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Abstract

A simple derivation of the Euler formula for the angular velocity of a rigid body is formulated.

A matrix method for solving problems of kinematics of resultant motion of a particle and plane motion of a rigid body is proposed.

Sets of interactive excel problems on the kinematics of a particle, resultant motion of a particle and plane motion of a body are presented. They allow the student to independently correct errors noted by the program at each step of the solution, and freeing the teacher from the work of checking the task.

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Kinematics of particle

Kinematics- part of Mechanics that explores the ways of description of the motion of a particle and rigid body. The motion is studied over time and relative to some reference frame - "hard" 3D-space in which the observer is able to measure the distances and time. The reference frame may have multiple coordinate systems, but they will all belong to the same reference system.

Time *t* is a scalar, monotonically increasing value from $t = 0$ known as ***initial moment***. In classical mechanics, time is considered to be the same in all frames of reference.

Methods of description of particle motion.

To describe the particle motion means to determine the position of the particle in space at any given moment of time. Let us consider three ways of description of the particle motion: ***vector, coordinate and natural***.

Vector method.

This is the main method, since most of the motion characteristics are vector quantities. The particle position in the reference frame at a given moment of time *t* is given by the radius vector of the particle (Fig. 1).

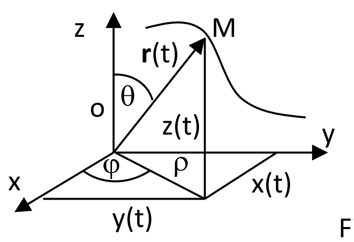


Fig.1

Vector-function $\boldsymbol{r}(t)$ of scalar argument *t* is a ***vector law*** of motion. Direction and module of radius-vector change over time and the particle M moves along the curve called a ***trajectory*** of the particle.

Hodograph of the vector function is a curve described by the vector end while the scalar argument changes, and the beginning of the vector is fixed. It is clear that the hodograph of the radius-vector of the particle is its trajectory.

Coordinate method

If we link a system of coordinates, for example, the Cartesian to the reference frame then the radius-vector can be described by its projections

$$x(t), y(t), z(t) \qquad (1)$$

the law of M motion in Cartesian coordinates. Motion equations (1) describe the particle trajectory (with time *t* parameter). If we eliminate the parameter *t*, we will get the equation of the trajectory:

$$f_1(x, y, z) = 0; \qquad f_2(x, y, z) = 0$$

The trajectory is the portion of this curve, which corresponds to the $t > 0$

In *cylindrical coordinates* (Fig. 1) the law of motion is

$$\rho(t), \quad \varphi(t), \quad z(t) \qquad (2)$$

In *spherical* coordinates (Fig. 1)

$$r(t), \ \varphi(t), \ \theta(t) \qquad (3)$$

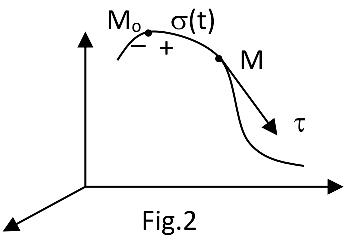


Fig.2

Natural method

It is useful when the trajectory of the particle is known beforehand (Fig. 2). Rails, for example, specify the path of the tram, so here we use the natural method.

To describe the position of the particle on the trajectory at a given moment of time we would need indicate on the trajectory a start point M_o, the positive direction (+), and the function of the ***curvilinear coordinate*** $\sigma(t)$ – the arc M_oM length with the appropriate sign.

Fig.3

It is convenient to choose the position of the particle at the initial moment $t = 0$, as a start point M_o and direction of its motion as a positive-direction.

Function σ(t) iscalled ***natural law of motion***. We should not confuse it with the path $s(t)$ that is a monotonically increasing function. Meanwhile coordinate $\sigma(t)$ can change its sign and go to zero. As for the tram, returning to the depot, coordinate $\sigma(t)$becomes zero, while the path $s(t)$ reaches its maximum value (Fig. 3).

Derivative of vector functions by scalar argument

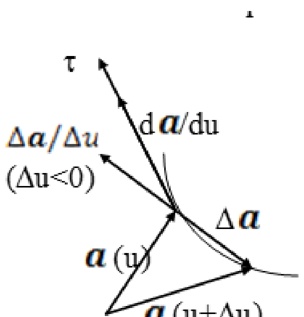


Рис.4

Consider a vector function $\boldsymbol{a}(u)$ of scalar argument u. With the change of the scalar u the end of the vector ***a*** describes a hodograph (Fig 4). An increment Δu of parameter u causes increment $\Delta\boldsymbol{a}$ of vector-function. Let the argument decrease

(Δ u < 0). Then vector $\Delta\boldsymbol{a}/\Delta u$ is directed opposite to the increment of $\Delta\boldsymbol{a}$. We call ***derivative of vector functions by scalar argument*** vector

$$\frac{d\boldsymbol{a}}{du} = \lim_{\Delta u \to 0} \frac{\Delta\boldsymbol{a}}{\Delta u} \qquad (4)$$

While the increment Δu is tending to zero the secant $\Delta\boldsymbol{a}$ is tending to tangent position. Thus, the vector derivative is always tangential to hodograph of vector-function.

Consider the basic properties of the vector derivative.

The derivative of vector-constant function is zero:

$$\boldsymbol{a} = \boldsymbol{Const} \rightarrow \qquad \frac{d\boldsymbol{a}}{du} = \boldsymbol{0} \qquad (5)$$

The derivative of vector function, constant by its module, is not zero since the vector still changes its direction. As to the derivative direction, since the hodograph of the function rests on the sphere of radius a, so the derivative is tangent to the vector itself.

$$a = Const \;\; \rightarrow \qquad \frac{d\boldsymbol{a}}{du} \perp \boldsymbol{a} \quad (6)$$

Such vectors are, for example, the vectors, connecting any two particles of a rigid body.

Next, go the properties deduced from linearity of differentiation operator

$$\frac{d}{du}(\boldsymbol{a}+\boldsymbol{b}) = \frac{d\boldsymbol{a}}{du} + \frac{d\boldsymbol{b}}{du}$$

$$\frac{d}{du}(\lambda\boldsymbol{a}) = \lambda\frac{d\boldsymbol{a}}{du}$$

$\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\boldsymbol{a}}{du}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\boldsymbol{b}}{du}$ (the order of factors can be changed!)

$\frac{d}{du}(\boldsymbol{a}\times\boldsymbol{b}) = \frac{d\boldsymbol{a}}{du}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{d\boldsymbol{b}}{du}$ (the order of factors cannot be changed!)

We will prove the most practically important property:
Projection of a derivative is equal to the derivative of the corresponding projection

$$\left(\frac{d\boldsymbol{a}}{du}\right)_x = \frac{d\boldsymbol{a}_x}{du} \qquad (7)$$

We will present the vector by its projections on the axis x, y, z with unit-vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

Take the time derivative, given that unit-vectors are constant:

$$\frac{d\boldsymbol{a}}{du} = \frac{da_x}{du}\boldsymbol{i} + \frac{da_y}{du}\boldsymbol{j} + \frac{da_z}{du}\boldsymbol{k}$$

On the other hand, you can submit derivative also via its projection

$$\frac{d\boldsymbol{a}}{du} = \left(\frac{d\boldsymbol{a}}{du}\right)_x\boldsymbol{i} + \left(\frac{d\boldsymbol{a}}{du}\right)_y\boldsymbol{j} + \left(\frac{d\boldsymbol{a}}{du}\right)_z\boldsymbol{k}$$

Comparing the two expansions, we conclude that property 7 is right. In Mechanics, for the sake of brevity the time derivative we mark by a dot over the letter:

$$\frac{d\boldsymbol{a}}{dt} \equiv \dot{\boldsymbol{a}}$$

Velocity and acceleration of particle with vector and coordinate methods

Vector method

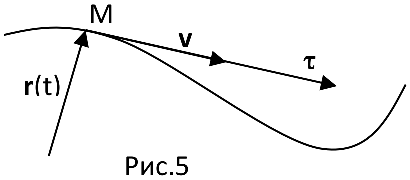


Рис.5

Velocity

Particle velocity and acceleration are vector quantities, so let's define them in vector method.

We call the ***velocity*** of the particle vector

$$\boldsymbol{V} \equiv \frac{d\boldsymbol{r}}{dt} \equiv \dot{\boldsymbol{r}} \qquad (8)$$

From the definition, it follows that the velocity is tangential to the hodograph of radius-vector **r**, i.e. to the trajectory of a particle. Velocity is directed toward the sense of motion of the particle on its trajectory.

Acceleration

We call **acceleration of the particle** vector

$$\boldsymbol{W} = \frac{d\boldsymbol{V}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} = \ddot{\boldsymbol{r}} \qquad (15)$$

Note that if the velocity of the particle is constant by module (uniform motion), the acceleration is normal to the velocity as the vector derivative. This will be confirmed in the natural method.

Coordinate method

Velocity

Deriving

$$\boldsymbol{r}(t) \;=\; x(t)\,\boldsymbol{i} \;+\; y(t)\,\boldsymbol{j} \;+\; z(t)\,\boldsymbol{k}$$

by time we get

$$\boldsymbol{V} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \qquad (9)$$

Thus, from the law of motion x (t), y (t) z (t) we can find the vector $\boldsymbol{V}$

$$V_x = \dot{x}, \quad V_y = \dot{y}, \quad V_z = \dot{z}; \qquad V = \sqrt{V_x^2 + V_y^2 + V_z^2};$$

$$Cos(x,\boldsymbol{V}) = \frac{V_x}{V}; \quad Cos(y,\boldsymbol{V}) = \frac{V_y}{V}; \quad Cos(z,\boldsymbol{V}) = \frac{V_z}{V}$$

Acceleration

Given the law of motion, we use the derivative properties to find the projections of acceleration vector

$$W_x = \dot{V}_x = \ddot{x}, \;\; W_y = \ddot{y}, \;\; W_z = \ddot{z} \qquad (16)$$

module and direction of the acceleration vector:

$$W = \sqrt{W_x^2 + W_y^2 + W_z^2};$$

$$Cos(x,\boldsymbol{W}) = \frac{W_x}{W}; \quad Cos(y,\boldsymbol{W}) = \frac{W_y}{W}; \quad Cos(z,\boldsymbol{W}) = \frac{W_z}{W} \qquad (17)$$

Velocity and acceleration of particle with natural method.
Frenet Formulas

Given the law of particle motion on its trajectory

$$\sigma(t)$$

It is obvious that the radius-vector of the particle is a function of the coordinate σ: $\boldsymbol{r}\ (\sigma)$. Frenet formulas define the ***natural basis*** of the three orthogonal unit vectors $\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b}$, via derivatives:

The ***1st Frenet formula*** specifies the ort of the ***tangent***

$$\boldsymbol{\tau} = \frac{d\boldsymbol{r}}{d\sigma} \qquad (10)$$

Direction.

It is tangent to the trajectory, as a derivative of the radius-vector.
Directed in positive direction of σ, regardless of the particle motion direction (sign of dσ).
Even if d**r** is directed to the start particle Mo, dσ is negative and the derivative is directed toward positive sense of σ
The module of the derivative is equal to 1 as the limit of the relationship of the chord to the arc $\Delta r\ /\Delta\sigma$

The ***2nd Frenet formula*** specifies the ort of the ***main normal n***

$$k\,\boldsymbol{n} = \frac{d\boldsymbol{\tau}}{d\sigma} \qquad (11)$$

Direction:

$\frac{d\tau}{d\sigma}$ is normal to **τ** as a derivative of a vector with permanent module. It specifies the direction of motion of the end of **τ** when the particle M moves.
$\frac{d\tau}{d\sigma}$ *is* directed toward the concavity of the trajectory. Even if the particle moves to the start point Mo ($d\sigma < 0$) and d**τ** is directed toward the bulge of trajectory, the derivative $\frac{d\tau}{d\sigma}$ still is directed toward a concavity in view of negativity of dσ.
Module k of the derivative $\frac{d\tau}{d\sigma}$ *is* called the curvature of the trajectory at the particle M.

Reciprocal value

$$\rho = \frac{1}{k}$$

is called the ***radius of curvature*** of the trajectory at the point M.

Ort ***b*** of ***binormal*** is directed so that $\boldsymbol{\tau},\ \boldsymbol{n}, \boldsymbol{b}$ would be right

$$\boldsymbol{b} = \boldsymbol{\tau} \times \boldsymbol{n} \qquad (12)$$

We call the plane (**τ, n)** the ***tangent plane*** to the path at M. The ***tangent plane*** can be obtained as a limit position of the plane of the circle through three points of M_0 M and M_1 on the trajectory when M_0 and M_1 tend to M. The limit value of the radius of such circle strives toward the radius of curvature ρ.

Velocity

Deriving

$$\boldsymbol{V} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{d\sigma}\frac{d\sigma}{dt} = \dot{\sigma}\boldsymbol{\tau} \qquad (13)$$

Thus

$$\boldsymbol{V} = V_\tau \boldsymbol{\tau}, \qquad V_\tau = \dot{\sigma} \qquad (14)$$

As we see, the velocity is tangent to the trajectory, and its projection onto the tangent is equal to the first derivative of law of motion

Acceleration

Deriving

$$\boldsymbol{W} = \frac{d\boldsymbol{V}}{dt} = \ddot{\sigma}\boldsymbol{\tau} + \dot{\sigma}\dot{\boldsymbol{\tau}} = \ddot{\sigma}\boldsymbol{\tau} + \dot{\sigma}\frac{d\boldsymbol{\tau}}{d\sigma}\frac{d\sigma}{dt} = \ddot{\sigma}\boldsymbol{\tau} + \frac{\dot{\sigma}^2}{\rho}\boldsymbol{n} \qquad (18)$$

Thus, the acceleration of the particle has two components (Fig.6) ***tangent and normal***

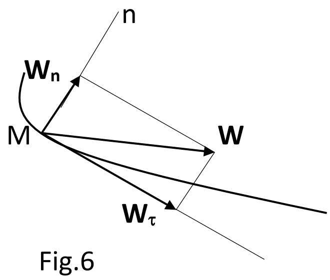


Fig.6

$$\boldsymbol{W} = \boldsymbol{W}_{\boldsymbol{\tau}} + \boldsymbol{W}_{\boldsymbol{n}} \qquad \boldsymbol{W}_{\boldsymbol{\tau}} = \ddot{\sigma}\boldsymbol{\tau} \qquad \boldsymbol{W}_{\boldsymbol{n}} = \frac{\dot{\sigma}^2}{\rho}\boldsymbol{n} \qquad (19)$$

$$W = \sqrt{W_\tau{}^2 + W_n{}^2}$$

Uniform motion is called the case of constant velocity module:

$$V = Const \qquad (\dot{\sigma} = Const).$$

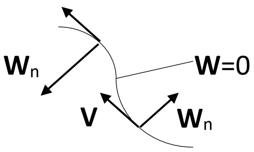


Fig.7

Under uniform motion (Fig. 7) the tangent acceleration is zero. Thus, the tangent acceleration $\boldsymbol{W}_{\boldsymbol{\tau}}$ characterizes the ***change of velocity module***.

Full acceleration is normal to the velocity. It disappears in the inflection points of the trajectory and is zero when the particle in on a straight line. Therefore, normal acceleration $\boldsymbol{W}_{\boldsymbol{n}}$ characterizes the ***change of direction*** of the velocity vector.

As is known, the acceleration is created by a force. This can be the active force or the force of reaction. When tramway is turning its normal acceleration created by the rails' reaction depends on the radius of curvature of the trajectory.

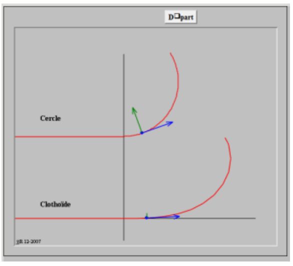


Fig.8

If we dock the straight stretch of rails with the rails of radius R, the normal acceleration of tram (and of passengers) instantly changes from zero to full scale value (the upper curve in Fig 8). The same way changes the reaction of the rails. The final and instant change of force is called a ***hit***.

The passengers also feel this shock. To avoid the shock, the curvature radius of the rail on the turn decreases smoothly (the lower curve in Fig.8).

Uniform accelarate motion is a motion with constant tangent acceleration:

$$\ddot{\sigma} = Const = W_\tau$$

Integrating, we get:

$$\dot{\sigma} = W_\tau t + C_1 \qquad (20)$$

where C_1 is the constant of integration, which should be found from the initial conditions:

$$t = 0: \quad \sigma = \sigma_0 \quad \dot{\sigma} = V_0 \qquad (21)$$

Find: $C_1 = V_0$ Repeated integration gives the law of the motion of the particle along the curve:

$$\sigma = W_\tau \frac{t^2}{2} + V_0 t + \sigma_0 \qquad (22)$$

Example of solving a problem on the kinematics of particle

The particle moves in the xy plane according to the law

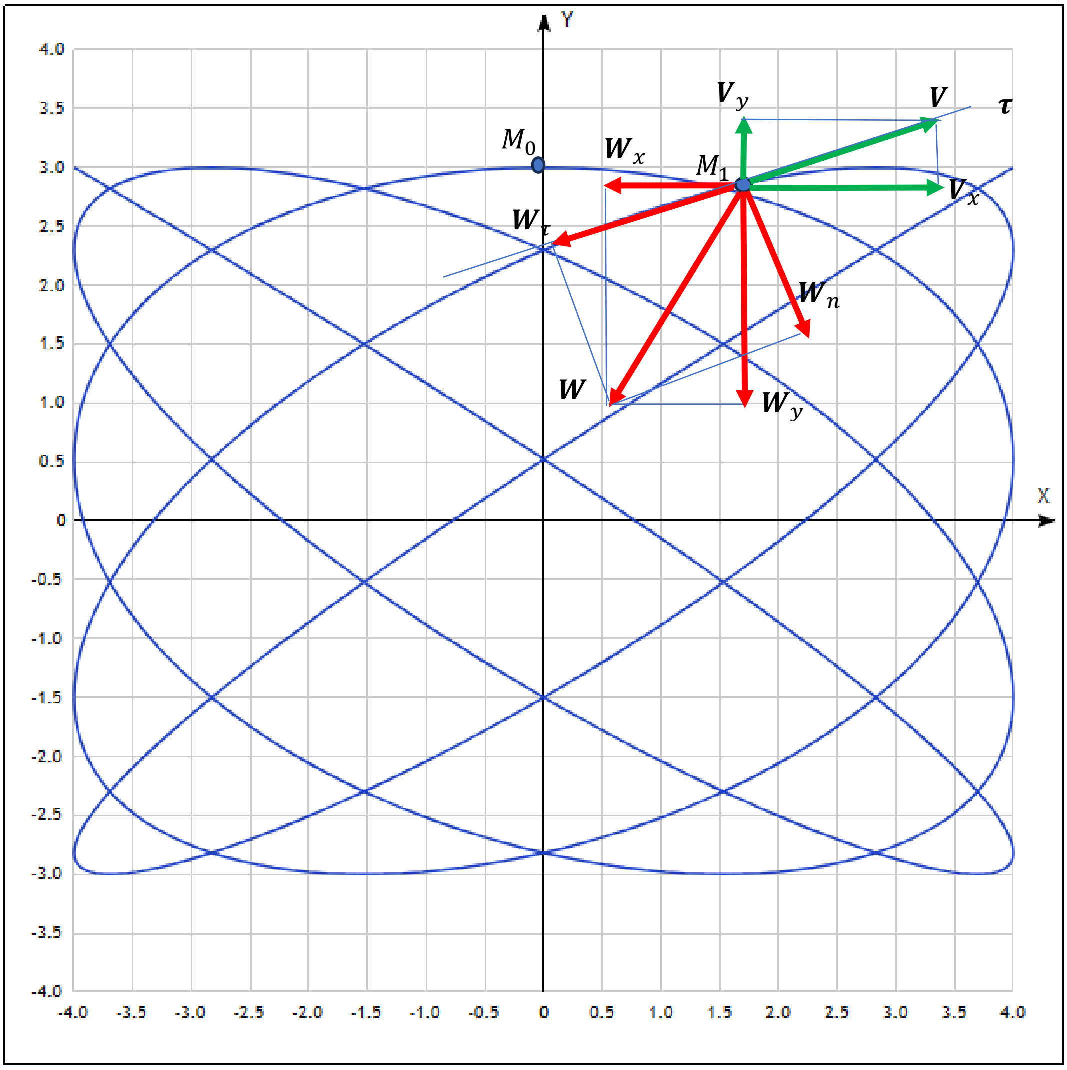
$$x = 4Sin\left(\frac{3\pi t}{4}\right), \; y = 3Cos\left(\frac{2\pi t}{3}\right)$$

Find the velocity and acceleration of the particle, the radius of curvature of the trajectory at the moment $t_1 = 1c$

Solution

Initial position at t=0 $\qquad x(0) = 0м, \quad y(0) = 3м$

At $t_1 = 2{,}8c$ $\qquad x(2{,}8) = 1{,}24м, \quad y(2{,}8) = 2{,}74м$



$x(t) = 4sin(\frac{3\pi t}{4}) \quad y(t) = 3cos(\frac{2\pi t}{3})$

Velocity

$$V_x = \dot{x} = 3\pi Cos\left(\frac{3\pi t}{4}\right)_{t=2,8} = 8{,}96м/с; \qquad V_y = \dot{y} = -2\pi Sin\left(\frac{2\pi t}{3}\right)_{t=2,8} = 2{,}56м/с$$

$$V = \sqrt{V_x{}^2 + V_y{}^2} = 9{,}3м/с \quad (24)$$

Acceleration

$$W_x = \ddot{x} = -\frac{9\pi^2}{4} Sin\left(\frac{3\pi t}{4}\right)_{t1} = -\frac{6{,}86м}{с^2};$$

$$W_y = \ddot{y} = -\frac{4\pi^2}{3} Cos\left(\frac{2\pi t}{3}\right)_{t1} = -12{,}02\text{м}/\text{с}^2$$

$$W = \sqrt{{W_x}^2 + {W_y}^2} = \frac{13{,}8\text{м}}{\text{с}^2} \qquad (26)$$

$$W_\tau = \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{V} = -9{,}9\text{м}/\text{с}^2 \qquad W_n = \sqrt{W^2 - {W_\tau}^2} = \frac{9{,}6\text{м}}{\text{с}^2}$$

$$\rho = \frac{V^2}{W_n} = 9\text{м} \qquad (28)$$

Excel assignments on particle kinematics can be downloaded from the link https://disk.yandex.ru/d/KCqL8Qh3170FHQ

Kinematics of rigid body

in development of Mikhail Valentinovich Mironov ideas

Euler's Formula.
Angular velocity of body

We will name **body vector** any vector ***a***, connecting two points of the body. All body vectors are constant in module and change only their directions, turning with the body. It is clear that the column of projections of vector ***a*** on the axis of stationary system of coordinates

$$a = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (1)$$

can be associated with the column of projections of its derivative

$$\dot{a} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad (2)$$

by 3 x 3 matrix Ω in the countless ways

$$\dot{a} = \Omega a \qquad \Omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \qquad (3)$$

We are interested in whether there exist among these matrixes one matrix Ω common to all body vectors, in other words, describing the movement of the entire body. As is known, the time derivative of a vector with permanent modulo is perpendicular to the vector. It means for an arbitrary body vector ***a***:

$$\boldsymbol{a} \cdot \dot{\boldsymbol{a}} = a^T \dot{a} = a^T \Omega\, a = 0 \qquad (4)$$

So

$$(x \quad y \quad z) \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= x^2\omega_{11} + y^2\omega_{22} + z^2\omega_{33} + xy(\omega_{12} + \omega_{21}) + yz(\omega_{23} + \omega_{32}) + zx(\omega_{31} + \omega_{13}) = 0 \qquad (5)$$

Matrix Ω will be independent of body vector, if all coefficients standing by the projections of the vector are equal to zero.

$$\omega_{11} = \omega_{22} = \omega_{33} = 0 \qquad (6)$$

$$\omega_{21} = -\omega_{12} = \omega_z \qquad \omega_{32} = -\omega_{23} = \omega_x \qquad \omega_{13} = -\omega_{31} = \omega_y$$

Denote the three non-zero elements of the matrix as in the attached matrix of the vector in the right oriented space.

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \qquad (7)$$

The elements of the matrix Ω have a simple geometrical meaning. They present the projections of the velocity of the end of the first index ort in its rotation around the second index ort on the third axis in right oriented space.
So

$$\omega_{12} = -\omega_z \qquad (8)$$

It means that the end of ort ***i*** moves against the z axis when it rotates around the y axis. It is clear why the elements with duplicate indices are zero.
Thus, common for the all the body-vectors common matrix Ω exists and it is skew-symmetric. Let's call it the ***matrix of angular velocity*** of the body. With its three elements, we can build of the column of projections of body's ***angular velocity vector***.

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad (9)$$

Thus, we come to the ***Euler formula in a matrix*** form

$$\dot{a} = \Omega a \qquad (10)$$

which corresponds to the ***vector Euler's formula***

$$\dot{\boldsymbol{a}} = \boldsymbol{\omega} \times \boldsymbol{a} \qquad (11)$$

Formula (11) shows that the time derivatives of all vectors in the body are expressed through a single and common angular velocity vector.

Theorem on distribution of velocities in a rigid body
Pole method.

Euler's formula gives the opportunity to express the characteristics of movement for all points of the body by the same characteristics of one, specially chosen point of the body, called ***a pole***. This method is called ***method of pole***.

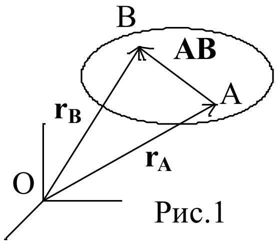


Рис.1

Consider an arbitrary point B in the body. The main in method of pole is the expression of the radius-vector of an arbitrary point of the body by the radius-vector of the pole A:

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{AB} \qquad (12)$$

Differentiating (12) in time, we find

$$\frac{d\boldsymbol{r_B}}{d\boldsymbol{t}} = \frac{d\boldsymbol{r}_{\mathrm{A}}}{d\boldsymbol{t}} + \frac{d\boldsymbol{AB}}{d\boldsymbol{t}}; \qquad \boldsymbol{V}_B = \boldsymbol{V}_A + \frac{d\boldsymbol{AB}}{d\boldsymbol{t}} \qquad (13)$$

To the body vector **AB** the Euler's formula must be applied

$$\frac{d\boldsymbol{AB}}{d\boldsymbol{t}} = \boldsymbol{\omega} \times \boldsymbol{AB} \qquad (14)$$

So, we come to the ***theorem of velocities*** in a solid

$$\boldsymbol{V}_B = \boldsymbol{V}_A + \boldsymbol{\omega} \times \boldsymbol{AB} \qquad (15)$$

Matrix form of this theorem in an arbitrary coordinate system has the form of:

$$V_B = V_A + \Omega(\mathrm{AB}) \qquad (16)$$

Consequences of the theorem

If velocities of two points A and B are equal the vector of angular velocity is parallel to AB. For example, in rotation of the body around a fixed axis the velocities of the points on this axis are equal to zero. Therefore, the angular velocity vector is parallel to the axis of rotation Z.

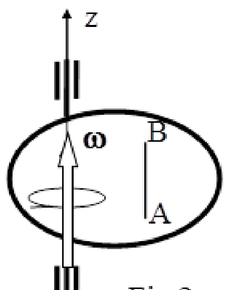


Fig.2

Usually, we place it on the axis (Fig. 2) and always direct it according to the rule of the right screw.
Inverse is also correct. The velocities of the points on the line, parallel to the angular velocity, are equal at this moment

$$if\ \boldsymbol{AB} \parallel \boldsymbol{\omega} \quad \rightarrow \quad \boldsymbol{V_B} = \boldsymbol{V_A} \qquad (4)$$

Theorem of projections. Projections of velocities of two points on the axis that passes through these points are equal. To prove the theorem design z axis that passes through both points. In view of the reciprocal perpendicularity $\boldsymbol{AB}$ and cross product $\boldsymbol{\omega} \times \boldsymbol{AB}$ we get:

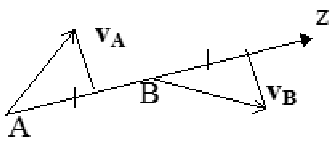


Рис.3

$$pr_{AB}\,\boldsymbol{V_A} = pr_{AB}\,\boldsymbol{V_B} \qquad (5)$$

This theorem is a natural requirement of that the distance between points of a rigid body should rest constant.

Example:
Find the velocities ratio for the points A and B of the rod of the crank mechanism. Point A belongs to crank OA, rotating around the axis and it moves on a circle, so its velocity is perpendicular to OA. Point B velocity is directed along line OB. With the theorem of velocity projections, we have

$$V_A Cos\alpha = V_B Cos\beta \qquad (6)$$

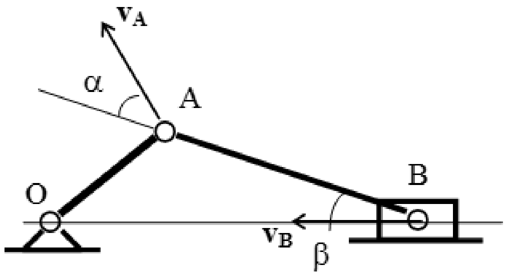


Рис.4

Translational motion of a rigid body

Instant translation is the movement when the body angular velocity instantly turns to zero

$\boldsymbol{\omega} = \boldsymbol{0}$ (1)

In this case,

$$\boldsymbol{V}_B = \boldsymbol{V}_A + \boldsymbol{\omega} \times \boldsymbol{AB} = \boldsymbol{V}_A = \boldsymbol{V} \qquad (2)$$

That means that at this moment the velocity of all points are equal. For example (Fig.1), at the moment when the crank OA ⊥ AB, ω = 0, velocity of points A and B are equal.

If the angular velocity is equal to zero for some period of time, the movement is called ***translation***. For example, the slide B (Fig. 1) moves at translation.

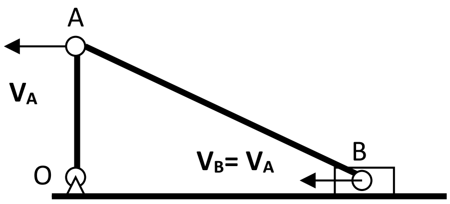


Fig.1

$$\frac{d\boldsymbol{AB}}{dt} = \boldsymbol{\omega} \times \boldsymbol{AB} \equiv \boldsymbol{0}$$

means that

$$\boldsymbol{AB} = \boldsymbol{Const} \quad (3)$$

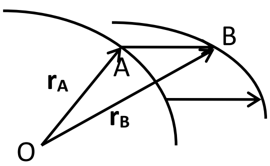


Fig.2

Thus, in translation any body-vector remains parallel to itself. The trajectory of any two points A and B are the same and are shifted to the constant vector **AB** (Figure 2).

Figure 3 shows a Ferris wheel, which cabin makes circular translation motion. All points of the cabin, including the points A and B are moving along the similar circles with AB centers offset.

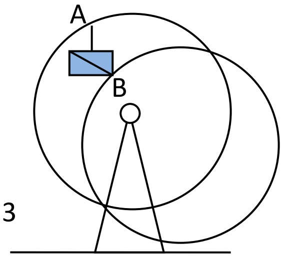


Fig.3

In general, all points of the body have different velocities, so the terms "velocity" and "acceleration" refer only to the point of the body, and the terms "angular velocity" and "angular acceleration" refer only to the body. Only at transition **V** can be called the velocity of the body (but better not to do so).

Differentiating (2), we find that at any given time accelerations are equal for all the points

$$\boldsymbol{W}_B = \boldsymbol{W}_A = \boldsymbol{W} \qquad (4)$$

The transition of the body is described by formulas of particle kinematics, since all points move the same way. As it is known, the movement of a particle in space is defined by three scalar functions of coordinates. Thus, in transition the body has 3 degrees of freedom.

Rotational motion
Angular velocity and angular acceleration of body.

Let the body rotate around a fixed z axis. It is convenient to describe the body position by angle of rotation (fig. 4)

$$\varphi = \varphi(t) \qquad (5)$$

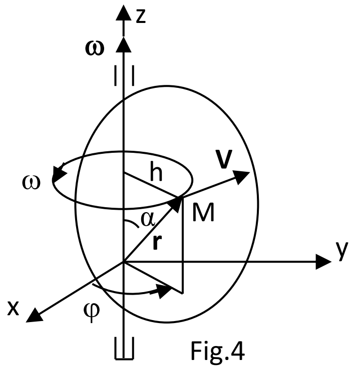


Fig.4

This is the ***law*** of body rotation. Thus, in rotational motion body has one degree of freedom.

As has been shown, the angular velocity **ω** of rotating body is directed along the axis of rotation. So $\omega_x = \omega_y = 0$ and matrix of angular velocity is:

$$\Omega = \begin{pmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let us find the projection ω_z of the angular velocity on axis z. The column of radius vector r projections

$$r = \begin{pmatrix} hCos\,\varphi \\ hSin\varphi \\ a \end{pmatrix} \qquad h = rSin\alpha \qquad a = rCos\alpha \qquad (6)$$

is connected with its derivative

$$\dot{r} = h\dot{\varphi}\begin{pmatrix} -Sin\,\varphi \\ Cos\varphi \\ 0 \end{pmatrix}$$

By Euler formula

$$\dot{r} = \Omega r = h\omega_z\begin{pmatrix} -Sin\,\varphi \\ Cos\varphi \\ 0 \end{pmatrix} \qquad (7)$$

We get

$$\omega_z = \dot{\varphi} \qquad (8)$$

Thus, the angular velocity is the velocity of change of the rotation angle φ . Hence the name of angular velocity. Body angular velocity vector is directed so that the right screw, rotating with the body, moves toward the axis z of rotation.

Angular acceleration of the body is the vector

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt} \qquad (9)$$

Since hodograph of vector $\boldsymbol{\omega}$ lays on the axis of rotation, the angular acceleration vector is directed along the axis of rotation. Differentiating (8) in time, we find:

$$\boldsymbol{\varepsilon} = \ddot{\varphi}\boldsymbol{k} = \varepsilon_z\boldsymbol{k};$$

Thus, projection of angular acceleration on z axis is equal to the second derivative of the law of rotation.

$$\varepsilon_z = \ddot{\varphi} \qquad (10)$$

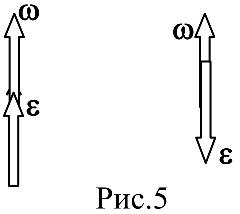


Рис.5

Accelerated is rotation with increasing angular velocity modulus. It is obvious that it will be the case if vectors of angular velocity and acceleration (left Fig. 5) have the same direction. Thus, the rotation will be accelerated if $\ddot{\varphi}\dot{\varphi} > 0$ 0 and decreasing when $\ddot{\varphi}\dot{\varphi} < 0$

Velocity and acceleration of a point of rotating body

According to the consequences from the theorem on velocity distribution the velocities of the points on the straight line parallel to the axis of rotation are equal. So, let's look at how the velocity is distributed on the cross-section line perpendicular to the axis of rotation.

Since the radius-vector of the point M is a body-vector, the velocity of the point comes from Euler's formula

$$\boldsymbol{V} = \dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r} \qquad (11)$$

Matrix form of this formula in any coordinate systems

$$V = \Omega\, r \qquad (12)$$

In accordance with Fig.4 velocity module V equals

$$V = \omega\, r\, Sin\, a = \omega h \qquad (13)$$

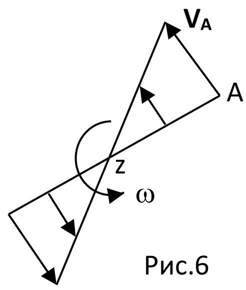


Рис.6

We see that the velocity module linearly depends on the distance h from the axis of rotation. Picture of the distribution of velocity on the line, perpendicular to the axis is presented in Fig. 6.

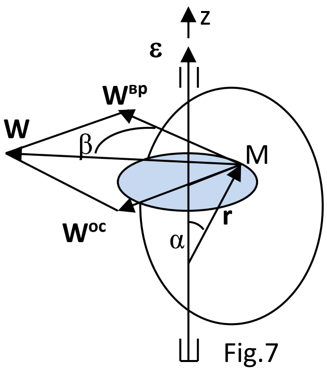


Fig.7

Let us find the acceleration of a point of rotating body. Differentiating (11) in time, we find

$$\boldsymbol{W} = \dot{\boldsymbol{V}} = \frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{r}) =$$
$$= \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}} = \boldsymbol{\varepsilon} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{V} \qquad (14)$$

Thus, the acceleration of a point of rotating body has two components (Fig. 7).

$$\boldsymbol{W} = \boldsymbol{W}^{\text{вр}} + \boldsymbol{W}^{\text{ос}} \qquad (15)$$

Componenet

$$\boldsymbol{W}^{\text{вр}} \equiv \boldsymbol{\varepsilon} \times \boldsymbol{r} \qquad (16)$$

is the tangents acceleration, but here it is called the ***rotational acceleration*** of the point. Special name introduced because not at all body movements $\boldsymbol{\varepsilon} \times \boldsymbol{r}$ is a tangent to the trajectory of the point (see spherical movement). Rotational acceleration points toward the angular acceleration **ε** arrow. Modulo

$$W^{\text{вр}} = \varepsilon r\, Sina = \varepsilon h$$

Second component

$$\boldsymbol{W}^{\text{ос}} \equiv \boldsymbol{\omega} \times \boldsymbol{V} \quad (17)$$

is directed to the axis of rotation, regardless of the direction of rotation (vector **ω**) and is therefore called the ***centripetal acceleration*** of the point. Vectors $\boldsymbol{\omega}$ and $\boldsymbol{V}$ change the direction together, so their vector product does not change its direction with changing of direction of body rotation. Modulo

$$W^{\text{ос}} = \omega V = \omega^2 h$$

The acceleration modulo W and the angle β that it makes with the direction to the axis:

$$W = \sqrt{W^{\text{вр}2} + W^{\text{ос}2}} = h\sqrt{\varepsilon^2 + \omega^4}; \qquad tg\beta = \frac{W^{\text{вр}}}{W^{\text{ос}}} = \frac{\varepsilon}{\omega^2}$$

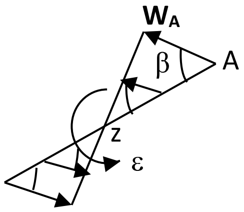


Fig.8

We see that the acceleration modulo linearly depends on the distance h of the point from the rotation axis, and that angle β is the same for all points of the body.

Now it is easy to draw a picture of distribution of accelerations in a rotating body. Since on the line parallel to the axis of rotation velocities are the same, then the same are accelerations. So, in all planes perpendicular to the axis of rotation, distributions of velocities and accelerations are the same. One of them is shown in Fig. 8.

Let us calculate the projections of a point M acceleration (Fig.7) on the axis rotating with the body by matrix method. Differentiating (12) by the time we get:

$$W = \dot{V} = \dot{\Omega} r + \Omega \dot{r} = \mathcal{E} r + \Omega\Omega r = (\mathcal{E} + \Omega^2) r \qquad (18)$$

Here Ɛ is a skew-symmetric matrix of angular acceleration

$$\mathcal{E} = \dot{\Omega} = \ddot{\varphi}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (19)$$

Plane motion of body

Motion law of a plane figure

The movement of the body is called plane, if the velocities of all its points remain parallel to some fixed plane. An example of such a movement is a cylinder rolling on the plane (Fig. 9). Velocity vectors of all points of the cylinder are parallel to the plane П. Multiplying velocity distribution formula

$$\boldsymbol{V}_B = \boldsymbol{V}_A + \boldsymbol{\omega} \times \boldsymbol{AB}$$

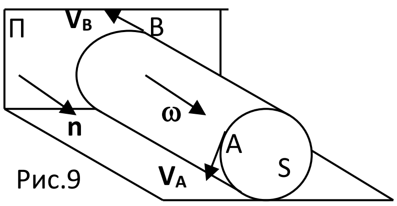


Рис.9

scalarly by the ort **n** of the normal to the plane П, we get:

$$0 = \boldsymbol{n} \cdot (\boldsymbol{\omega} \times \boldsymbol{AB}) = (\boldsymbol{n} \times \boldsymbol{\omega}) \cdot \boldsymbol{AB} \qquad (20)$$

Since $\boldsymbol{AB}$ – is an arbitrary body vector

$$\boldsymbol{n} \times \boldsymbol{\omega} = 0$$

Thus, in the plane motion angular velocity vector $\boldsymbol{\omega}$ stay parallel to **n**.

We know that the velocities on the line parallel to $\boldsymbol{\omega}$ are equal. Since $\boldsymbol{\omega}$ does not change its direction, it is true all the time. It means that acceleration is also the same.

So, there's no point in studying the distribution of velocities and accelerations in the entire body. It is enough to understand how they are distributed in the section S parallel to the plane of motion П.

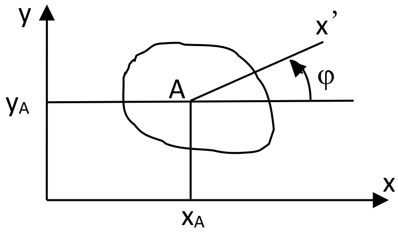


Fig.10

This section is called ***plane figure***. In all parallel sections the distribution of velocities and accelerations will be similar.

Usually, we combine the plane figure with the drawing plane xy (Fig.10). The figure position on the plane is determined by three coordinates:

$$x_A(t), y_A(t), \varphi(t) \qquad (21)$$

They represent the law of plane movement of the body, which therefore has three degrees of freedom.

Velocity and acceleration of a plane figure point

Given the law of motion of the plane figure

$$x_A(t), y_A(t), \varphi(t)$$

we can find the vectors of angular velocity $\boldsymbol{\omega}$ and acceleration $\boldsymbol{\varepsilon}$, velocity $\boldsymbol{V}_A$ and acceleration $\boldsymbol{W}_A$ of the pole A.

Then, by theorem on distribution of velocity, we can find the velocity of any arbitrary point B of the plane figure.

$$\boldsymbol{V}_B = \boldsymbol{V}_A + \boldsymbol{\omega} \times \boldsymbol{AB}$$

All three vectors lie in the plane of the figure. The last term $\boldsymbol{\omega} \times \boldsymbol{AB}$ is perpendicular to AB and directed toward the rotation of the figure (Fig. 1). Therefore, this term is called here ***velocity*** of the point ***B around the pole A -*** $\boldsymbol{V}_{\mathrm{BA}}$.

$$\boldsymbol{V}_B = \boldsymbol{V}_A + \boldsymbol{V}_{\mathrm{BA}}; \qquad \boldsymbol{V}_{\mathrm{BA}} \equiv \boldsymbol{\omega} \times \boldsymbol{AB} \qquad (1)$$

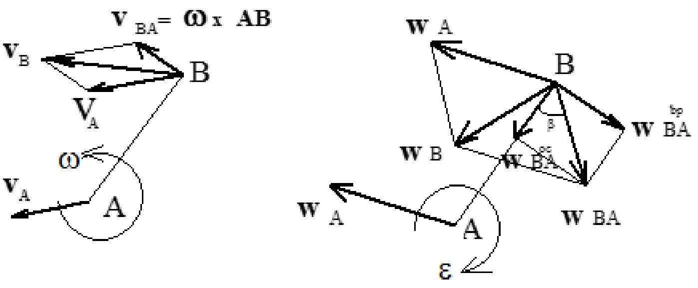


Рис.1

Differentiating (1) we find acceleration of the point B

$$\dot{\boldsymbol{V}}_B = \dot{\boldsymbol{V}}_A + \dot{\boldsymbol{\omega}} \times \boldsymbol{AB} + \boldsymbol{\omega} \times \dot{\boldsymbol{AB}}$$

Or

$$\boldsymbol{W_B} = \boldsymbol{W_A} + \boldsymbol{W}_{\mathrm{BA}};$$

$$\boldsymbol{W_{AB}} = \boldsymbol{W}_{\boldsymbol{AB}}^{\text{вр}} + \boldsymbol{W}_{\boldsymbol{AB}}^{\text{ос}} \qquad (2)$$

$$\boldsymbol{W}_{\boldsymbol{AB}}^{\text{вр}} = \boldsymbol{\varepsilon} \times \boldsymbol{AB}; \qquad \boldsymbol{W}_{\boldsymbol{AB}}^{\text{ос}} = \boldsymbol{\omega} \times \boldsymbol{V}_{\mathrm{BA}};$$

We see that acceleration of an arbitrary point B of the plane figure consists of acceleration of the pole $\boldsymbol{W_A}$ and acceleration $\mathbf{W_{BA}}$ of the point in rotation around the pole A. Acceleration $\mathbf{W_{BA}}$, as it should be, has the rotary component $\boldsymbol{W}_{\boldsymbol{AB}}^{\text{вр}}$, directed perpendicular to AB in the direction of the angular acceleration $\boldsymbol{\varepsilon}$ and centripetal component $\boldsymbol{W}_{\boldsymbol{AB}}^{\text{ос}}$, always directed to the pole A (Fig. 1).

Given that vectors $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ are directed perpendicular to the plane figure, all above components are in the plane of the figure and have modules:

$$W_{AB}{}^{bp} = \varepsilon\, AB \qquad W_{AB}{}^{oc} = \omega^2 AB \qquad (3)$$

Acceleration $\mathbf{W_{BA}}$ module

$$W_{AB} = \sqrt{W_{AB}{}^{bp^2} + W_{AB}{}^{oc^2}}$$

Angle β of $\mathbf{W_{AB}}$ to AB is the same for all points

$$tg\beta = \frac{W_{AB}{}^{bp}}{W_{AB}{}^{oc}} = \frac{\varepsilon}{\omega^2} \qquad (4)$$

Equivalent matrix formula (1) and (2) in any coordinate system have the form:

$$V_B = V_A + \Omega(AB), \qquad W_B = W_A + (\mathcal{E} + \Omega^2)(AB) \qquad (5)$$

Instantaneous Center of velocities. Velocity distribution in the plane figure.

It is difficult to understand from equation (1) how are distributed velocities in the plane figure. The picture will become clearer, if we introduce the notion of instantaneous Center of velocities (**ICV**).

ICV is a point $\mathcal{P}$ of the infinite extension of the plane figure, which velocity is zero at the moment.

$$\boldsymbol{V}_{\mathcal{P}} = \boldsymbol{0}$$

We will show that **ICV** exists if the angular velocity $\boldsymbol{\omega}$ is not zero at the moment. To do this, we multiply from the left side by vector $\boldsymbol{\omega}$ the velocity formula for $\mathcal{P}$

$$\boldsymbol{V}_{\mathcal{P}} = \boldsymbol{V}_A + \boldsymbol{\omega} \times \boldsymbol{A\mathcal{P}} = 0$$

Remembering the formula of the double cross product

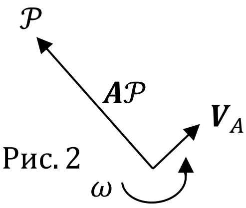


Рис. 2

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

we find

$$\mathbf{0} = \boldsymbol{\omega} \times \boldsymbol{V}_A + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{A\mathcal{P}}) =$$

$$= \boldsymbol{\omega} \times \boldsymbol{V}_A + \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \boldsymbol{A\mathcal{P}}) - \boldsymbol{A\mathcal{P}}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) = \boldsymbol{\omega} \times \boldsymbol{V}_A - \boldsymbol{A\mathcal{P}}\omega^2 ;$$

$$(\boldsymbol{\omega} \cdot \boldsymbol{A\mathcal{P}} = \mathbf{0})$$

Thus (Fig.2):

$$\boldsymbol{A\mathcal{P}} = \frac{\boldsymbol{\omega} \times \boldsymbol{V}_A}{\omega^2}; \qquad \mathrm{A}\mathcal{P} = \frac{V_A}{\omega} \qquad (8)$$

If now we assume $\mathcal{P}$ as the pole, formula of velocity will look familiar to rotational motion:

$$\boldsymbol{V}_B = \boldsymbol{V}_{\mathcal{P}} + \boldsymbol{\omega} \times \boldsymbol{\mathcal{P}B} = \boldsymbol{\omega} \times \boldsymbol{\mathcal{P}B}$$

$$\boldsymbol{V}_B = \boldsymbol{\omega} \times \boldsymbol{\mathcal{P}B} \qquad (9)$$

Thus, at the given moment velocities are distributed in the plane figure, as if it is revolving around **ICV** $\mathcal{P}$.

This means that the velocity of any point A of the flat figure is perpendicular to the line AP and the following relations are true (Fig.3):

$$V_{\mathrm{A}} = \omega A\mathcal{P}; \quad V_{\mathrm{B}} = \omega \mathrm{B}\mathcal{P}; \qquad \omega = \frac{V_{\mathrm{A}}}{A\mathcal{P}} = \frac{V_B}{B\mathcal{P}}; \qquad \frac{V_{\mathrm{A}}}{V_B} = \frac{A\mathcal{P}}{B\mathcal{P}} \qquad (10)$$

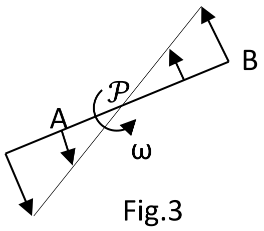


Fig.3

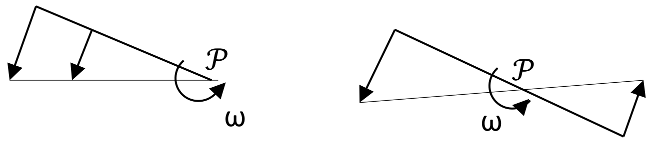


Fig.4

Figure 3 tells how to build the **ICV** $\mathcal{P}$ in various cases: Velocities of two points A,B are parallel to each other and perpendicular to AB. In this case, **ICV** $\mathcal{P}$ lies on the intersection of AB and the line drawn across the ends of the velocity vectors (Fig. 4).

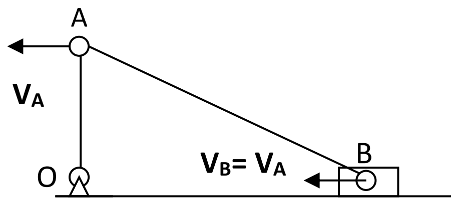
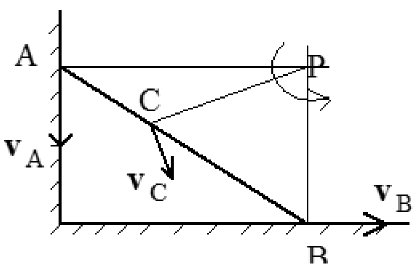


Fig.5

In case when the velocities of two points are parallel, but the points do not lie on the same perpendicular line Fig. 5 the perpendiculars to the velocities intersect in infinity and

$$\omega = \frac{V_{\mathrm{A}}}{A\mathcal{P}} = 0$$

Directions of velocity are known for two points: for example, for points A and B of rod AB moving along the axis (Fig. 6). According to Fig.3 **ICV** $\mathcal{P}$ is on the crossing of perpendiculars to these velocities. By the way, knowing the position of P, it is easy to determine the direction of velocity for an arbitrary point C of the rod: it is perpendicular to CP and in the direction of rotation.



Rolling without slipping of the plane figure on the curve, such as the wheel on the road.

Contact point P is the instantaneous Centre of velocity. Wheel circumference is often mistaken for the trajectory of the point A and its velocity mistakenly refer tangentially to the circle, while it is perpendicular to AP. As we can see, no any point of the wheel does have velocity, directed against the wheel center movement. So stone, separated from the wheel, is always moving forward

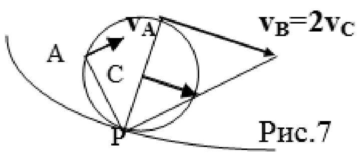


Рис.7

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Instantaneous Center of Acceleration (ICA).
Distribution of accelerations in the plane figure

ICA is the point Q, which acceleration is zero at the moment. We will show that **ICA** exists if ω, ε, are not equal to zero simultaneously. From vector $\boldsymbol{W_A}$ at the direction of ε we draw angle (Fig.8)

$$\beta = arctg\left(\frac{\varepsilon}{\omega^2}\right)$$

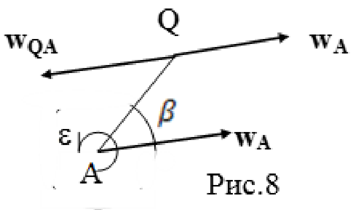


Рис.8

At this direction we draw

$$AQ = \frac{W_A}{\sqrt{\varepsilon^2+\omega^4}}$$

Acceleration of the point Q

$$\boldsymbol{W_Q} = \boldsymbol{W_A} + \boldsymbol{W_{QA}}$$

It is clear that $\boldsymbol{W_{QA}}$ is opposite to $\boldsymbol{W_A}$. It's modulo equals

$$W_Q = AQ\sqrt{\varepsilon^2+\omega^4} = W_A$$

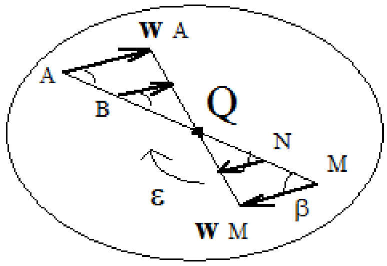


Fig.9

It means that

$$\boldsymbol{W_A} = -\boldsymbol{W_{AQ}} \quad \text{and} \quad \boldsymbol{W_Q} = \boldsymbol{0}$$

i.e. Q is the instantaneous Center of acceleration. If now we choose the pole in Q, the formula of acceleration of an arbitrary point A will be the same as for the rotational motion:

$$\boldsymbol{W_A} = \boldsymbol{W_{AQ}} = \boldsymbol{W_{AQ}^{\text{вр}}} + \boldsymbol{W_{AQ}^{\text{ос}}}; \qquad W_A = AQ\sqrt{\varepsilon^2+\omega^4} \qquad (11)$$

This means that accelerations in the plane figure are distributed as if it is revolving around the **ICA** Q (Fig. 9). On the straight line passing through the Q, acceleration are parallel and have the angle β with the direction to Q. Module of acceleration linearly depends on the distance from the Q.

It should be emphasized that, in general, the **ICV** and the **ICA** do not coincide. So, for the wheel moving uniformly and without slippage, **ICV** is at the point of its contact with the road, and the **ICA** is at the center of the wheel. Since $\varepsilon = 0$ and $\beta = 0$, the accelerations of all points are directed to the center of the wheel (Fig.10).

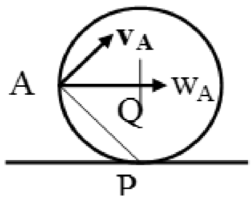


Fig.10

Another example is the rod which end A slides uniformly along the wall and the end B along the floor. It is obvious that Q is the **ICA**, accelerations of all points are horizontal (as $\mathbf{W}_B$) and linearly depend on the distance from Q (fig. 11).

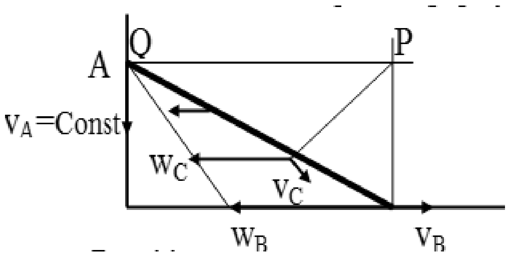


Fig.11

Thus, the formulas for velocities and accelerations show that plane movement of the body can be thought of as a result of addition of two movements: translation with the pole plus rotation around the pole

Example of solving a problem about a plane mechanism using vector and matrix methods

The slider-crank mechanism (Fig. 23), consisting of a crank OA, a connecting rod AB and a slider B, moves in the plane of the drawing. The straight-line x, along which the slider B moves, does not pass through the axis of rotation O of the crank and in this case the slider-crank mechanism is called non-central. Lengths of links: OA = AB = 1 m,

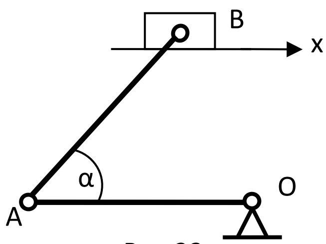


Рис.23

The slider B moves according to the law x = - 6/π Sin(πt/6)

The mechanism is shown at the moment of time t_1 = 1c when α=60°
In this position of the mechanism, determine:
the velocities and accelerations of the points B and A,
the angular velocities and accelerations of the crank and connecting rod.

Vector method.
Speeds

Slider B.
In projection on the x-axis

$$V_{Bx} = \dot{x} = -Cos\left(\frac{\pi t}{6}\right), \quad \text{при } t = t_1: \quad V_{Bx} = -0{,}87 \text{ м/с}$$

The direction of the vector VB is determined by the sign of the projection.

Connecting rod AB.
The position of the MCV (point P) of connecting rod AB is found using the known velocity $\boldsymbol{V}_{\text{B}}$ and the known direction of the velocity of joint A. To do this, we draw perpendiculars through points A and B to the directions of the velocities of these two points until they intersect at point P.
The angular velocity ω_{AB} of the connecting rod AB is determined by velocity of point B using the formula

$$\omega_{AB} = \frac{V_B}{BP} = \frac{V_B}{ABSin\alpha} = \frac{0{,}87}{0{,}87} = 1\ c^{-1}$$

Hinge A
Knowing ω_{AB} and the position of the MCV P, we find the velocity of hinge A

$$V_A = \omega_{AB}\ AP = \omega_{AB} ABCos\alpha = 0{,}5 \text{ м/с}$$

Crank OA
The angular velocity ω_{OA} of the crank OA is determined through the known velocity of point A using the formula

$$\omega_{OA} = \frac{V_A}{OA} = 0{,}5\ c^{-1}$$

Accelerations.
Slider B.
In projection on the x-axis

$$W_{Bx} = \ddot{x} = \frac{\pi}{6} Sin\left(\frac{\pi t}{6}\right), \quad \text{при } t = t_1: \quad W_{Bx} = 0{,}26 \text{ м/с}^2$$

The direction of the vector $\boldsymbol{W}_B$ is determined by the sign of the projection.

Hinge A.
Hinge A moves in a circle around the axis O. Therefore, its acceleration consists of rotational $\boldsymbol{W}_A^{rot}$ and centripetal $\boldsymbol{W}_A^{oc}$ accelerations. The latter is always directed toward the center O.

$$W_A^{\text{ос}} = \omega_{OA}{}^2 \text{OA} = 0{,}25 \text{ м/с}^2$$

$\boldsymbol{W}_A^{rot}$ is perpendicular to OA. But its direction is unknown. A similar situation was encountered with reactions of bilateral constraints. For them, the line of action is known, but the direction is unknown. We

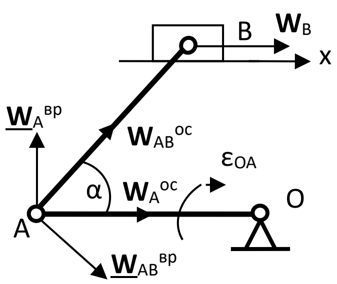


Рис.24

direct such a vector arbitrarily and highlight it by underlining. The direction of such a vector is opposite to the shown direction if its value in the answer is negative.

Let us express the acceleration of the hinge A through the acceleration of the pole B

$$\boldsymbol{W}_A^{\text{вр}} + \boldsymbol{W}_A^{\text{ос}} = \boldsymbol{W}_B + \boldsymbol{W}_{AB}^{\text{вр}} + \boldsymbol{W}_{AB}^{\text{ос}} \quad (10)$$

In this vector equation there are two scalar unknowns: rotational accelerations $W_A^{\text{вр}}$ and $W_{AB}^{\text{вр}}$. The acceleration modulus $W_{AB}^{\text{ос}}$ can be calculated

$$W_{AB}^{\text{ос}} = \omega_{AB}{}^2 AB = 1 \text{ м/с}^2$$

To find the acceleration modulus $W_A^{\text{вр}}$, we project equation (10) onto the direction AB**.**

$$W_A^{\text{вр}} Sin\alpha + W_A^{\text{ос}} Cos\alpha = W_B Cos\alpha + W_{AB}^{\text{ос}}$$

Thus

$$W_A^{\text{вр}} = (W_B Cos\alpha + W_{AB}^{\text{ос}} - W_A^{\text{ос}} Cos\alpha)/Sin\alpha$$

$$= (0{,}26 \cdot 0{,}5 + 1 - 0{,}25 \cdot 0{,}5)/0{,}87 = 1{,}16 \text{ м/с}^2$$

A positive result means that Fig. 24 shows the correct vector direction.

Crank OA

The angular acceleration of the crank OA is directed in accordance with the direction of the rotational acceleration $\boldsymbol{W}_A^{\text{вр}}$ clockwise. Its modulus

$$\varepsilon_{0\text{A}} = W_A^{\text{вр}}/OA = 1{,}16\text{с}^{-2}$$

Connecting rod AB

The angular acceleration of connecting rod AB can be found through the rotational acceleration $\boldsymbol{W}_{AB}^{\text{вр}}$.
Projecting equation (10) onto y, we obtain:

$$W_A^{\text{вр}} = -W_{AB}^{\text{вр}} Cos\alpha + W_{AB}^{\text{ос}} Sin\alpha$$

Thus

$$W_{AB}^{\text{вр}} = \left(-W_A^{\text{вр}} + W_{AB}^{\text{ос}} Sin\alpha\right)/Cos\alpha = 2 \cdot (-1{,}16 + 1 \cdot 0{,}87) = -0{,}58 \text{ м/с}^2$$

The negative result indicates that the actual direction of the vector $\boldsymbol{W}_{AB}^{\text{вр}}$ is opposite to the direction shown in Fig.24.
The angular acceleration of the connecting rod corresponds to the direction of the vector $\boldsymbol{W}_{AB}^{\text{вр}}$ clockwise. Its modulus

$$\varepsilon_{\text{AB}} = \boldsymbol{W}_{AB}^{\text{вр}}/AB = 0{,}58\text{с}^{-2}$$

Matrix Solution

Velocity

Velocity Formula

$$V_B = V_A + \Omega_{AB}(AB)$$

Or

$$\begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ V_{Ay} \end{pmatrix} + \begin{pmatrix} 0 & -\dot{\varphi}_{AB} \\ \dot{\varphi}_{AB} & 0 \end{pmatrix} \begin{pmatrix} AB\ Cos\alpha \\ AB\ Sin\alpha \end{pmatrix}$$

In expanded form

$$V_{Bx} = \dot{x} = -\dot{\varphi}_{AB} AB\ Sin\alpha$$

$$0 = V_{Ay} + \dot{\varphi}_{AB} AB\ Cos\alpha$$

Thus

$$\dot{\varphi}_{AB} = -\frac{\dot{x}}{AB\ Sin\alpha} = \frac{0{,}87}{0{,}87} = 1\ c^{-1}$$

Counterclockwise. The result coincided with the vector method.
Since

$$V_{Ay} = -\dot{\varphi} OA$$

We have

$$\dot{\varphi} = -\frac{V_{Ay}}{OA} = 0{,}5\ \mathrm{c}^{-1}$$

The result coincided with the vector method.

Accelerations

Acceleration Formula

$$W_B = (\in +\Omega^2)(OA) + \left(\in_{AB} + \Omega_{AB}{}^2\right)(AB)$$

The matrices have the form

$$\in +\Omega^2 = \begin{pmatrix} -\dot{\varphi}^2 & -\ddot{\varphi} \\ \ddot{\varphi} & -\dot{\varphi}^2 \end{pmatrix} \qquad \in_{AB} + \Omega_{AB}{}^2 = \begin{pmatrix} -\dot{\varphi}_{AB}{}^2 & -\ddot{\varphi}_{AB} \\ \ddot{\varphi}_{AB} & -\dot{\varphi}_{AB}{}^2 \end{pmatrix}$$

$$(OA) = \begin{pmatrix} -OA \\ 0 \end{pmatrix} \qquad (AB) = \begin{pmatrix} AB\ Cos\alpha \\ AB\ Sin\alpha \end{pmatrix}$$

Substituting, we find

$$\begin{pmatrix} \ddot{x} \\ 0 \end{pmatrix} = \begin{pmatrix} -\dot{\varphi}^2 & -\ddot{\varphi} \\ \ddot{\varphi} & -\dot{\varphi}^2 \end{pmatrix}\begin{pmatrix} -OA \\ 0 \end{pmatrix} + \begin{pmatrix} -\dot{\varphi}_{AB}{}^2 & -\ddot{\varphi}_{AB} \\ \ddot{\varphi}_{AB} & -\dot{\varphi}_{AB}{}^2 \end{pmatrix}\begin{pmatrix} AB\ Cos\alpha \\ AB\ Sin\alpha \end{pmatrix}$$

Having expanded, we obtain two equations with two unknowns $\ddot{\varphi}$ and $\ddot{\varphi}_{AB}$

$$\ddot{x} = OA\dot{\varphi}^2 - AB\ Cos\alpha\ \dot{\varphi}_{AB}{}^2 - AB\ Sin\alpha\ \ddot{\varphi}_{AB}$$

$$0 = -OA\ddot{\varphi} + AB\ Cos\alpha\ \ddot{\varphi}_{AB} - AB\ Sin\alpha\ \dot{\varphi}_{AB}{}^2$$

Thus

$$\ddot{\varphi}_{AB} = \frac{1}{AB\ Sin\alpha}\left(OA\dot{\varphi}^2 - \ddot{x} - AB\ Cos\alpha\ \dot{\varphi}_{AB}{}^2\right) =$$

$$= \frac{1}{0{,}87}(0{,}25 - 0{,}26 - \ 0{,}5) = -0{,}58\ c^{-2}$$

$$\ddot{\varphi} = \frac{1}{OA}\left(AB\ Cos\alpha\ \ddot{\varphi}_{AB} - AB\ Sin\alpha\ \dot{\varphi}_{AB}{}^2\right) =$$

$$= -0{,}29 - 0{,}87\ = -1{,}16\ c^{-2}$$

The result coincided with the vector method.

Interactive excel problems
on the kinematics of a plane mechanism

At each step of the solution, the incorrectly filled cell is colored pink, which allows the student to independently find the error.
The problems and an example of its solution can be downloaded from the link
https://disk.yandex.ru/d/-VD2F0MKgSd31A

Resultant rotation of rigid body

Theorem on composition of body angular velocities

Let the body rotate with angular velocity $\boldsymbol{\omega_r}$ relative to moving coordinate system x y z, which in turn rotates with angular velocity $\boldsymbol{\omega_e}$ relative to conventionally fixed coordinate system X Y Z (Fig. 1).

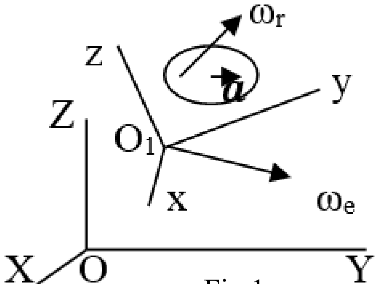


Fig.1

Consider a body vector **a**. Observer O_1 in the mobile system will write the formula Eulerian for relative derivative of the vector **a.**

$$\frac{d_r\boldsymbol{a}}{dt} = \boldsymbol{\omega_r} \times \boldsymbol{a} \quad (1)$$

The observer O in the motionless system Euler's formula will write the formula Eulerian for absolute derivative of the vector **a.**

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{\omega_a} \times \boldsymbol{a} \quad (2)$$

As we know, both derivatives are related by

$$\frac{d\boldsymbol{a}}{dt} = \frac{d_r\boldsymbol{a}}{dt} + \boldsymbol{\omega_e} \times \boldsymbol{a} \quad (3)$$

g.1

Thus,

$$\boldsymbol{\omega_a} \times \boldsymbol{a} = \boldsymbol{\omega_e} \times \boldsymbol{a} + \boldsymbol{\omega_r} \times \boldsymbol{a} \quad (4)$$

Since $\boldsymbol{a}$ is an arbitrary body vector, then it follows from (4) the ***theorem of angular speeds composition:***

$$\boldsymbol{\omega_a} = \boldsymbol{\omega_e} + \boldsymbol{\omega_r} \qquad (5)$$

Generalisation. If we consider a sequence of N moving coordinate systems, the formula (5) can be summarized:

$$\boldsymbol{\omega_a} = \sum_{\boldsymbol{k=1}}^{\boldsymbol{N}} \boldsymbol{\omega_k} + \boldsymbol{\omega_r} \quad (6)$$

where $\boldsymbol{\omega_k}$ - angular velocity of the system number k in relation to the system number ом $k-1,$ and $\boldsymbol{\omega_r}$ - angular velocity of the body in relation to the system number N.

Composition of body rotations around the parallel axes.

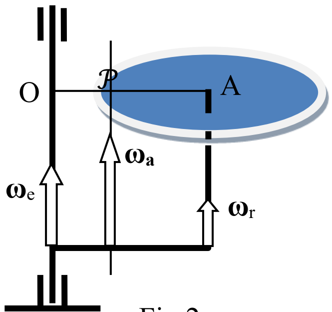


Fig.2

Consider a mechanism consisting of a driver, rotating around a fixed axis z, with an angular velocity of $\boldsymbol{\omega_e}$ and the disk rotating relative to the driver, with an angular velocity of $\boldsymbol{\omega_r}$. Obviously, the disk makes a plane movement

Let first investigate the case when the angle velocities have the same direction (Fig.2). In this case, the absolute angular speed

$$\boldsymbol{\omega_a} = \boldsymbol{\omega_e} + \boldsymbol{\omega_r} \qquad (7)$$

is different from zero. This means that there is an instantaneous center of velocity $\mathcal{P}$, whose speed is zero at this moment:

$$\boldsymbol{V_{\mathcal{P}}} = \boldsymbol{0}$$

The point $\mathcal{P}$ is in a composite movement, so its speed is equal to the sum of the transitional and relative velocities

$$\boldsymbol{V_{\mathcal{P}}} = \boldsymbol{V_{\mathcal{P}}^e} + \boldsymbol{V_{\mathcal{P}}^r} = \boldsymbol{0}$$

So,

$$\boldsymbol{V_{\mathcal{P}}^e} = -\boldsymbol{V_{\mathcal{P}}^r}$$

Transitional and relative velocity are directed oppositely only at the points of the line OA. Among them, there is a point P for which:

$$\omega_e O\mathcal{P} = \omega_r A\mathcal{P}$$

We found the location of the instantaneous center of velocity

$$\frac{O\mathcal{P}}{A\mathcal{P}} = \frac{\omega_r}{\omega_e} \qquad (8)$$

Thus, in this case, the body makes a plane movement in which the instantaneous center of velocity $\mathcal{P}$ divides "internally" the distance AB back proportionally to angular velocities.

Now let the directions of the spins be opposite (Fig.3). In this case, the absolute angular velocity is equal by modulo to the difference

$$\omega_a = \omega_r - \omega_e \qquad (\omega_r > \omega_e)$$

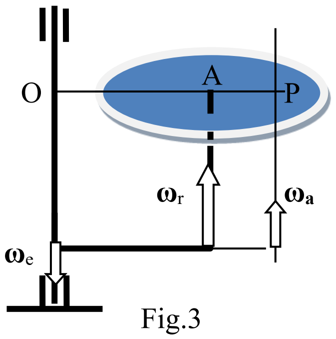


Fig.3

First, suppose that $\omega_a \neq 0$

Then again there is instantaneous center of velocity $\mathcal{P}$. But now it's outside of the segment OA, from the side of the greater angular velocity. Still

$$\frac{O\mathcal{P}}{A\mathcal{P}} = \frac{\omega_r}{\omega_e}$$

The instantaneous center of velocity divides the distance OA also back proportionally but "externally".

Spins couple

Is called the case when the directions of the spins are opposite and velocity modules are equal $\omega_r = \omega_e$

The disk does not rotate since

$$\omega_a = 0$$

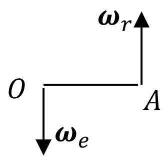


Fig.4

It executes a circular transitional motion. Just like the cabin of the Ferris wheel. The speeds of all points are equal

$$V = V_A = \omega OA \quad (9)$$

to the "moment" of spin pair.

Differential and Planetary gears. Villis method

Mechanism, shown in Figure 5, consisting of two wheels in gearing, which are on the ends of the crank OA, is called the ***differential*** if the central wheel rotates.

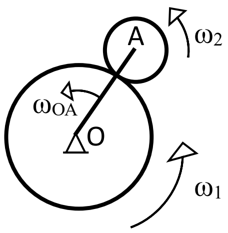


Fig.5

$$\omega_1 \neq 0$$

and ***planetary*** if central wheel does not rotate

$$\omega_1 = 0$$

Let us find by Villis method the angular velocity ω_2 of the little wheel if the angular velocities ω_{OA} and ω_1 are known.

The Villis method consists in giving the whole mechanism the angular velocity $-\omega_{OA}$. According to theorem of angular velocities composition, crank OA will stop. The mechanism will become a common external gearing of two wheels with new angular velocities

$$\widetilde{\omega}_1 = \omega_1 - \omega_{OA}; \qquad \widetilde{\omega}_2 = \omega_2 - \omega_{OA}$$

The new angular velocities are opposite in direction and in inverse proportion to the radii of the wheels

$$\frac{\widetilde{\omega}_2}{\widetilde{\omega}_1} = -\frac{r_1}{r_2} \quad (10)$$

Thus

$$\frac{\omega_2 - \omega_{\mathrm{OA}}}{\omega_1 - \omega_{\mathrm{OA}}} = -\frac{r_1}{r_2}$$

Or

$$\omega_2 = \omega_{\mathrm{OA}} - \frac{r_1}{r_2}(\omega_1 - \omega_{\mathrm{OA}}) = \frac{1}{r_2}(\omega_{\mathrm{OA}}OA - \omega_1 r_1) \qquad (11)$$

For the planetary mechanism

$$\omega_1 = 0; \qquad OA = r_1 + r_2$$

We get the obvious result

$$\omega_2 = \omega_{0A}\frac{\mathrm{OA}}{r_2}$$

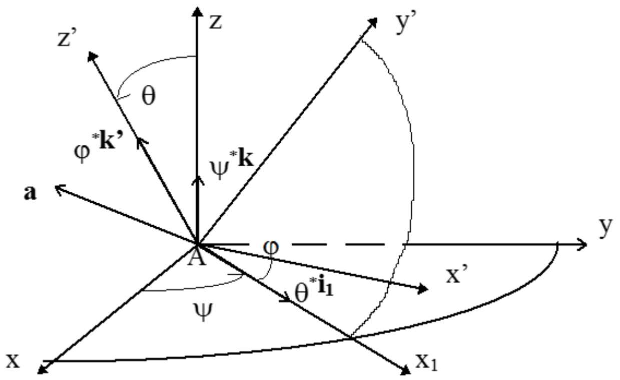
It is not much harder to find the same result with the plane motion formulas:

$$V_A = \omega_{0A}OA; \quad V_B = \omega_1 r_1; \quad \omega_2 = \frac{1}{r_2}(V_A - V_B) = \frac{1}{r_2}(\omega_{0A}OA - \omega_1 r_1)$$

Spherical motion of the body

Euler Angles. The law of motion.

Spherical is called the motion of the body in which one point of the body is fixed. The name reflects the fact that at this motion all points of the body move on spheres. The full name of this motion is ***rotation around a fixed point.***



We will show that the position of the body, can be specified by three angular coordinates. In classical mechanics most often, we use Euler angles: angles of **precession** ψ**, nutation** ϑ and **rotation** φ (Fig. 6).

To do so we will build the body position in the space using the given Euler angles values. In other words, we will superpose axis (x y z) with axes (x ', y ', z ') by three successive turns.

First, we turn (x y z) around the z-axis at the angle of ψ to axis (x_1 y_1z) (y_1 not shown on Fig. 6). The axis x_1 is called the ***line of nodes***. Next, we turn (x_1 y_1z) around the x_1 -axis at the angle θ to axes (x_1 y_2 z ‘) (y_2 not shown).

The third and the last rotation we make around the axis z '. The axis (x_1 y_2 z ') superpose with axes (x ', y ', z '):

We have shown that the Euler angles determine the position of the body. Thus, the three functions

$$\psi(t), \theta(t), \varphi(t)$$

are the ***law of spherical body motion***. Therefore, we say that such body has 3 degrees of freedom.

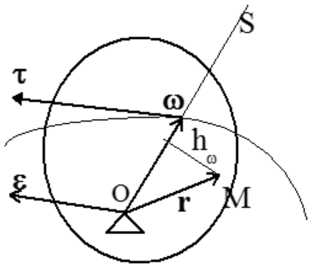
Angular velocity and acceleration of the body

Let us find the angular velocity of the body, using the theorem of angular velocities composition. It can be used since Euler angles specify the position of each of the coordinate systems relative to the previous coordinate system.

Body undergoes three rotations with angular velocities: $\dot{\psi}\boldsymbol{k}$ around the z-axis, $\dot{\vartheta}\,\boldsymbol{i_1}$ around the axis x_1 and $\dot{\varphi}\boldsymbol{k'}$ around the z'- axis. According to theorem of angular velocities composition:

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{k} + \dot{\vartheta}\,\boldsymbol{i_1} + \dot{\varphi}\boldsymbol{k'} \quad (12)$$

Projecting this expression of the fixed axis, we find:



$$\omega_x = \dot{\vartheta} Cos\psi + \dot{\varphi} Sin\psi Sin\vartheta;$$
$$\omega_y = \dot{\vartheta} Sin\psi - \dot{\varphi} Cos\psi Sin\vartheta;$$
$$\omega_z = \dot{\psi} + \dot{\varphi} Cos\vartheta; \quad (12)$$

Unlike the rotary motion of the body, where the angular velocity vector is all the time directed along the fixed axis, here vector $\boldsymbol{\omega}$ changes module, and direction. Therefore, the angular acceleration vector

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt} \quad (13)$$

directed tangential to the vector hodograph has different direction than $\boldsymbol{\omega}$.

Velocity and acceleration of the body point

Let us choose the fixed point O as a pole. Then the velocity of an arbitrary point M of the body can be found using the formula

$$\boldsymbol{V} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \times \boldsymbol{r} \quad (14)$$

It follows that the velocity is distributed in the body as if the body is rotating around the instantaneous axis of S (Fig. 7). This means that the velocities of the points on the axis S are equal to zero. Modulo of velocity

$$V = \omega h$$

Acceleration of an arbitrary point M

$$\boldsymbol{W} = \boldsymbol{\varepsilon} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad (15)$$

consists of rotational and centripetal components:

$$\boldsymbol{W} = \boldsymbol{W}^{\text{вр}} + \boldsymbol{W}^{\text{ос}}; \qquad \boldsymbol{W}^{\text{вр}} = \boldsymbol{\varepsilon} \times \boldsymbol{r}; \qquad \boldsymbol{W}^{\text{ос}} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad (16)$$

It will be shown that in spherical motion $\boldsymbol{W}^{\text{вр}}$ and $\boldsymbol{W}^{\text{ос}}$ are not mutually perpendicular. Centripetal acceleration $\boldsymbol{W}^{\text{ос}}$ is directed to the instantaneous axis of rotation, and $\boldsymbol{W}^{\text{вр}}$ is perpendicular to the plane ($\boldsymbol{\varepsilon}$ $\mathbf{r}$).

To vector formulas correspond the matrix expressions of speed and acceleration, with which it is easy to calculate them at any moment of time.

$$V = \Omega r; \qquad W = (\mathcal{E} + \Omega^2) r \quad (17)$$

Example

The movable cone rotates on the still cone without slipping. Set: angle α, length OA and speed V_C of the point C.

Determine the velocity and acceleration of the top point A of the rolling cone.

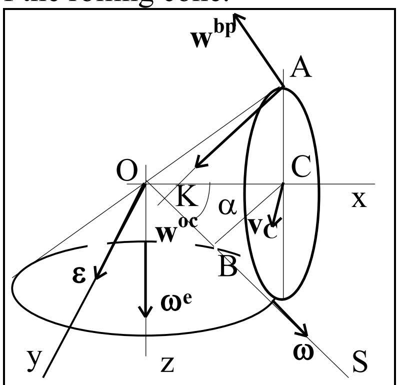


Рис.3

Due to the lack of slip, velocity of the points on the line S of contact are zero now. It is the instantaneous axis and vector of angular velocity $\boldsymbol{\omega}$ is directed along it.

$$V_c = \omega CB = \omega OA Sin\alpha Cos\alpha \quad (18)$$

So

$$\omega = \frac{2V_c}{OASin2\alpha}; \quad V_A = \omega AK = 2V_c \qquad (19)$$

Centripetal acceleration $\boldsymbol{W}^{oc}$ of the point A is directed to S:

$$W^{oc} = \omega^2 AK = \frac{4V_c^{\,2}}{OASin\,2\alpha} \qquad (20)$$

Constant by module angular velocity $\boldsymbol{\omega}$ rotates together with the instantaneous axis S around the vertical axis z. Speed of rotation is equal to

$$\omega_e = \frac{V_c}{OC} = \frac{V_c}{ACCos\alpha} \qquad (21)$$

Angular acceleration **ε** is tangential to the hodograph of vector **ω** so parallel to $\boldsymbol{V_c}$. Thus, in spherical motion angular acceleration and velocity are not collinear. Find **ε** with Euler formula:

$$\boldsymbol{\varepsilon} = \boldsymbol{\omega_e} \times \boldsymbol{\omega} \qquad (22)$$

Thus

$$\varepsilon = \omega_e \omega Cos\alpha = \frac{2{V_c}^2}{OA^2 Sin\ 2\alpha} \qquad (23)$$

Rotational acceleration $\boldsymbol{W}^{\text{вр}}$ of the point A is directed as the cross product

$$\boldsymbol{W}^{\text{вр}} = \boldsymbol{\varepsilon} \times \boldsymbol{OA} \qquad (24)$$

perpendicular to OA in the xz plane.

$$W^{\text{вр}} = \varepsilon OA = \frac{2{V_c}^2}{OASin2\alpha} \qquad (24)$$

We see that in spherical motion rotational and centripetal accelerations are not perpendicular.

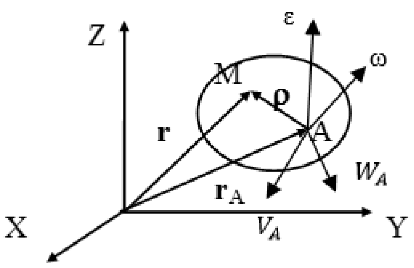
Finally

$$W^2 = {W^{\text{вр}}}^2 + {W^{\text{ос}}}^2 - 2W^{\text{вр}}W^{\text{ос}}Cos2\alpha \qquad (25$$

Free movement of body

Velocity and acceleration of body point

Consider a free body moving relative to the frame of reference with axes X, Y, Z (Fig. 12). Body movement is set, if we know the method of determining its position at any time t. It is enough to set the motion of the pole and rotation of the body around the pole. As will know, the rotation can be set by three Euler angles $\psi(t), \Theta(t), \varphi(t)$.. Thus, six functions



$$X_A(t), Y_A(t), Z_A(t) \qquad (12)$$
$$\psi(t), \Theta(t), \varphi(t)$$

represent the ***law of free movement*** of a rigid body. This means that the body has 6 degrees of freedom

Note that the first three functions give us the velocity $\boldsymbol{V_A}$ and acceleration $\boldsymbol{W_A}$ of the pole. The Euler angles, let find the angular velocity **ω** and angular acceleration $\boldsymbol{\varepsilon}$ of the body.

Velocity of arbitrary point of the body can be found with the velocity distribution theorem.

$$\boldsymbol{V} = \boldsymbol{V_A} + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

Differentiating theorem, we find

$$\dot{\boldsymbol{V}} = \dot{\boldsymbol{V}}_{\boldsymbol{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}$$

Bearing in mind that

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon}$$

is angular acceleration of the body, and for body vector $\boldsymbol{\rho}$ by Euler's formula

$$\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$$

we get the formula of acceleration for the arbitrary point of the body,

$$\boldsymbol{W} = \boldsymbol{W_A} + \boldsymbol{\varepsilon} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

We have already met the last two terms in the plane movement. As there, let's call them rotational and centripetal accelerations at its spinning around the pole.

$$\boldsymbol{W} = \boldsymbol{W_A} + \boldsymbol{W}_{\boldsymbol{MA}}^{\text{вр}} + \boldsymbol{W}_{\boldsymbol{MA}}^{\text{ос}}$$

Resultant motion of particle

Absolute, relative and translational motions

We know that the laws of Mechanics are only performed in inertial reference system. So, as we know, can be considered a heliocentric system. Let's call this system the ***absolute*** and associate it with X, Y, Z. Movement of the particle M relative to the absolute system is described by radius vector **r** (t) and is called ***absolute***. Will mark the speed and acceleration of the particle in absolute motion with "a" index:

$$\boldsymbol{V_a}, \ \boldsymbol{W_a}$$

Sometimes it is more convenient to describe the particle motion in relation to the carrying body, on which moves the particle (Fig.1). For example, the motion of the car is more naturally described in relation to Earth, and not to the Sun.

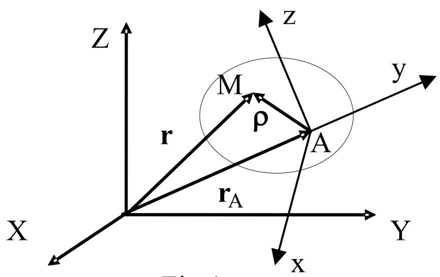


Fig.1

Likewise, we describe the movement of the passenger in relation to the tram (to the body) and not to the ground. Movement in relation to the carrying body is called ***relative***. We will mark speed and acceleration of relative motion with "r" index:

$$\boldsymbol{V_r}, \ \boldsymbol{W_r}$$

Associate with the carrying body axes x, y, z. Relative motion is set by projections of relative radius-vector $\boldsymbol{\rho}(t)$ on the axis

$$x(t), y(t), z(t)$$

Let the movement of the carrying body in relation to the "absolute" system of reference be specified by coordinates of the pole and by Euler angles:

$$X_A(t), Y_A(t), Z_A(t)$$
$$\psi(t), \Theta(t), \varphi(t)$$

These laws define the speed and acceleration of the pole $\boldsymbol{V_A}$, $\boldsymbol{W_A}$ and angular speed $\boldsymbol{\omega}$ and acceleration $\boldsymbol{\varepsilon}$ of the carrying a body.

We call ***translational speed and acceleration***

$$\boldsymbol{V_e}, \ \boldsymbol{W_e}$$

of the particle M the speed and acceleration of that particle of the carrying body, with which coincides at the given moment the particle M. In other words, the speed and acceleration of the particle M, fixed on the carrying body.

Let us find the absolute speed and acceleration of the particle M with the help of the given characteristics of translation and relative movements.

$$\boldsymbol{V_a}, \boldsymbol{W_a} \ (\boldsymbol{V_A}, \boldsymbol{W_A}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}) \qquad (1)$$

Absolute and relative derivatives relationship

From Fig.1 we have

$$\boldsymbol{r} = \boldsymbol{r_A} + \boldsymbol{\rho} \qquad (2)$$

Fig.1 and formula (2) are the same as for the free body movement, but with one crucial difference. Here the vector **ρ** is not a body vector. Its module is changing, because the particle M moves in relation to the body. For this reason, we cannot apply Euler's formula to the vector **ρ**.

Representing vector **ρ** in the moving reference system with the relative motion law:

$$\boldsymbol{\rho} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k} \qquad (3)$$

Here **i, j, k**- unit vectors of the moving system, rotating with the body.
Differentiating (2) in time, we find

$$\boldsymbol{V_a} = \boldsymbol{V_A} + \dot{\boldsymbol{\rho}} \qquad (4)$$

Differentiating (3) in time, we find:

$$\dot{\boldsymbol{\rho}} \equiv \frac{d\boldsymbol{\rho}}{dt} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} + x\,\frac{d\boldsymbol{i}}{dt} + y\,\frac{d\boldsymbol{j}}{dt} + z\,\frac{d\boldsymbol{k}}{dt}$$

Basis vectors i, j, k are the body vectors, so their derivatives we find with Euler formulas

$$\frac{d\boldsymbol{i}}{dt} = \boldsymbol{\omega} \times \boldsymbol{i}; \qquad \frac{d\boldsymbol{j}}{dt} = \boldsymbol{\omega} \times \boldsymbol{j}; \qquad \frac{d\boldsymbol{k}}{dt} = \boldsymbol{\omega} \times \boldsymbol{k} \qquad (5)$$

Thus,

$$\frac{d\boldsymbol{\rho}}{dt} = \frac{d_r\boldsymbol{\rho}}{dt} + \boldsymbol{\omega} \times \boldsymbol{\rho} \qquad (6)$$

Here we design the ***relative derivative***

$$\frac{d_r\boldsymbol{\rho}}{dt} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \qquad (7)$$

It describes the change of vector $\boldsymbol{\rho}$ in relation to the carrying body.

Formula (6) expressed the ***theorem of derivatives***: absolute derivative of the vector specified in the mobile system, equals relative derivative plus the cross product of angular velocity by vector.

Note that at translation motion of the carrying body ($\omega = 0$) derivatives are similar.

$$\frac{d\boldsymbol{\rho}}{dt} = \frac{d_r\boldsymbol{\rho}}{dt} \quad \text{при} \quad \boldsymbol{\omega} = \boldsymbol{0}$$

Velocities composition theorem.

Formula (4) takes the form

$$\boldsymbol{V_a} = \boldsymbol{V_A} + \boldsymbol{\omega} \times \boldsymbol{\rho} + \frac{d_r\boldsymbol{\rho}}{dt} \qquad (8)$$

In mobile system, the column of relative derivative projections has a simple form

$$\frac{d_r\boldsymbol{\rho}}{dt} \quad \to \quad \dot{\rho} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

Therefore, the matrix form of formula (8) in the moving axes

$$V_a = V_A + \Omega\rho + \dot{\rho} \qquad (9)$$

If we fix the particle on the body at the given moment, then

$$\frac{d_r\boldsymbol{\rho}}{dt} = 0$$

absolute speed, by definition, becomes the speed of translation.

$$\boldsymbol{V_e} = \boldsymbol{V_A} + \boldsymbol{\omega} \times \boldsymbol{\rho} \qquad (10)$$

Let us find the relative speed by fixing the body ($\boldsymbol{V_A} = \boldsymbol{0}$; $\boldsymbol{\omega} = \boldsymbol{0}$)

$$\boldsymbol{V_r} = \frac{d_r\boldsymbol{\rho}}{dt} \qquad (11)$$

Thus, we come to the ***theorem of velocities composition*** in vector form

$$\boldsymbol{V_a} = \boldsymbol{V_e} + \boldsymbol{V_r} \qquad (12)$$

Absolute speed is equal to the sum of translational and relative speeds.

Example

The disc rotates evenly around the z-axis with angular velocity $\omega = 2c^{-1}$.

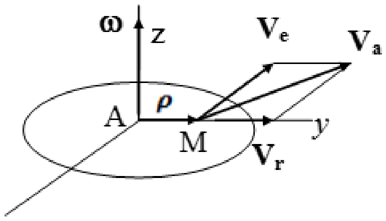


Fig.2

The particle M moves along the disc radius according to the law

$$y = 3t^2 - 2t \ (\text{м}).$$

Find the absolute speed of the particle at moment of time t_1=1 s. First, let's solve a problem by ***method of stopping***. Method consists in examining the relative motion by stopping translation mentally, and vice versa. This is consistent with the definitions of these movements.

Relative motion (ω = 0)

We mentally stop the disk rotation and find the projection of the relative velocity of the moving axis y, deriving the law of relative movement:

$$V_{ry} = \dot{y} = (6t-2)|_{t=1} = 4 \text{ м/сек}$$

Translational motion (y=Const)

Fixing the particle M at distance $0\text{M} = y|_{t=1} = 1\text{м}$, we find its speed in rotation

$$Ve = \omega OM = 2\text{м/сек}$$

Theorem of speed composition

$$\boldsymbol{V_a} = \boldsymbol{V_e} + \boldsymbol{V_r}$$

in projections on the moving axis gives

$$V_{ax} = -Ve = -2\frac{\text{м}}{\text{с}}, \qquad V_{ay} = V_r = 4\text{м/с}, \qquad V_{az} = 0$$

Let us find the absolute velocity by ***matrix method.***

$$V_A = 0; \quad \Omega = \omega\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \rho = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}; \quad \dot{\rho} = \begin{pmatrix} 0 \\ \dot{y} \\ 0 \end{pmatrix}$$

Find the projections of absolute velocity of the moving axis:

$$\begin{pmatrix} V_{ax} \\ V_{ay} \\ V_{az} \end{pmatrix} = \Omega\rho + \dot{\rho} = 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 3t^2-2t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 6t-2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}\text{м/с} \qquad (13)$$

We see that the results are the same with the stopping method.

Acceleration composition theorem

Differentiating the velocity composition theorem in vector form (10), we find

$$\boldsymbol{W_a} = \boldsymbol{W_A} + \boldsymbol{\varepsilon}\times\boldsymbol{\rho} + \boldsymbol{\omega}\times\dot{\boldsymbol{\rho}} + \dot{\boldsymbol{V}}_{\boldsymbol{r}} \qquad (14)$$

Vectors ρ and $\mathbf{V_r}$ are set in mobile system, so their absolute derivatives are given by the derivatives theorem

$$\dot{\boldsymbol{\rho}} = \boldsymbol{\omega}\times\boldsymbol{\rho} + \frac{d_r\boldsymbol{\rho}}{dt} = \boldsymbol{\omega}\times\boldsymbol{\rho} + \boldsymbol{V_r} \qquad (15)$$

$$\boldsymbol{\omega}\times\dot{\boldsymbol{\rho}} = \boldsymbol{\omega}\times\boldsymbol{V_r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}); \qquad \dot{\boldsymbol{V}}_{\boldsymbol{r}} = \boldsymbol{\omega}\times\boldsymbol{V_r} + \frac{d_r\boldsymbol{V_r}}{dt}$$

It is remarkable that in these both expressions the component $\boldsymbol{\omega}\times\boldsymbol{V_r}$ is derived from two different formulas: $\boldsymbol{\omega}\times\dot{\boldsymbol{\rho}}$ and $\dot{\boldsymbol{V}}_{\boldsymbol{r}}$**.** In the first case, the product of $\boldsymbol{\omega}\times\boldsymbol{V_r}$ characterizes the change of the transitional speed $\boldsymbol{\omega}\times\boldsymbol{\rho}$ due to changes in the relative position of the particle.

In the second case, the product of $\boldsymbol{\omega}\times\boldsymbol{V_r}$ characterize the change of direction of the vector of relative speed of $\boldsymbol{V_r}$ by turning carrying body with an angular velocity $\boldsymbol{\omega}$.

Thus, two products $\boldsymbol{\omega}\times\boldsymbol{V_r}$ characterize the mutual influence of relative motion on rotary transitional speed and of transition rotation on the relative speed. It is amazing what these effects are identical!

We get

$$\boldsymbol{W_a} = \boldsymbol{W_A} + \boldsymbol{\varepsilon}\times\boldsymbol{\rho} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho} + \boldsymbol{V_r}) + \boldsymbol{\omega}\times\boldsymbol{V_r} + \frac{d_r\boldsymbol{V_r}}{dt}$$

Combining the similar terms, we find

$$\boldsymbol{W_a} = \boldsymbol{W_A} + \boldsymbol{\varepsilon}\times\boldsymbol{\rho} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}) + 2(\boldsymbol{\omega}\times\boldsymbol{V_r}) + \frac{d_r\boldsymbol{V_r}}{dt} \qquad (16)$$

It is better to write the matrix form of formula (16) in the mobile reference system in which the last component looks most simple:

$$W_a = W_A + (\mathcal{E} + \Omega^2)\rho + 2\Omega\dot{\rho} + \ddot{\rho} \qquad (17)$$

To find the transitional acceleration, we fix, by definition, the particle on the carrying body. Then $\boldsymbol{V_r}, \boldsymbol{W_r} = \boldsymbol{0}$ and absolute acceleration becomes the transitional one by definition.

$$\boldsymbol{W_e} = \boldsymbol{W_A} + \boldsymbol{\varepsilon} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \qquad (18)$$

We see that formula (18) is the same as the formula for the acceleration of the body particle, as it should be by definition. Stopping the carrying body ($\boldsymbol{W_A}, \boldsymbol{\omega}, \boldsymbol{\varepsilon} = 0$), we find the relative acceleration

$$\boldsymbol{W_r} = \frac{d_r \boldsymbol{V_r}}{dt} \qquad (19)$$

Component

$$\boldsymbol{W_c} = 2(\boldsymbol{\omega} \times \boldsymbol{V_r}) \qquad (20)$$

is called additional or ***Coriolis acceleration***

We come to ***Coriolis theorem***

$$\boldsymbol{W_a} = \boldsymbol{W_e} + \boldsymbol{W_r} + \boldsymbol{W_c} \qquad (21)$$

We see that unlike the speeds, the sum of transitional and relative accelerations does not give, in general, the absolute acceleration. That is why ***Coriolis*** acceleration is called additional.

This acceleration is named after the French scientist Gustave Gaspard Coriolis, who described it for the first time in 1833 (by Gauss in 1803, and by Euler in 1765 year (!)).

The necessity of Coriolis acceleration is evident from the following simple example. Platform of radius R rotates evenly with angular velocity ω (Fig. 3). The man runs on the edge of the platform against the rotation with relative speed

$$V_r = \omega R,$$

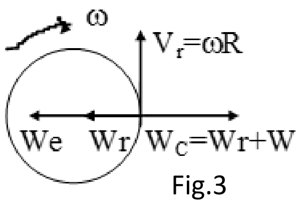


Fig.3

Thus, in relation to the Earth the man is motionless, and its absolute acceleration is zero. However, the sum of transition and relative accelerations is not zero.

Really, the relative acceleration $\boldsymbol{W_r}$ being a normal acceleration of the particle, is directed toward the center of the platform and is equal to:

$$W_r = \frac{V_r^2}{R} = \omega^2 R$$

Transitional acceleration of the particle, being centripetal acceleration particle is also directed toward the center of the platform and is equal to the relative acceleration

$$W_e = \omega^2 R = W_r$$

The sum of accelerations

$$\boldsymbol{W_e} + \boldsymbol{W_r}$$

is directed to the center and is not equal to zero.

$$W_e + W_r = 2\omega^2 R$$

Only the ***Coriolis*** acceleration $\boldsymbol{W}_C$ ensures the absence of absolute acceleration. Vector of angular velocity $\boldsymbol{\omega}$ is into the drawing, so $\boldsymbol{W}_C$ is directed from the center and by modulo it equals

$$W_c = 2\omega V_r = 2\omega^2 R$$

Now, according to acceleration theorem absolute acceleration $\boldsymbol{W_a}$ becomes zero. In projections on the radius:

$$W_a = W_c - W_e - W_r = 0$$

Coriolis acceleration

$$\boldsymbol{W_c} = 2(\boldsymbol{\omega} \times \boldsymbol{V_r}) \qquad (22)$$

It is directed according to the rule of the right screw and is zero in three cases

$$\boldsymbol{W_c} = \boldsymbol{0}:$$

1. Carrying body in translation or reverses the direction of rotation ($\boldsymbol{\omega} = \boldsymbol{0}$)
2. Relative velocity $\boldsymbol{V_r}$ is parallel to the angular velocity of the body **ω**. So $\boldsymbol{W_c} = \boldsymbol{0}$ when driving along the meridian at the intersection of the Earth's equator.
3. Particle stopped on the carrying body ($\boldsymbol{V_r} = \boldsymbol{0}$)

On this basis, we conclude that Coriolis acceleration describes:

1. The change of the transitional speed $\boldsymbol{\omega} \times \boldsymbol{\rho}$ due to changes in the relative position of the particle.
2. The change of direction of the vector of relative speed of $\boldsymbol{V_r}$ by turning carrying body with an angular velocity $\boldsymbol{\omega}$.

Example of solving a problem using vector and matrix methods

Let's take the same example as in for theorem of velocities composition

$$\omega = 2c^{-1}. \quad y = 3t^2 - 2t \text{ (м)}.$$

First, we use the ***stopping method*** (Fig. 4)

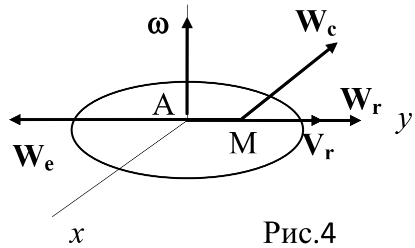


Рис.4

Relative motion (ω = 0)

$$W_{rx} = \ddot{x} = 6 \text{ м/с}^2$$

Translational motion ($\boldsymbol{V_r = 0}$)

$$W_e = \omega^2 x|_{t=1} = 4 \text{ м/с}^2$$

Coriolis acceleration

$$W_c = 2\omega V_r = 16 \text{ м/с}^2$$

Coriolis's theorem in projections on rotating axis

$$W_{ax} = -W_c = -16 \text{м/с}^2; \qquad W_{ay} = W_r - W_e = 2 \text{ м/с}^2; \quad W_z = 0$$

The same result we get by ***matrix method***.

$$W_a = W_A + (\mathcal{E} + \Omega^2)\rho + \ddot{\rho} + 2\Omega\dot{\rho}$$

In moving axis:

$$W_A = 0; \qquad \mathcal{E} = \dot{\Omega} = 0$$

$$\Omega^2 = -\omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \rho = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}; \quad \dot{\rho} = \begin{pmatrix} 0 \\ \dot{y} \\ 0 \end{pmatrix}; \quad \ddot{\rho} = \begin{pmatrix} 0 \\ \ddot{y} \\ 0 \end{pmatrix}$$

$$W_a = (\Omega^2)\rho + \ddot{\rho} + 2\Omega\dot{\rho}$$

$$\begin{pmatrix} W_{ax} \\ W_{ay} \\ W_{az} \end{pmatrix} = -\omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3t^2 - 2t \\ 0 \end{pmatrix} + 2\omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 6t - 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}_{t=1}$$

$$= \begin{pmatrix} -16 \\ 2 \\ 0 \end{pmatrix} \text{м/с}^2$$

We see that the results are the same.
Advantage of matrix method is the possibility to get the result for any moment of time without drawing vectors.

Interactive excel problems
on kinematics of resultant motion of particle

At each step of the solution, the incorrectly filled cell is colored pink, which allows the student to independently find the error.

The problems and an example of its solution can be downloaded from the link https://disk.yandex.ru/d/KUS3FkBiatmT_A