \quad M_O \equiv M_z = 0; \qquad (10)$

It can be shown that another two forms of equations exist for the plane system of forces:

II) $V_x = 0; \quad M_A = 0; \quad M_B = 0 \quad (AB \not\perp x)$ (11)

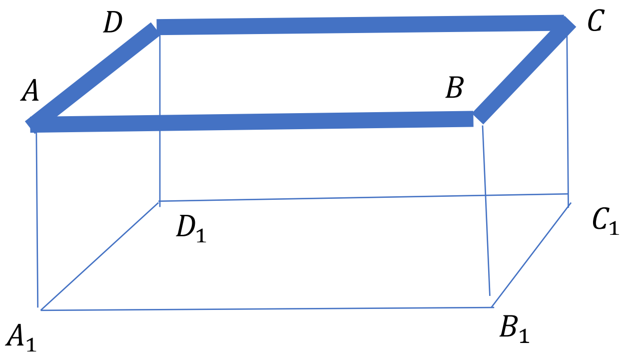
III) $M_A = 0; \quad M_B = 0; \quad M_C = 0$ (ABC- not on the same line) (12)

An example of setting up determinate supports in the form of rods on hinges

It is required to support a rectangular frame at its corners A, B, C, D on weightless rods with hinges at their ends so that the supports are statically determinate, that is, their reactions would be absent in the absence of load (the determinant of the equation matrix would not be equal to zero).

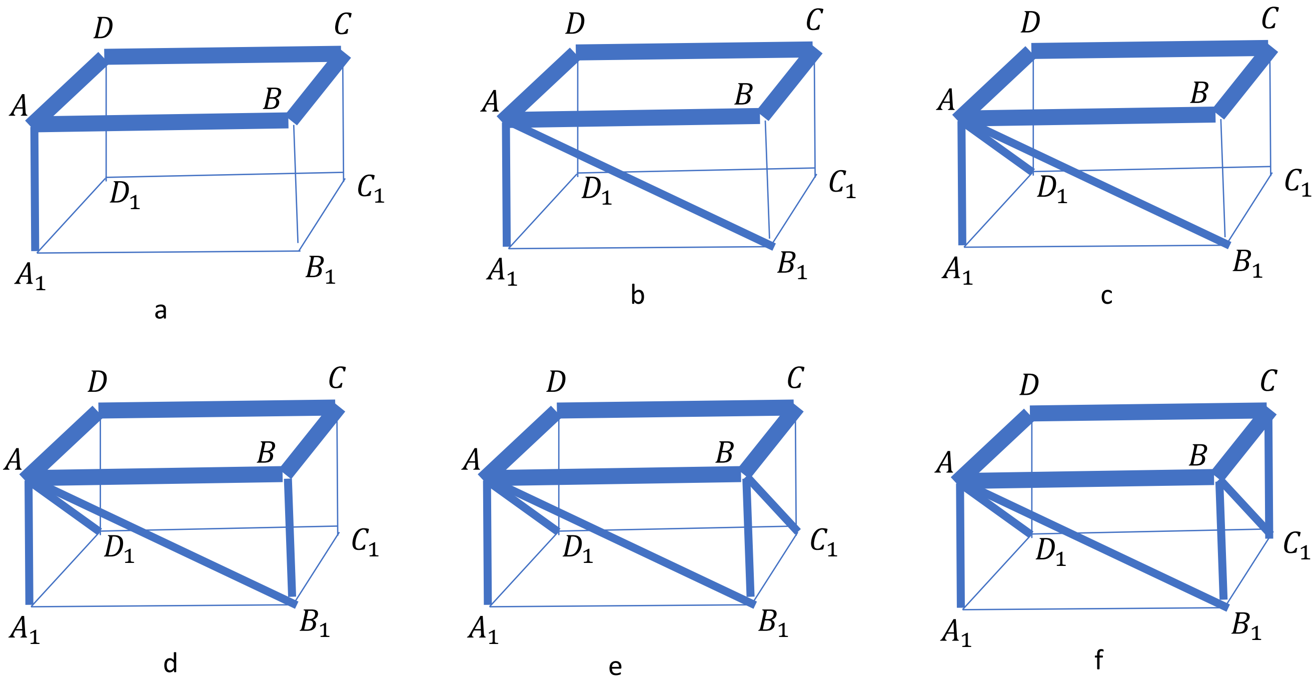
As is known, the reaction of such a rod is directed along the rod. It follows that in each rod one unknown arises, and there must be 6 rods.

By placing rods, we will gradually deprive the frame of all degrees of freedom, ensuring that each new reaction would not appear on the same line with the reactions of the already constructed supports.



Let's start with point A. If we place a vertical rod AA_1 in it, then point A will be able to rotate around the hinge A_1 (a). Let's place a second rod AB_1 in point A. After that, point A will still be able to rotate around the axis A_1B_1 (b). To stop this movement, let's place a third rod in point A. Its reaction should create a moment relative to the axis A_1B_1, which means it cannot lie in the same plane as this axis.

Let's put the rod AD_1 (c). Now the point A is fixed in space. There are still 3 rods to install.

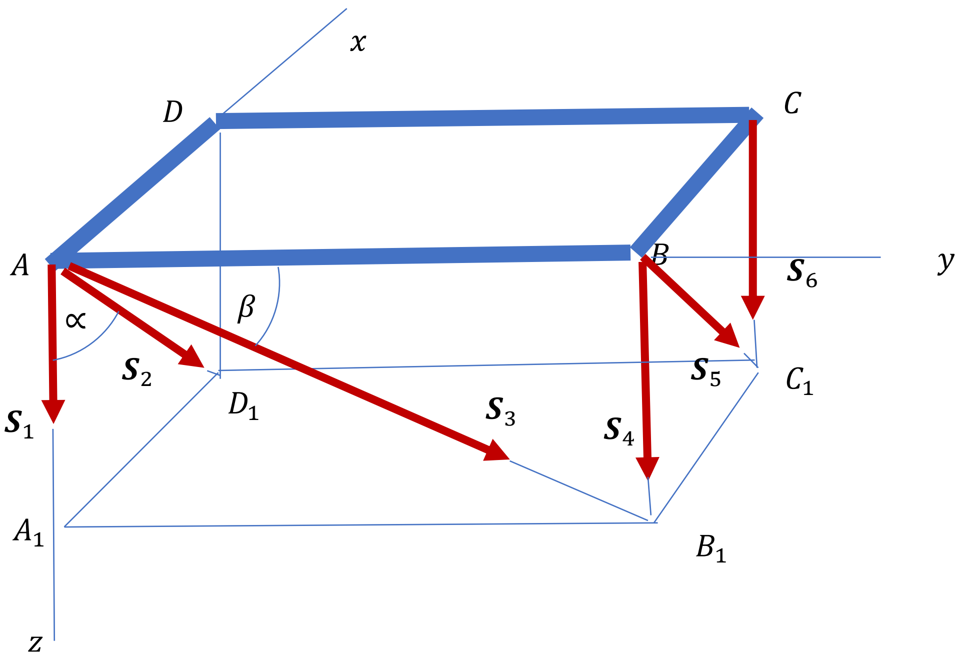


The question arises: is it possible to place them all at point B? The answer is obvious - it is impossible. On the one hand, the supports will become redundant in the direction AB, since 3 rods at one point can create a reaction arbitrarily directed in space, which means that the reactions at points A and B can end up on one straight line AB, which is prohibited.

On the other hand, the connections will become insufficient, since the frame will still be able to rotate around the axis AB.

At point B, we can place a vertical rod BB_1 (d). Its reaction cannot pass through point A, which is required. Now point B can still move in the direction BC. To stop this movement, we put the rod BC_1 (e). The resultant of the reactions of the two rods at point B cannot pass through point A, which is required.

Now the straight-line AB is fixed, but the frame can rotate around AB. To stop this rotation, it is enough to put the rod CC_1 (f).



Let us show that the determinant of the matrix of equilibrium equations is not equal to zero. That is, in the absence of load, the reactions are equal to zero, which means that the supports are determinate**.**

	S_1	S_2	S_3	S_4	S_5	S_6
$\sum F_{kx}$	0	$Sin\propto$	0	0	$Sin\propto$	0
$\sum F_{ky}$	0	0	$Cos\beta$	0	0	0
$\sum F_{kz}$	1	$Cos\propto$	$Sin\beta$	1	$Cos\propto$	1
$\sum m_x(F_k)$	0	0	0	AB	$ABCos\propto$	AB
$\sum m_y(F_k)$	0	0	0	0	0	$-BC$
$\sum m_z(F_k)$	0	0	0	0	$-ABSin\propto$	0

After subtracting the first column from the fourth and sixth, then the fourth from the sixth, we find that the determinant is different from zero, that is, the connections are determinate:

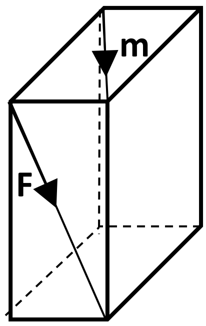
$$\Delta = 1 * \boldsymbol{Sin} \propto * \boldsymbol{Cos}\beta * AB * AB * BC * Sin \propto \neq 0$$

if $\propto \neq 0$ and $\beta \neq \pi/2$

An example of setting up determinate supports
and solving a problem on statics in space using vector and matrix methods

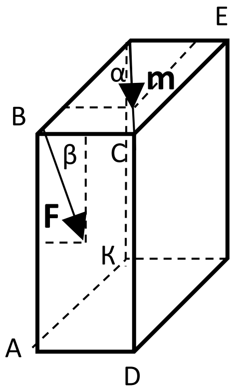
Problem statement

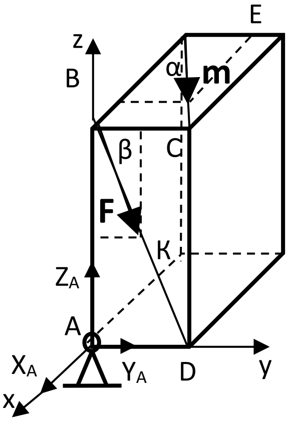
The parallelepiped is loaded with force ***F*** and moment ***m.***
1. Prove that they are not balanced.
2. Designate the vertices, angles and axes.
3. Set dterminate connections: spherical and cylindrical hinges, a rod on two hinges.
4. Write the equations of equilibrium of the body
a) geometrically and b) matrix-wise



Solution

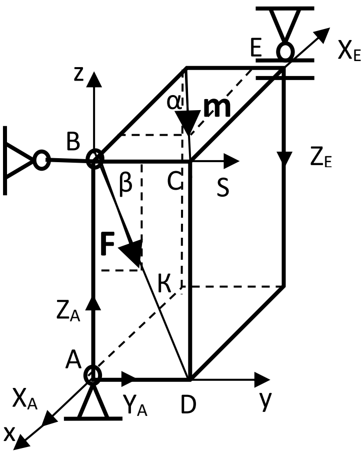
1. The force and moment cannot be balanced, since at the point of application of the force the main vector of the system is equal to the force F, and the main moment is equal to the moment m, and both are not equal to zero.





2. Let us designate the angles α and β and the vertices A, B, E, at which we will place the supports. Let us add the notations of the vertices C and D to designate the dimensions of the body.
3. Let us place the determinate supports. Let us start with the spherical hinge at the vertex A. Since the greatest number of unknowns (3) arise in it, it is reasonable to choose the origin of coordinates here. The reaction of the hinge is directed arbitrarily and can pass through the points B and E of the other supports.

At the point E let us place a cylindrical hinge. In order for the supports to remain determinate, we must direct its axis so that its reaction would not pass through the vertex A. The reaction of the cylindrical hinge is directed arbitrarily in the plane perpendicular to the hinge axis. Therefore, its axis cannot be perpendicular to the straight-line AE. Otherwise, the connections will become redundant in the direction AB and insufficient for rotation around the axis perpendicular to AE.



We direct the axis of the cylindrical hinge along the y axis.

It remains to place the rod on hinges at the vertex B. It cannot be directed in the plane ABE. Otherwise, it will be redundant in its direction and will not keep the body from turning around AE. Let's direct the rod along the y axis.

a) Geometric method

The projection equations are straightforward.

$$V_x: \; X_A - X_E = 0$$

$$V_y: \; Y_A + S + F\,Cos\beta = 0$$

$$V_z: \; Z_A - Z_E - F\,Sin\beta = 0$$

When calculating moments of forces, we use a simple algorithm.

1. First, we ask whether the moment of force is equal to zero (if the force is parallel or intersects the axis). For example, F intersects the y-axis.

2. If the answer is negative, then we look to see if the force is perpendicular to the axis. Then the moment is found using the formula, for example

$$m_x(F) = \pm F_y h_z$$

It is better to choose the sign by imagining where the screw is moving under the action of the force. If the screw is moving along the axis, then plus, otherwise minus.

To find the arm, we use the rule of three indices. h_z must be perpendicular to both F_y and axe x.

3. If the force is directed arbitrarily, then it should be decomposed into components along the axes. For the components, we use the second case.

$$\begin{aligned} &M_x:\ m\,Cos\alpha\ - F\,Cos\beta\,AB - S\,AB - Z_E\,BC = 0 \\ &M_y:\ m\,Sin\alpha - X_E\,CD - Z_E\,CE = 0 \qquad (15) \\ &M_z:\ X_E\ BC = 0 \end{aligned}$$

b) Matrix method

Equilibrium equations in matrix form

$$R_A + R_B + R_E + F = 0$$
$$AR_A + BR_B + ER_E + BF + M = 0 \qquad (16)$$

Columns of projections of forces and moments:

$$R_A = \begin{pmatrix} X_A \\ Y_A \\ Z_A \end{pmatrix},\quad R_B = \begin{pmatrix} 0 \\ S \\ 0 \end{pmatrix},\quad R_E = \begin{pmatrix} -X_E \\ 0 \\ -Z_E \end{pmatrix},\ F = \begin{pmatrix} 0 \\ FCos\beta \\ -FSin\beta \end{pmatrix},\ M = \begin{pmatrix} mCos\alpha \\ mSin\alpha \\ 0 \end{pmatrix}$$

Adjoint matrices of vectors of points of application of forces

$$A = 0,\quad B = \begin{pmatrix} 0 & -AB & 0 \\ AB & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\quad E = \begin{pmatrix} 0 & -AB & AD \\ AB & 0 & -CE \\ -AD & CE & 0 \end{pmatrix}$$

Substituting the matrices into expressions (16), we arrive at equations (15).

Set of interactive excel problems on determining the reactions of supports of a plane system of two bodies

At each step of the solution, the incorrectly filled cell is colored pink, which allows the student to independently find the error.

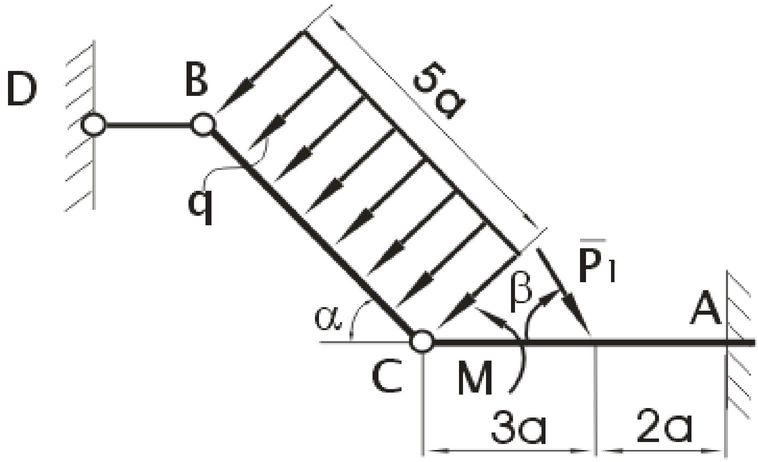
The problems are completed by the student until the correctness of the solution is confirmed by substituting the found solutions into the equilibrium equations of the entire system. Therefore, it is not the correctness of the solution that is assessed, but the deadline for completing the task.

The tasks and an example of its solution can be downloaded from the link https://disk.yandex.ru/d/CjdfFXuXTHvPnw

Example of completing a task about a two-body system

A mechanical system consists of two beams BC and AC, connected to each other by a cylindrical hinge C. At point A, beam CA is fixed in a vertical wall using a rigid fixture, at point B -

using a weightless rod BD. The system is loaded with a uniformly distributed load of intensity q, a pair



of forces with a moment M and a force P_1.

Determine the reactions at points A, B, C. Neglect the weight of the beams and friction in the hinges.

Given:

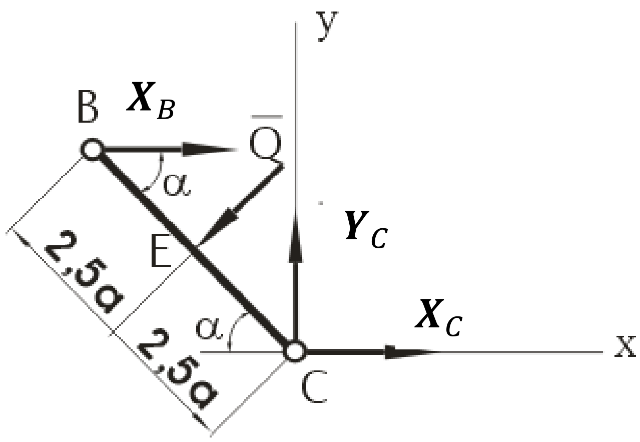
a	α	β	q	P1	M
м	град	град	н/м	н	нм
1,2	60	30	4,8	1,5	1

Make up equations

BC			CA			ABC		
V_x=0	V_y=0	M_c=0	V_x=0	V_y=0	M_c=0	V_x=0	V_y=0	M_c=0

Calculation schemes and equilibrium equations
Let's consider the equilibrium of each of the bodies of the system.

Beam BC



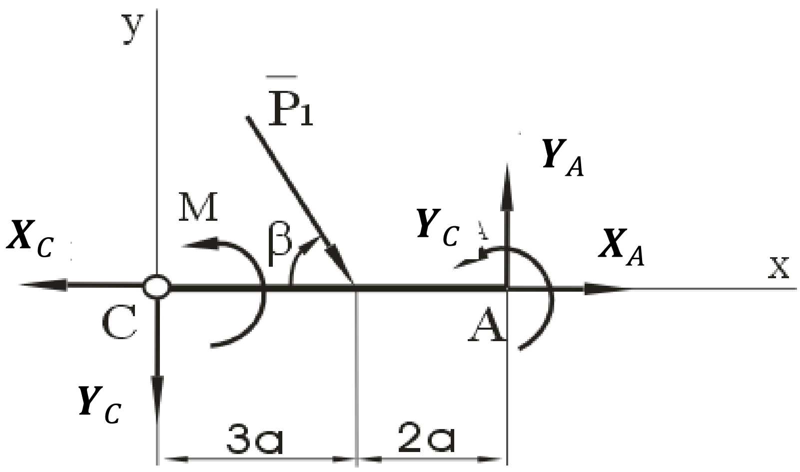
The beam is acted upon by:
resultant Q of uniformly distributed load at point E;
the modulus of force is equal to Q = 5q a = 5 x 4.8 x 1.2 = 28.80; (N)

We will direct the reaction of the weightless rod XB and the components X_C, Y_C of the reaction of the cylindrical hinge positively.

Since the beam BC is at rest, the following conditions are satisfied:

1) $V_X = X_B + X_C - Q \cdot \sin\alpha = 0$,
2) $V_Y = Y_C - Q \cdot \cos\alpha = 0$,
3) $M_C = Q \cdot 2{,}5a - X_B \cdot 5a \cdot \sin\alpha = 0$.

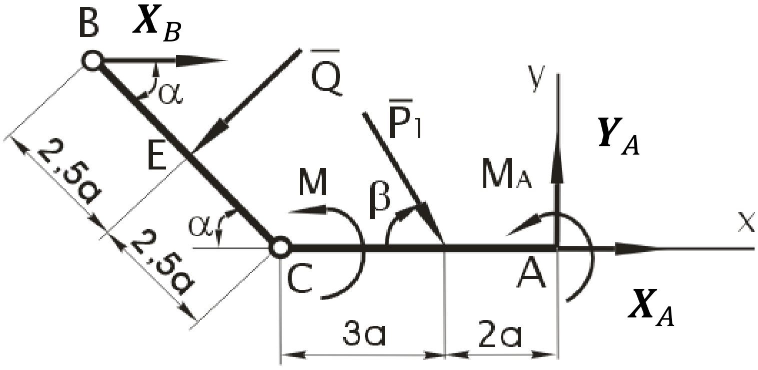
Beam CA



The beam is acted upon by:
- force $\boldsymbol{P_1}$ and moment $\boldsymbol{M}$;
- at point A – reactions $\boldsymbol{X_A}, \boldsymbol{Y_A}$ and moment M_A of rigid fixing are directed positively.
at point C – reactions X_C, Y_C of the cylindrical hinge, according to Newton's third law, are directed opposite to the corresponding reactions of beam BC.
Since beam AC is at rest, the following conditions are met:

4) $V_X = X_A - X_C + P_1 \cdot \cos\beta = 0$,
5) $V_Y = Y_A - Y_C - P_1 \cdot \sin\beta = 0$,
6) $M_C = M + M_A - (P_1 \cdot \sin\beta) \cdot 3a + Y_A \cdot 5a = 0$.

To check the obtained results, it is necessary to make the equilibrium equations of the entire system and substitute the solutions there. If the sums turn to zero, then the solution is correct.



7) $V_X = X_B - Q \cdot \sin\alpha + X_A + P_1 \cdot \cos\beta = 0$?

8) $V_Y = Y_A - Q \cdot \cos\alpha - P_1 \cdot \sin\beta = 0$?

9) $M_C = Q \cdot 2{,}5a - X_B \cdot 5a \cdot \sin\alpha + M + M_A - (P_1 \cdot \sin\beta) \cdot 3a + Y_A \cdot 5a = 0$?

Set of interactive excel problems on statics in space

The problems are made for independent student preparation for the test and for conducting tests by professor in the classroom on individual students' computers.

In the preparation mode, the student is given a code to check the solution. If the student enters the code before starting the solution, he sees the mistakes made at each step of the solution and tries to correct them.

In the self-control mode, the student enters the code after finishing the solution and sees all his mistakes and the final grade, which allows him to understand whether he is ready to write the test.

When conducting a test in the classroom professor sends random tasks to the students. The student returns the solution to the teacher, into which the teacher enters a secret code. The student sees his mistakes and the final grade.

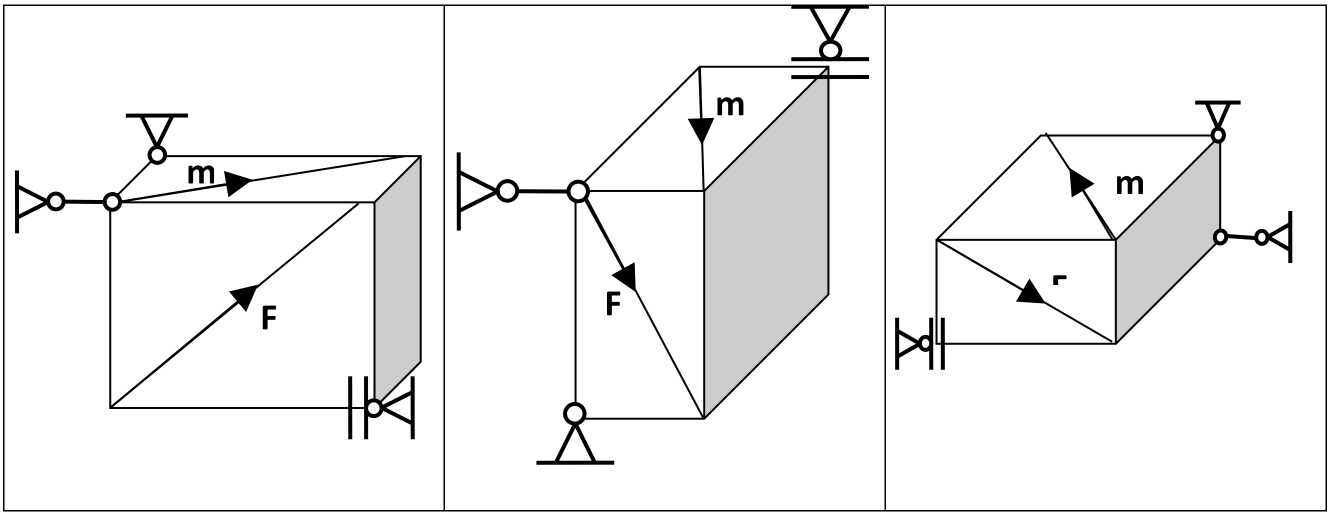
The teacher sends a scan of the screen to the student to work on his mistakes. The student writes down the correct equations at home, which he hands over to the teacher before rewriting the test.

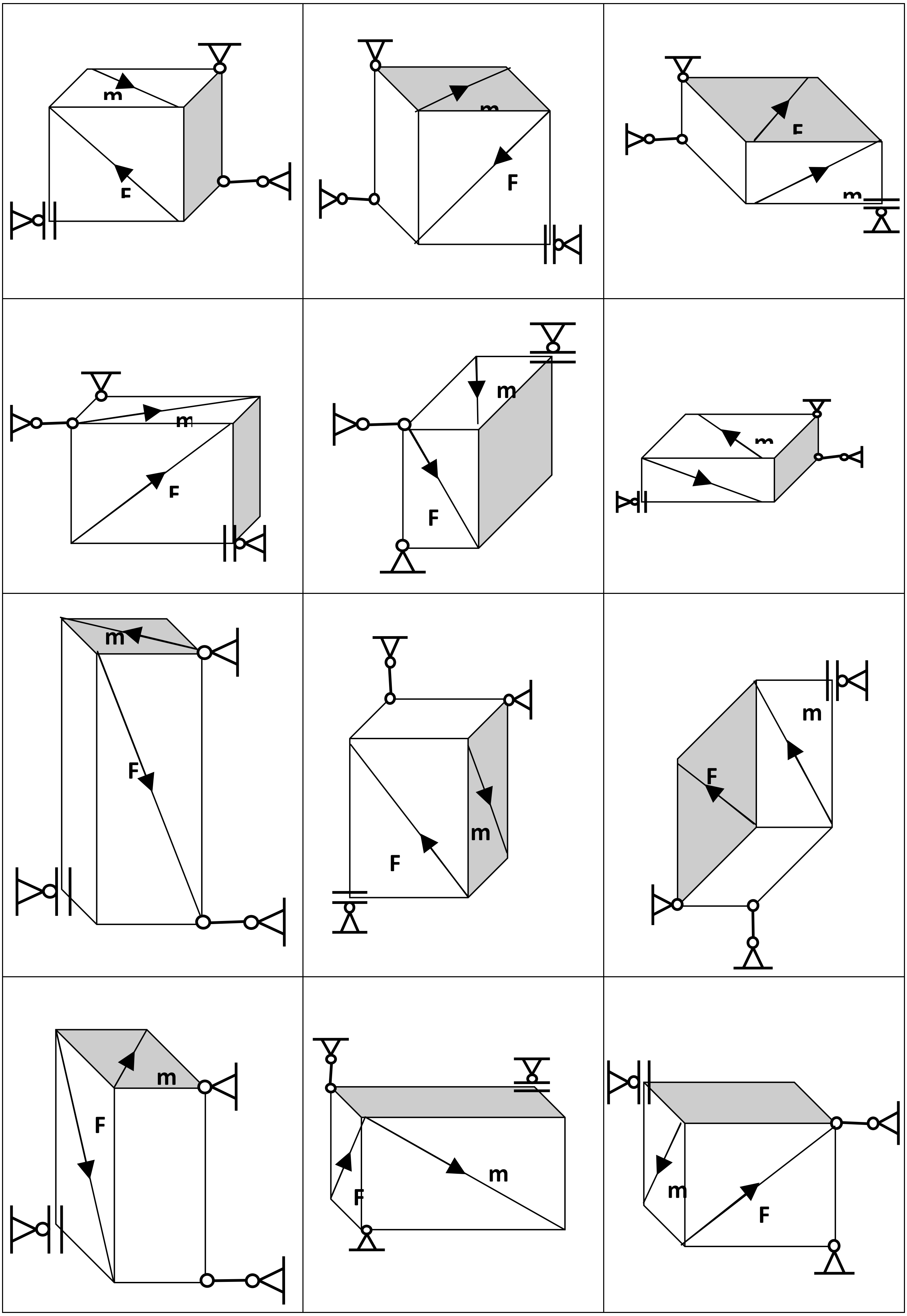
The tasks and a video of an example solution can be downloaded from the link https://disk.yandex.ru/d/1BEvxfJJMTK-IA

Set of equally complex problems on statics in space

A rectangular parallelepiped is subject to force ***F*** and moment ***m***.
Denote the vertices and angles. Draw the coordinate axes. Write down the equilibrium equations. Indicate the directions of the axis of the cylindrical hinge and the rod, at which the supports become redundant in one direction and insufficient in the other.

1. What is the principal vector of reactions and their principal moment relative to the point of application of the force?
2. Do the reactions have a resultant?

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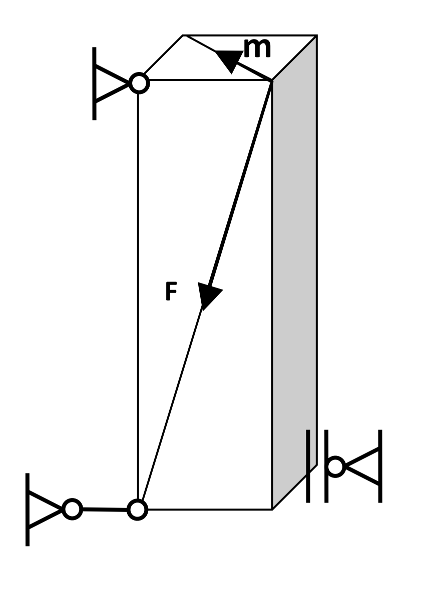
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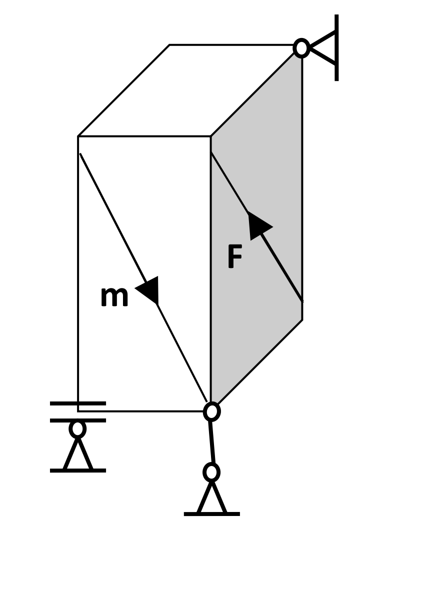
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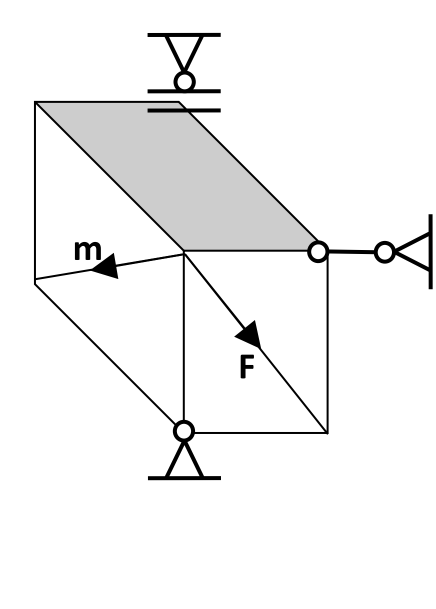
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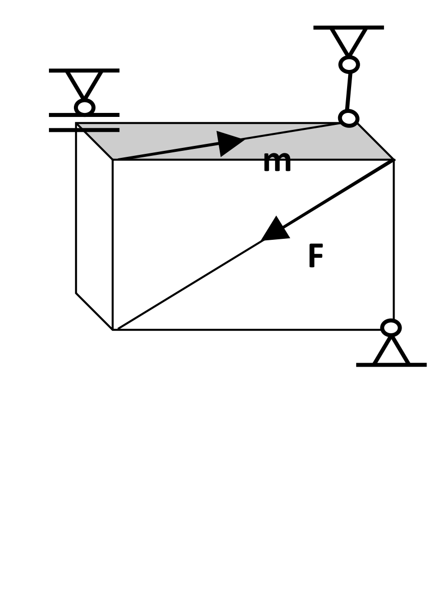
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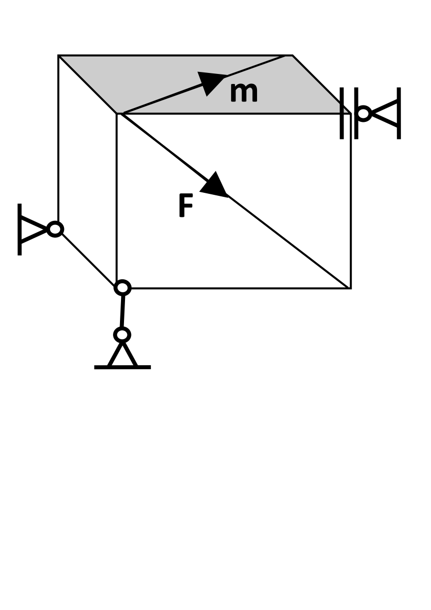
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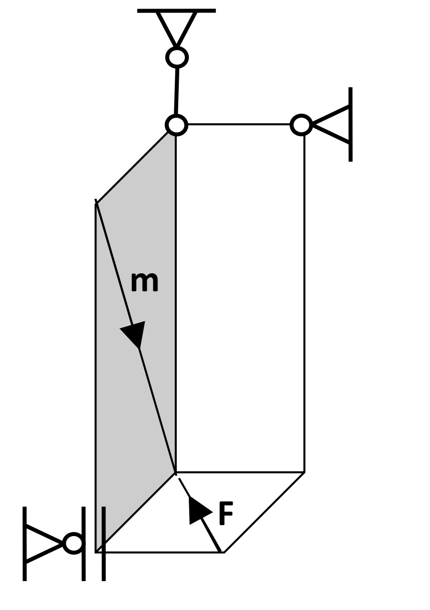
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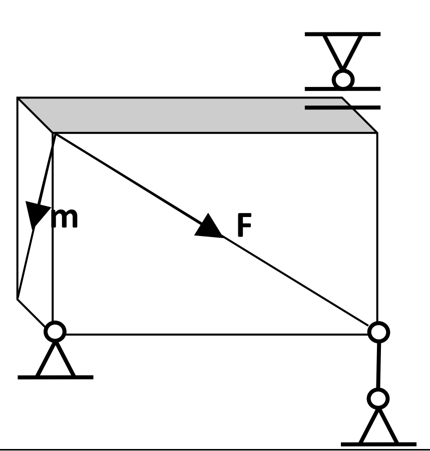
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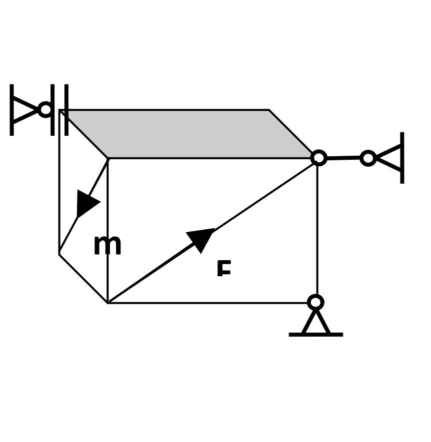
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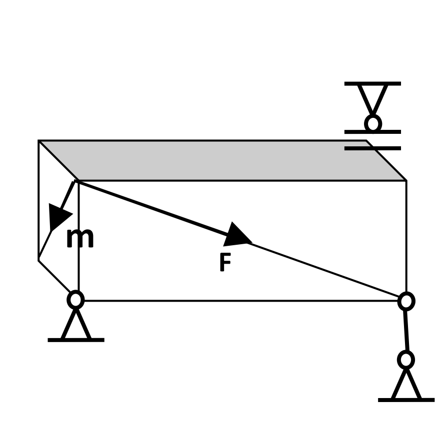
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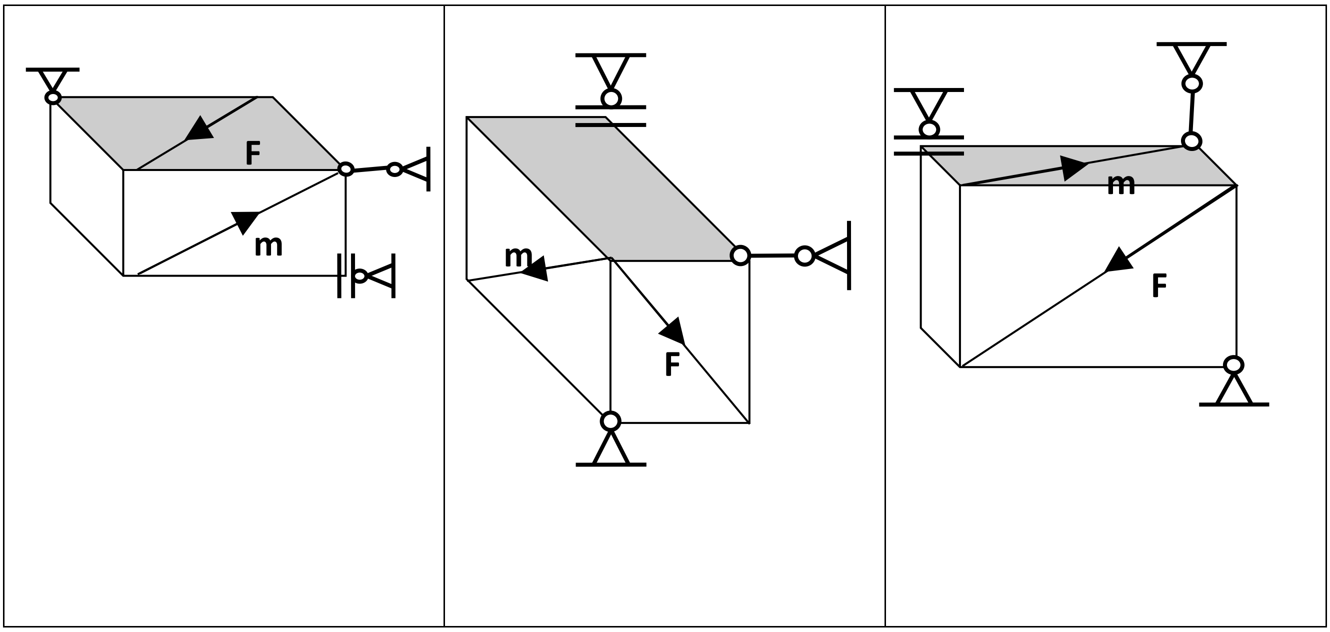

Statics Colloquium Questions

Moment of force about a point. Axial vectors.
Theorem on dependence and theorem on projections of moments.
Matrix calculation of vector product.
Moment of force about an axis. Calculation. When it equals zero?
Algebraic moment of force about the center for a coplanar force system.
The principal vector and principal moment of the system about the center.
Dependence of the principal moment on the center.
Rotational system of forces. Couple of forces.
Four principles (axioms) of mechanics. Corollaries.
Conditions for maintaining the rest of a point.
Conditions for maintaining the rest of an arbitrary discrete system.
Necessary conditions for equilibrium of external forces of an arbitrary system.
Load and constraints reactions. Direct and inverse problems of statics.
Statically determinate constraints. Definability conditions for the matrix of a system.
Rule for constructing determinate constraints
Proof of sufficiency of equilibrium conditions of external forces for maintaining the rest of a rigid body.
Scalar equilibrium conditions for particular systems of forces.
Theorem on static equivalence of 2 loads.
Equivalent transformations of force and force couples in a solid
Poinsot's theorem.
Conditions for the existence of a resultant.
Varignon's theorem
Reduction of constraint reactions to force and moment.

Equivalent transformation of forces in a solid

Theorem on Static equivalence of loads.
Equivalent transformation of force and couple.

As you know, a resultant force can replace all forces, applied to a point.

$$\boldsymbol{R} \sim \{\boldsymbol{F}\} \qquad \boldsymbol{R} = \sum \boldsymbol{F_k}$$

For an arbitrary mechanical system thus exist only one- point reduction of forces into their resultant.

For a rigid body the equivalent transformations of forces are much broader. We have shown that the body will remain at rest under any balanced system of forces. Thus, all balanced systems are ***statically equivalent*** between themselves and are equivalent to zero (an empty system).

Since in Statics we consider the bodies fixed at rest under arbitrary load, all applied forces are necessarily balanced.

$$\boldsymbol{V^R} + \boldsymbol{V^a} = \boldsymbol{0}; \qquad \boldsymbol{M_o^R} + \boldsymbol{M_o^a} = \boldsymbol{0}$$

This means that support reactions change with the load change. During this change the complete system of forces remains balanced, undergoing ***the equivalent transformation***.

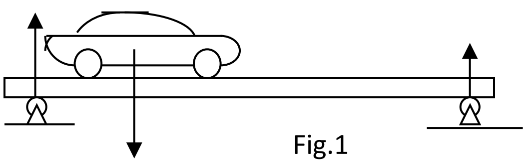

Fig.1

For example, when the car is moving on the bridge its weight (load) moves and changes the reactions of supports. Complete system of three forces, applied to the bridge, stay balanced and equivalent to itself and to zero.

Two loads $\{\mathbf{F}\}$ and $\{\mathbf{Q}\}$ applied to a fixed body are called ***statically equivalent***, if they cause the same reactions of support.

$$\{\boldsymbol{F}\} \sim \{\boldsymbol{Q}\} \quad if \quad \{\boldsymbol{R}\}_F = \{\boldsymbol{R}\}_Q$$

Replacement of $\{\mathbf{F}\}$ by $\{\mathbf{Q}\}$ in this case is called ***statically equivalent transformation*** of the load $\{\mathbf{F}\}$.

On Fig.1) the car moves and changes the reactions of supports. It means that ***parallel shift of force is not a statically equivalent force transformation.***

Theorem of Static equivalence of two force systems.

Reactions of statically determinate supports are uniquely determined from equilibrium equations. Since in the right parts of equilibrium equations we find projections of the main vector and the main moment of the load, we arrive to ***Theorem of Static equivalence of two loads:***

A necessary and sufficient condition for static equivalence of loads {F} ~ {Q} is the equality of their main vectors and main moments.

$$\{\boldsymbol{F}\} \sim \{\boldsymbol{Q}\} \quad \Leftrightarrow \quad \boldsymbol{V}\{\boldsymbol{F}\} = \boldsymbol{V}\{\boldsymbol{Q}\}; \;\; \boldsymbol{M_o}\{\boldsymbol{F}\} = \boldsymbol{M_o}\{\boldsymbol{Q}\} \qquad (1)$$

Equivalent transformations of a force and of a couple applied to a solid.

Force.

According to Theorem of equivalence, two vectorlike equal forces will be equivalent if they have a common line of action. ***Thus, the force can be moved only along its line of action.*** Example (Fig. 1) shows that parallel force shift changes the reactions, therefore, is not equivalent transformation.

A couple of forces

The main vector of the couple is equal to zero, so its main moment does not dependent on the center and is equal to the couple moment. Thus, equivalent are all the couples with the same vector of moment. It means that we can:

1. change the forces and arm of the pair, without changing their composition;
2. rotate and move the pair in its plane and

3. carry the couple in a parallel plane.

Conditions of resultant existence.
Varignon's theorem

If the forces system **{F}** has one equivalent force **R**, then **R** is called ***resultant*** of system **{F}**.

$$\boldsymbol{R} \sim \{\boldsymbol{F}\}.$$

It was shown that all force systems applied to a point, always have resultant. For a rigid body it is not true.

Suppose that the given system **{F}** has a resultant **R**. Then according to the theorem of equivalence, it must be equal to the system main vector

$$\boldsymbol{R} = \boldsymbol{V}\{\boldsymbol{F}\}. \qquad (3)$$

Therefore, the first condition of resultant existence is that the main vector of the system $\boldsymbol{V}$ is not zero:

$$\boldsymbol{V} \neq \boldsymbol{0}. \qquad (4)$$

We see that a couple of forces cannot have resultant.
The second term of theorem of equivalence leads to ***Varignon theorem*** :

If Resultant exists its moment about arbitrary point equals the main moment of the force system about the same point.

Since the resultant moment (and thus the system main moment $\mathbf{M_o}$) is perpendicular to the resultant (main vector **V**) the second condition of resultant existence will be the perpendicularity of $\mathbf{M_o}$ and **V**.

$$\boldsymbol{M_o} \perp \boldsymbol{V}$$

Thus, having built at point O the main vector $\boldsymbol{V}$ and main moment $\boldsymbol{M_o}$, we can state that the force system has a resultant if:

$$\boldsymbol{V} \neq \boldsymbol{0}; \qquad \boldsymbol{M_o} \cdot \boldsymbol{V} = \boldsymbol{0} \qquad (M_x V_x + M_y V_y + M_z V_z = 0) \qquad (5)$$

Note that the resultant has practical sense if it can be applied to the body. Another way, if the line of resultant action crosses the body. For example, the force of gravity of a bagel in horizontal plane, can be drawn, but not applied to it.

It has been shown that the main vector of plane and parallel force systems is perpendicular to the main moment. Thus, such systems always have a resultant, if $\mathbf{V} \neq \mathbf{0}$.

Couple has no resultant. However, the minimal change of one of the couple forces will make $\mathbf{V} \neq \mathbf{0}$ though small. The resultant will exist but will pass far away from the body.

It is obvious that support reactions of the body, loaded by one force **F,** have a resultant equal to **–F** on the same line of action.

Poinsot's theorem.

An arbitrary force system {F}, applied to a solid body, is equivalent to "screw" {Po; m}, consisting of:

one force $\boldsymbol{P_o} = \boldsymbol{V}\{\boldsymbol{F}\}$ applied in an arbitrary point of the body O and a couple {Q, Q '} with $\boldsymbol{m} = \boldsymbol{M_o}\{\boldsymbol{F}\}$

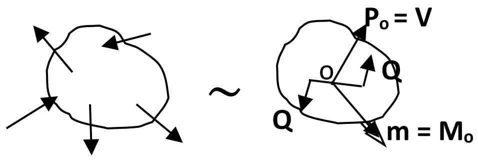

Fig.2

$$\{\boldsymbol{F}\} \sim (\boldsymbol{P_o}; \boldsymbol{m})$$

$$\boldsymbol{P_o} = \boldsymbol{V}\{\boldsymbol{F}\}; \quad \boldsymbol{m} = \boldsymbol{M_o}\{\boldsymbol{F}\}$$

To prove the theorem let us compare the main vectors and the main moments of $\{\boldsymbol{F}\}$ and of the "screw":

$$\boldsymbol{V}\{\boldsymbol{P_o}; \boldsymbol{Q}, \boldsymbol{Q'}\} = \boldsymbol{P_o} + \boldsymbol{Q} + \boldsymbol{Q'} = \boldsymbol{P_o} = \boldsymbol{V}\{\boldsymbol{F}\} \qquad (\boldsymbol{Q} + \boldsymbol{Q'} = \boldsymbol{0}) \qquad (2)$$

$$\boldsymbol{M_o}\{\boldsymbol{P_o}; \boldsymbol{Q}, \boldsymbol{Q'}\} = \boldsymbol{m_o}(\boldsymbol{P_o}) + \boldsymbol{m} = \boldsymbol{m} = \boldsymbol{M_o}\{\boldsymbol{F}\} \qquad (\boldsymbol{m_o}(\boldsymbol{Po}) = \boldsymbol{0})$$

We see that they are respectively equal. Thus, the theorem is proved. This equivalent transformation of any force system $\{\boldsymbol{F}\}$ to a ***"screw"*** is called ***system reduction to the point***.

Poinsot's theorem is valid only for the rigid body. Moving, for example, a force along a vertically suspended gum band, we are not changing the reaction, but changing the length of the band. The lower the force, the longer is the band.

Reduced contact reactions

Real bodies contact on some surfaces. For example, the cantilever, welded in the wall (painted surfaces on Fig.3) where contact forces are the unknown reactions. It is impossible to find their distribution from only three statics equations. This support is usually called as "fixed end"

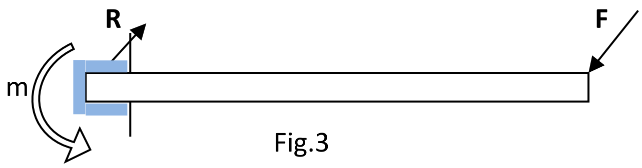

Fig.3

Using Poinsot theorem we can reduce the number of unknowns to one force R with 2 projections and one moment m (3 unknowns).

Reactions for more "weak" supports can be deduced from the reactions of the "fixed end" by removing restrictions on the movement or turn in a particular direction. The corresponding reaction component disappears.

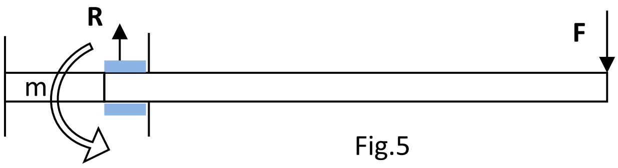

Fig.5

If, for example, we make a cross hole in the wall (Fig. 5), then the bar will be able to move horizontally, and the horizontal component of the reaction disappears. Thus, we come to support known as "rolling cantilever ".

Reduction of distributed reactions and loads

Distributed load q(x) may also be reduced by Poinsot's theorem to one force with modulo equal to the square of the "plot", and applied at the "Center of gravity" of the plot. Figure 6 provides examples of uniform and triangular loads.

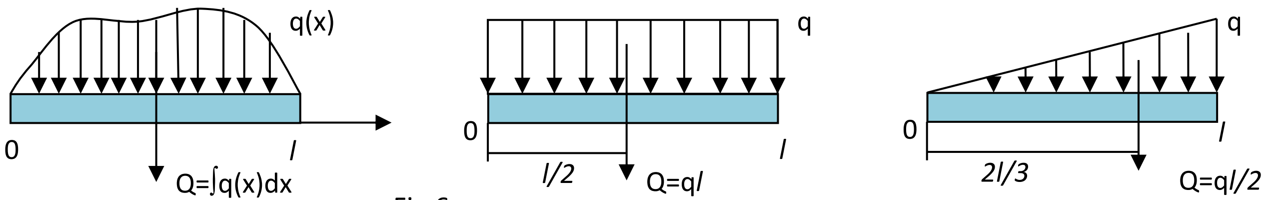

Fig.6

Limit states of equilibrium

There are supports, whose reaction is limited by modulo. These include the friction. In statics we are interested in the limit state of the body in rest when the reaction reaches its limit value and the body is about to start moving.

External friction.

Coulomb Law.

We call *external* the friction between the surfaces of the investigated body and of the body of support. Friction is a complex physical phenomenon, which has both negative and positive properties. Friction is subject of the science called Tribology. In engineering calculations we usually operate with the models reflecting the physical phenomenon with some degree of accuracy.

Consider a body at rest on the horizontal rough plane (Fig.1). It is under the action of the force of gravity $\boldsymbol{P}$ and normal reaction $\boldsymbol{N} = -\boldsymbol{P}$. To budge the body along the plane, we should apply some horizontal force $\boldsymbol{F}$. This means that force $\boldsymbol{F}$ causes resistance. Horizontal component F_{Tp} of the plane reaction $\boldsymbol{R}$ is called ***force of friction***.

As long as the body remains at rest, the conditions of equilibrium are true

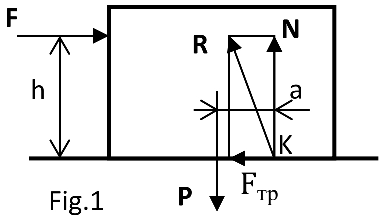

Fig.1

$$V_x:\ F + F_{\text{тр}x} = 0;\quad F_{\text{тр}x} = -F$$

$$V_y:\ N - P = 0;\quad N = P\quad (1)$$

$$M_K:\ Pa - Fh = 0;\quad a = \frac{F}{P}h$$

From the first equation (1) we see that at rest the force of friction $F_{\text{тр}}$ is equal by modulo and opposite by direction to the active force ***F.***

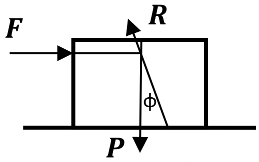

Fig.2

The last equation (1) shows that the force F causes the displacement of the point K of application of reaction ***R*** towards the force ***F***. Thus, the body at rest is under the action of two force couples: {**F**,**Fтр**} and {**P**,N}. They are equal by modulo and opposite by direction.

On the other hand, the body is in equilibrium under the action of three forces: ***F, R, P***. It means that they all intersect at a single point (fig. 2).

Increase of the driving force ***F*** increases the force of friction $F_{\text{тр}}$. At a certain value of the driving force ***F*** the body starts to slide over the surface. This means that at the limit state of equilibrium module of the friction force reaches the limit value $F^*_{\text{тр}}$.

Friction angle.

Experience shows that, at first approximation, the limit module of the force of friction $F^*_{\text{тр}}$ depends on the module N of normal reaction under ***Coulomb law***

$$F^*_{\text{тр}} = fN\qquad (2)$$

Constant *f* is called ***friction coefficient.***

Friction coefficient f is a dimensionless value, depending on the materials and the surfaces in contact (roughness, temperature, humidity, etc.). Coefficient of friction f is determined by experience. In a pretty wide range it does not depend on the area of the contact surfaces. In the references, the following values of the friction coefficients can be found: tree on tree 0.4 -0.7; metal on metal 0.25 - 0.15; steel on ice 0.027.

Thus, the module of friction force may vary within

$$0 < F_{\text{тр}} < F^*_{\text{тр}} = fN$$

Full reaction of support

$$\mathbf{R} = \mathbf{N} + \mathbf{F_{\text{тр}}}$$

makes with the normal some angle φ. Figures 1 and 2 show that

$$\text{tg}\varphi = \frac{F_{\text{тр}}}{N}$$

At the moment when the body starts to slip $F_{\text{тр}} = F^*_{\text{тр}}$ and angle φ reaches its maximum value ∝ called ***friction angle***

$$tg \propto = \frac{F^*_{\text{тр}}}{N} = f\qquad (3)$$

Tangent of friction angle ∝ is equal to the friction coefficient f.

At the rest of the body angle φ (Fig.2) may vary within

$$0 \le \varphi \le \propto$$

Consider a simple experiment, which allows practically measure the friction angle. Put the body on a turntable. Let us increase the angle of the tilt table to the value of $\varphi_{\max}$, in which the body will start to slide down.

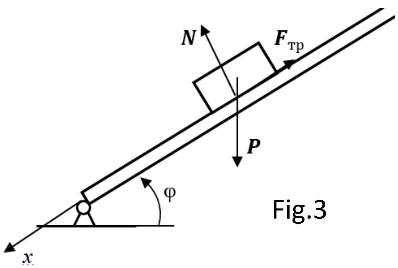

Fig.3

We will show that $\varphi_{\max}$ is equal to the angle of friction ∝.

Projecting on the x axis all the forces that act on the body, we obtain

$$P\text{Sin}\varphi_{max} - fP\text{Cos}\varphi_{\max} = 0$$

From here

$$tg\varphi_{max} = f = tg \propto;\quad \varphi_{max} \equiv \propto\qquad (4)$$

Friction cone. Self-locking.

Consider a weightless body on a horizontal rough plane with friction coefficient of f (Fig. 4). Construct a cone with a vertical axis and angle of friction $\propto$ at top two corners, and call it ***friction cone***.

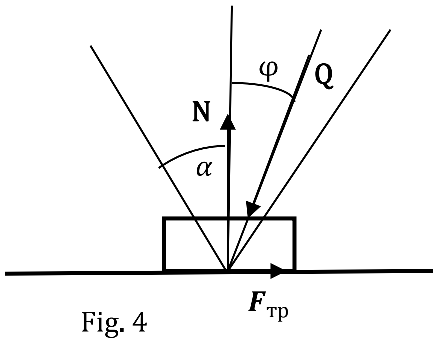

Fig. 4

Let us show that no any force **Q** applied to the body inside the friction cone $\varphi < \propto$ would shift the body. Shear force

$$F = QSin\varphi < QSin \propto = Qtg \propto Cos \propto < QfCos\varphi = Nf = F^*_{\text{тр}}$$

or

$$F < F^*_{\text{тр}} \quad (5)$$

for any value of force Q

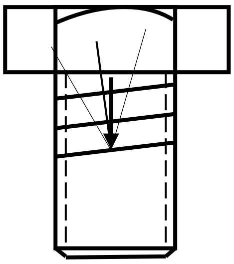

Fig. 5

It means that, if the force is applied inside the friction cone, the body remains at rest. This property is called the phenomenon ***self-locking***.

It is one of the many beneficial properties of friction. Without friction we would not be able to walk, the cars would not be able to move.

At the same time when we run up the icy hill, we intuitively try to shove from the ice at an angle that is less than the angle of friction.

All the screws (Fig. 5) have a thread angle less than the angle of friction of the screw on the nut, which prevents the unscrewing at vibration. The fine thread screws have the minimum angle of the thread, and the maximum self-locking.

Friction of rope on cylinder (Euler problem)

Let the cylinder be held by the rope, one end of which is tensioned by force $\boldsymbol{F_1}$ (fig. 6). We will define the force $\boldsymbol{F_2}$ at the other end of the rope which can counterbalance the force $\boldsymbol{F_1}$, if the angle of the rope contact is γ, and coefficient of friction of the rope on the cylinder is f.

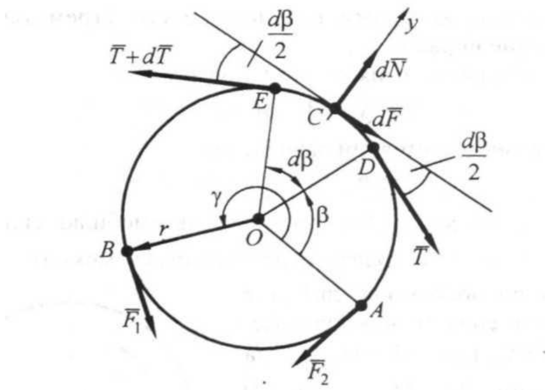

Fig.6

Consider the balance of elemental plot of rope DE of length $ds = rd\beta$, where *r* is the radius of the cylinder. In points D and E the tension of the rope will be $\boldsymbol{T}$ and $(\boldsymbol{T} + \boldsymbol{dT})$. The rope plot also undergo the force of normal pressure of cylinder $\boldsymbol{dN}$ and friction force $\boldsymbol{dF}$ at the middle point C.

The sum of projections of all forces on the tangent to the cylinder is zero.

$$dT = dF$$

The smallest value of force $\boldsymbol{F_2}$ will match the limit case of balance, i.e. the Coulomb's law

$$dT = fdN.$$

Projecting all forces on the normal y, we find

$$dN = (T + T + dT)sin\frac{d\beta}{2} = 2T\frac{d\beta}{2} = Td\beta$$

Now ***:***

$$dT = fdN = fTd\beta$$

Separate the variables and integrate over the equation from F_2 to F_1 and from 0 to γ:

$$\int_{F_2}^{F_1} \frac{dT}{T} = f\int_0^{\gamma} d\beta$$

Hence

$$ln\frac{F_1}{F_2} = f\gamma$$

or

$$F_2 = F_1 e^{-f\gamma} \qquad (6)$$

When f = 0, $F_2 = F_1$, as expected. Increasing the angle of γ, we can significantly reduce the force $\boldsymbol{F}_2$. For example, if f = 0.5 and γ = 4π: $F_2 = 0,002F_1$.

The significant reduction in the tension of the rope, wrapped on a pillar, is used for mooring ships to berth.

Capsizing

Suppose you want to move a wardrobe of weight P along the floor with coefficient of friction f, pushing it by horizontal force ***F*** (Fig. 7). Experience tells us that if we push the wardrobe at its legs, it will begin to slide on the floor. If we push the wardrobe on the linoleum (high friction) at high point, then the wardrobe will not slide, but will begin to keel over. Let us find out when and why this happens.

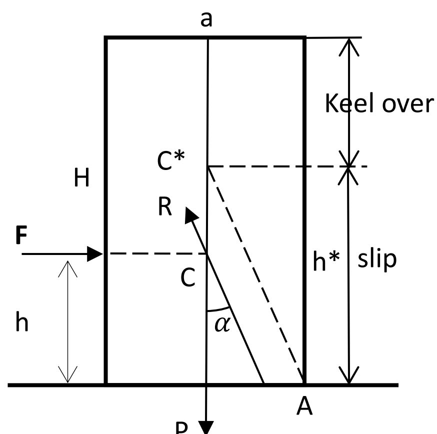

Fig.7

At the limit state of rest, before the start of keel over, the floor reaction ***R*** deviates from the vertical by angle of friction $\alpha = arctgf$. As is known, all three force ***P, F, R*** intersect at the same point C.

Picking up the point C of force application, we move the reaction ***R*** to the corner A, where it will come when

$$h^* = \frac{a}{2}\text{ctg}\,\alpha = \frac{a}{2f} \qquad (7)$$

Thus, the mode of the wardrobe movement after the limit equilibrium state depends on the ratio of the height of the wardrobe H and the distance h*::

1. If the coefficient of friction *f,* and thus the friction angle α are so small that

$$f < \frac{a}{2H}; \qquad (h^* = \frac{a}{2f} > H)$$

the wardrobe will slip forward at any h

2. With big enough coefficient of friction ($f > \frac{a}{2H}$ и $H > h^*$) there are two sections of the wardrobe:

If we apply the force ***F*** at $h < h^*$ the wardrobe will start transition.

The limit equilibrium state before the rollover is at $h = h^*$ when moments of forces ***F*** and ***P*** about the point A are equal:

$$Fh^* = P\frac{a}{2}; \qquad \mu_{max} = \frac{F}{P} = \frac{a}{2h^*} = f$$

We find another practical way to determine the coefficient of friction *f* by measuring the distances a and h^*.

At rollover when $h > h^*$ the force ***F*** creates unbalanced moment $F(h - h^*)$ about the point C*, and the wardrobe begins to rotate around the point A.

Thus, if $f > \frac{a}{2H}$ tilting the wardrobe is easier the higher it is (the smaller is the coefficient μ). Force ***F*** will be minimal, if it is applied at H. Then:

$$\mu_{min} = \frac{F}{P} = \frac{a}{2H}; \qquad \frac{a}{2H} < \mu < f$$

Rolling

Let us consider the change of coefficient μ with increasing the number n of sides of isosceles section of the body. Denote the length of the side by a (fig. 9). Assume that friction of the body on the surface is strong enough to let us roll the body by horizontal force ***F:***

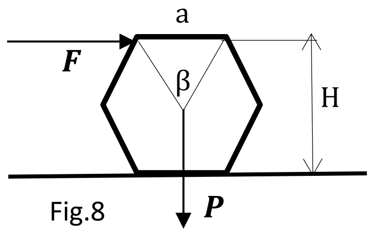

Fig.8

$$f > \frac{a}{2H}$$

From Figure 8 we find

$$\frac{a}{H} = tg\left(\frac{\beta}{2}\right) = \text{tg}\frac{\pi}{n}$$

Hence

$$\mu_{min} = \frac{1}{2}\text{tg}\frac{\pi}{n}$$

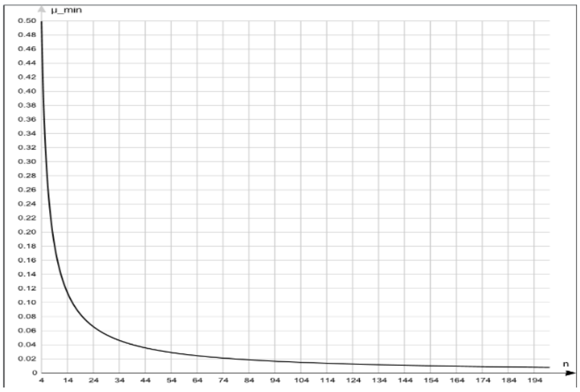

Fig.9

Wheel

It is clear that if the n → ∞, i.e. making section in circle μ_{min} and force ***F*** tend to zero. We come to the obvious conclusion that it is the easiest to roll the body with round section.

Dependence of μ_{min} (n) is shown in Figure 10.

Internal friction

The real body is not absolutely solid. When a wheel is rolling on the road, both the wheel and the road are being deformed. This deformation is viscoelastic, that is accompanied by internal friction. The result is a resistance to movement of the wheel on the road. Its nature can be understood if we consider the limit state of wheel at rest before it starts to drive under the action of a moment (***driving wheel***) or a central force (***driven wheel***).

Moment and coefficient of rolling friction of a soft wheel.

First, let us examine the resistance to rotation of a soft drive wheel on absolutely solid horizontal road caused by internal friction of the wheel's material. This may be a flat drive wheel of the car on asphalt (Fig.10).

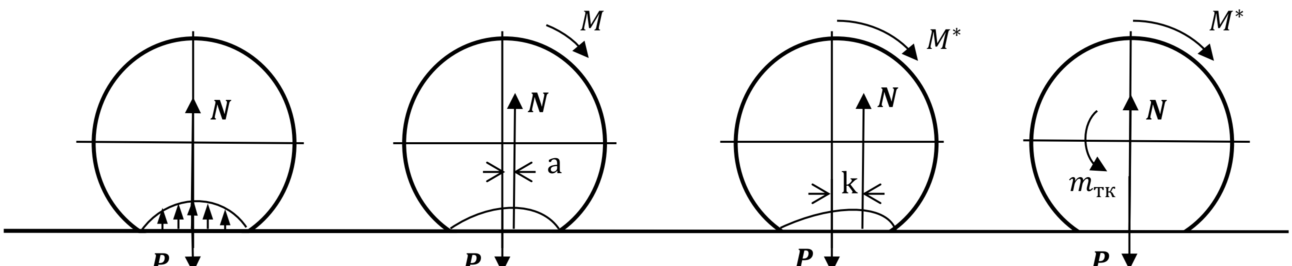

Fig.10 Fig.11 Fig.12 Fig.13

Distributed normal reactions of the road on the wheel and their resultant ***N*** are always normal to the road.

At the absence of the driving moment M plot of normal reactions is symmetrical, and the resultant N runs through the center of the wheel (Fig. 10).

If we apply to the wheel a small moment M, then the resultant ***N*** remaining vertical, moves towards the moment action by some amount *a* (Fig. 11). Just as this happens at rollover.

If we increase the moment M up to a certain value M*, corresponding to the limit state of the wheel at rest before it start to rotate offset *a* reaches the maximum value *k*, which is called the ***coefficient of rolling friction*** (Fig. 12).

At this point force of gravity ***P*** and reaction ***N*** form a force pair with moment

$$m_{тк} = Nk$$

called ***moment of rolling friction***.

Instead of the picture (Fig. 12) sometimes we draw an equivalent picture (Fig. 13).

It should be noted that the center of the wheel will start moving only when the friction force appears after the moment M exceeds the limit value of M*. Until that there is no force of friction.

Driven wheel

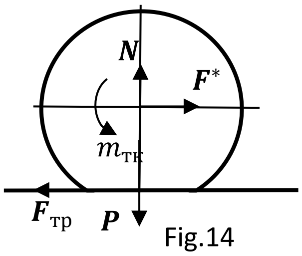

Fig.14

Picture of acting forces at the limit state of rest of the wheel is on the Figure 14. Contrary to the driving wheel the friction force $\boldsymbol{F}_{тр}$ is always present and together with the limit driving force $\boldsymbol{F}^*$ creates the turning couple $\{\boldsymbol{F}^*\boldsymbol{F}_{тр}\}$..

It should be noted that the wheel may not start to rotate, if the grip is not good enough. In this case, the friction force $\boldsymbol{F}_{тр}$ reaches the limit value $F^*_{тр} = fN$ before it does the *rolling friction moment*.

$$fNr < Nk \quad \text{or} \quad fr < k$$

The wheel will start to move in transition, without rotating. So, a flat driven wheel will be skidding, moved by the driving wheels. Spin resistance has nothing to do with translation of the body, only with its rotation.

Force and coefficient of resistance of a soft road

Consider the resistance of internal friction in a soft road to translation of an absolutely solid wheel. First consider the driven wheel, since the central driving force ***F*** is always able to overcome

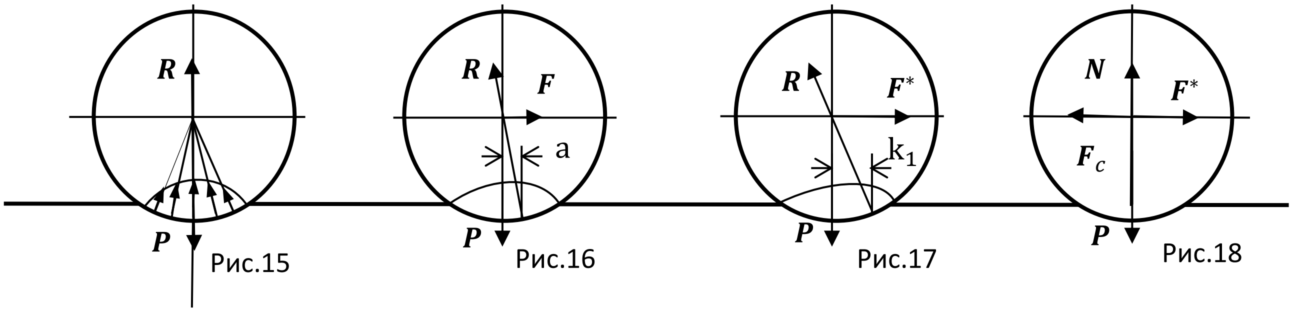

Рис.15 Рис.16 Рис.17 Рис.18

any resistance of the road.

Distributed normal reactions of the road and their resultant ***R*** always pass by the center of the wheel.

In the absence of the driving force ***F*** the plot of normal reactions is symmetrical, and the resultant $\boldsymbol{R} = -\boldsymbol{P}$ is vertical (Fig. 15).

If we apply a small central driving force ***F***, then the wheel will stay at rest, and the point of application of reaction ***R*** moves to a certain value *a* in direction of ***F*** (Fig. 16).

If we increase the force ***F*** to a certain limit $\boldsymbol{F}^*$ corresponding to the limit stay of rest of the wheel before moving the gap *a* reaches its maximum value k_1 (Fig. 17), which is called ***coefficient of road resistance***.

Meanwhile the vertical component ***N*** of reaction ***R*** balances the force of gravity of the wheel ***P***. Horizontal component $\boldsymbol{F}_c$ of reaction ***R*** , called ***force of road resistance***, has a module

$$F_c \cong N\frac{k_1}{r} \quad (8)$$

and balances the limit driving force $\boldsymbol{F}^*$. Instead of picture (Fig. 17) we often draw the figure (Fig. 18).

It is remarkable that the resistance force is inversely proportional to the radius of the wheel. This is why the off-road vehicles have large wheels.

In addition, not knowing the formule (8), peoples of the North have always made a maximum radius of curvature of the strips at the leading end of the sleds. Because the resistance force (8) has nothing to do with rotation, it is associated only with its transition movement.

It should be noted that the driven wheel will start rotating only when the friction force $\boldsymbol{F}_{тр}$ appears after the driving force ***F*** exceeds its limit $\boldsymbol{F}^*$.

Driving wheel

In the limit state of rest before the center of the driving wheel starts to move the force picture will look like in Figure 19.

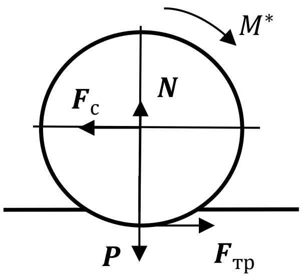

Fig.19

Unlike in the driven wheel, the friction force $\boldsymbol{F}_{\text{тр}}$ appears with any value of the driving moment M. At the rest forces $\boldsymbol{F}_{\text{c}}$ and $\boldsymbol{F}_{\text{тр}}$ make a couple of resistance $\{\boldsymbol{F}_{\text{c}}\boldsymbol{F}_{\text{тр}}\}$, balancing the driving moment M.

It should be noted that the center of the driving wheel can stay at rest if the grip with the road is not good enough. In this case, the friction force $\boldsymbol{F}_{\text{тр}}$ reaches the limit value of $F_{\text{тр}}^* = fN$ before it does the resistance force$\boldsymbol{F}_{\text{c}}$. The wheel starts to slip on the spot. So may happen when we drive in deep snow.

Ccondition of lack of slippage

$$fN < N\frac{k_1}{r}$$

or

$$fr < k_1$$

In general case, when both the wheel and the road are deformable we have both ***moment of rolling friction*** $m_{\text{тк}}$, and the ***force of road resistance*** of the road $\boldsymbol{F}_{\text{c}}$. Picture of the applied forces at an extreme state of rest before moving is presented in Figure 20. The resultant equivalent is in Figure 21.

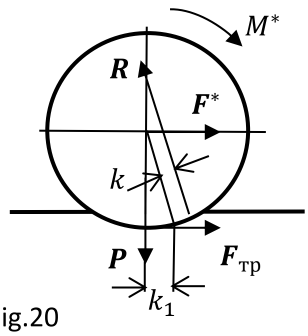

Fig.20

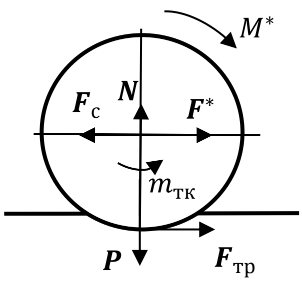

Fig.21