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ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ ПЕТРА ВЕЛИКОГО

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Dynamics of the particle

DYNAMICS OF THE PARTICLE

Dynamics is the main branch of mechanics. In this section, the laws of motion of a rigid body under the influence of applied forces are studied. The simplest object of dynamics is a material particle, i.e., a body whose dimensions can be neglected in comparison with the length of its trajectory. For example, the planet Earth can be taken as a material particle if we consider its motion around the Sun.

Returning once more to the principles of mechanics.

In statics we have already formulated the Principles of Mechanics. Now let's consider their interpretation and consequences for the dynamics.

1. Galileo's principle of inertia (Newton's first law).

There exists a reference system in which an isolated material particle maintains rest or uniform rectilinear motion (by inertia).

An isolated particle is a particle that does not interact with other particles. Obviously, the concept of an isolated particle is an abstraction, it is impossible to find such a particle. However, this fundamental concept allowed Galileo to understand that the action of forces is not required for motion by inertia. After all, before him, people believed that to move you need to apply force (the cart needs to be pushed), forgetting about the forces of resistance.

There is no experimental evidence for the existence of inertial frames of reference. Obviously, this concept is also an abstraction. However, systems have been found that are very close to inertial systems. The heliocentric frame of reference can be considered the "most" inertial frame of reference. Its center is in the Sun, and its axes are directed to distant stars.

It will be shown that the reference frame associated with the Earth is not inertial. However, the error in the implementation of Newton's laws on Earth is small.

2. Basic principle (Newton's second law).

Acceleration of a material particle is proportional to the applied force and inversely proportional to the mass of the particle

$$\boldsymbol{W} = \frac{1}{m}\boldsymbol{F}$$

Here m is the mass of the particle, a scalar constant.

The principle makes it possible to distinguish an inertial system from a non-inertial one. The motorcyclist feels a force that accelerates him in relation to a standing person. A person does not feel any force from acceleration in relation to a motorcyclist. The reason is that the frame of reference associated with a standing person is inertial, while the motorcyclist's system is not. In other words, the acceleration of a motorcyclist is caused by a force acting on the wheel, and the acceleration of a standing person is not caused by a force, but only by the acceleration of a motorcyclist.

Corollary 1: Impossible to create force without resistance, namely, the mass, which expresses the ability of a particle to resist a change in its velocity.

Corollary 2: For a given mass, the force is determined by acceleration. With a small mass of a particle, the magnitude of the force is limited by the ability to create a large acceleration of the particle. It is impossible to apply a large force to a fluff, since this would require the creation of a very large acceleration. If you hit a pillow lying on the scales with a stick, the arrow will swing less than when striking without a pillow. The reason is acceleration. The pillow reduces the acceleration of the stick (a large path of deformation of the pillow), which means that it reduces the force of impact for a given weight of the stick.

This is the only quantitative law of mechanics. It connects the three magnitudes m, **W,** and **F**, thus expressing one of them through the other two independent ones. Acceleration **W** with dimension [W] is always taken as an independent quantity, since it relates the basic quantities of length and time.

Depending on what is taken as the second independent quantity (m or F), ***Two types of system of mechanical units***:

1) Systems in which the mass m of dimension [m] is taken as the second independent quantity. An example is the SI system. In it, [W] = m/sec2, [m] = kg, and the derived unit of force is called Newton:
$$[F] = [m][w] = \text{Н} = \text{к}g\ \text{м}/s\text{ек}^2$$
2) Systems in which force is taken as the second independent quantity. An example is the Technical System. In it, [W] = m/sec2, [F] = kGf, and the derived unit of force is called the Technical Unit of Mass:
$$[m] = [F]/[w] = \text{TEM} = \text{kGf sec}^2/\text{m}$$

When solving problems, it is important to perform all calculations in the same system of units. Let us recall the ratio between the units of forces in the two systems. $1kGf = 9{,}8\text{H}$

3. The principle of equality of action and reaction.

Two particles interact with oppositely directed forces of the similar modulo. These forces are balanced only for particles of the same solid.

4. The principle of external additivity (the rule of addition of forces)

The action of the medium on the particle can be replaced by one force F, which is equal to the sum of the forces with which the particles of the medium act on the particle under study.

$$\boldsymbol{F} = \sum \boldsymbol{F}_{\mathrm{k}}$$

Unlike static, ***Forces in Dynamics*** can be functions of the position of a particle (its radius - vector **r**), velocity $\boldsymbol{V}$ and the independent variable - time t.

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{V};\ t)$$

Consider, for example, the forces acting on a rocket (Fig.1): the force of gravity $\boldsymbol{P}(\boldsymbol{r})$depends on the distance to the Earth, the thrust force of the engine $\boldsymbol{F}(t)$ is a function of time, the force of air resistance $\boldsymbol{R}(\boldsymbol{r}, \boldsymbol{V})$ depends on the velocity of the rocket and the density of the atmosphere (distance to the Earth)

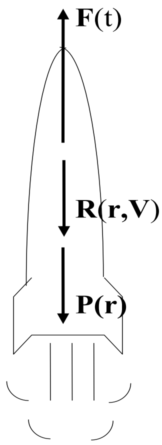


Рис 1

Differential equations of particle motion.

Let us write down Newton's second law, taking into account that the acceleration of a particle is the second time derivative of the radius - vector

$$m\ddot{\boldsymbol{r}} = \sum \boldsymbol{F}_k\,(\boldsymbol{r}, \dot{\boldsymbol{r}}; t) \qquad (1)$$

An expression that containing the ordinary derivatives of the desired function of the independent variable is called an ***ordinary differential equation***. The order of the higher derivative is called the order of the differential equation. Equation (1) is a second-order vector differential equation.$\boldsymbol{r}(t)$.

To solve problems, equation (1) must be written in scalar form, that is, in projections on the coordinate axis. Projecting (1) on the Cartesian axes, we find the differential equations of motion of a particle in Cartesian coordinates:

$$\begin{aligned} m\ddot{x} &= \sum F_{kx}\,(x, y, z; \dot{x}, \dot{y}, \dot{z}; t) \\ m\ddot{y} &= \sum F_{ky}\,(x, y, z; \dot{x}, \dot{y}, \dot{z}; t) \quad (2) \\ m\ddot{z} &= \sum F_{kz}\,(x, y, z; \dot{x}, \dot{y}, \dot{z}; t) \end{aligned}$$

This system of differential equations has a sixth order.

Equation (1) in projections on the axes τ, n, b gives three differential equations in the natural axes.

$$m\ddot{\sigma} = \sum F_{k\tau} \qquad \frac{m\dot{\sigma}^2}{\rho} = \sum F_{kn} \qquad 0 = \sum F_{kb}$$

Here it is taken into account that the projection of acceleration on binormal b is equal to zero.

Direct and inverse problems of particle dynamics

Differential equations, for example, in Cartesian coordinates (4), allow the formulation of two types of problems of particle dynamics:

1) ***The direct problem of particle dynamics*** is to determine the resultant forces applied to the particle according to a given law of its motion. Suppose the law of motion of a particle in Cartesian coordinates is given.

$$x(t), y(t), z(t)$$

Need to find resultant R(t).

The solution of this problem is associated with the differentiation of the law of motion. Projections and modulus of resultant forces are found according to the formulas:

$$R_x = m\ddot{x}; \quad R_y = m\ddot{y}; \quad R_z = m\ddot{z}$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

Example of a direct problem:

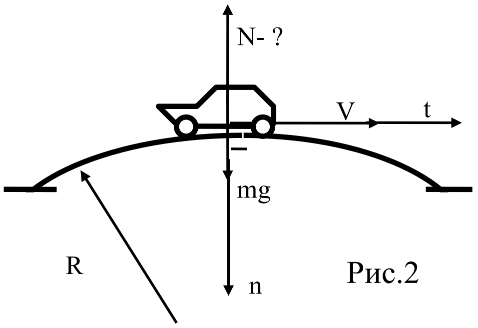


Рис.2

Find the reaction N of the bridge of radius R from the car of mass m moving at velocity V on the top of the bridge (Fig.2). Since the trajectory of movement is known, we need to use the equations in the natural axes:

In the projection on the normal

$$\frac{mV^2}{R} = mg - N$$

So

$$N = m\left(g - \frac{V^2}{R}\right)$$

2) ***The inverse problem of particle dynamics*** is the main one and consists in determining the law of motion of a particle under the given forces.

In this case, equations (2) are a system of differential equations for finding three unknown functions of time t

$$x(t), \quad y(t), \quad z(t)$$

The solution of the inverse problem is associated with the integration of system (4) of the sixth order. With integration, six constants arise and the solution (the second integral of the equations) will be:

$$x = x\,(t;\ C_1\, C_2\ \ C_3\, C_4\, C_5\, C_6)$$
$$y = y\,(t;\ C_1\, C_2\ \ C_3\, C_4\, C_5\, C_6)$$
$$z = z\,(t;\ C_1\, C_2\ \ C_3\, C_4\, C_5\, C_6)$$

The presence of integration constants indicates that system (4) has many solutions. This means that the forces do not unambiguously determine the motion of a particle. In other words, the same force causes different trajectories of the particle.

For example, the movement of a stone under the influence of the same gravitational force can move along different trajectories depending on how it is thrown (Fig. 3). Integration constants are determined from the initial conditions of motion.

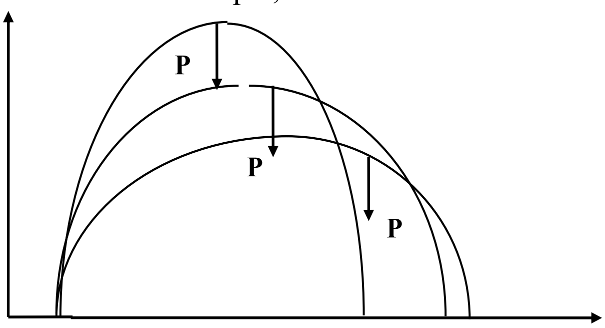


Рис.3

$$t = 0\text{:}\ \ x = x_0;\ \ y = y_0;\ \ z = z_0$$

$$\dot{x} = \dot{x}_0 \ \ \dot{y} = \dot{y}_0 \ \ \dot{z} = \dot{z}_0$$

To determine the integration constants, you need to substitute these conditions into solution (9) and its derivative (the first integral of the equations)

$\dot{x} = \dot{x}(t;\ C_1\, C_2\ \ C_3\, C_4\, C_5\, C_6)$

$\dot{y} = \dot{y}(t;\ C_1\ C_2\ \ C_3\ C_4\ C_5\ C_6)$

$\dot{z} = \dot{z}(t;\ C_1\ C_2\ \ C_3\ C_4\ C_5\ C_6)$

It gives an algebraic system with respect to the constants C_1,... C_6, which has a single solution.

DYNAMICS OF THE RELATIVE PARTICLE MOTION

Basic equation of dynamics of relative particle motion

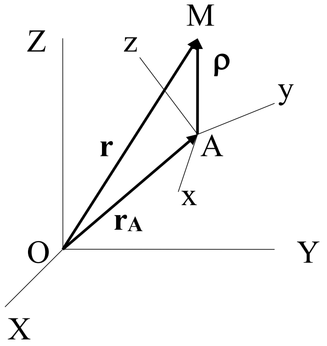
Newton's laws are valid only in an inertial frame of reference. Only one frame of reference is known, close to the inertial one. It is a heliocentric system, the axes of which originate in the Sun and are directed to distant stars. It has been experimentally established that the reference system associated with the Earth is not inertial.

A discouraging feeling arises: the only inertial frame of reference exists so far away from us that Newtonian mechanics is not suitable for studying the motion of bodies near the Earth. Let us show that there are innumerable inertial frames, and that Newton's like laws can be applied in non-inertial frames of reference if we know how these systems move.

First, let us find out how to construct the differential equations of motion of the particle with respect to a non-inertial frame of reference. Let the motion of a mobile system with axes x, y, z with respect to an inertial system with axes X, Y, Z be given by the functions of the coordinates of the origin A and Euler angles:

$$X_A(t), Y_A(t), Z_A(t) \qquad \psi(t), \Theta(t), \varphi(t) \qquad (3)$$

According to the law (3), it is possible to find velocity and acceleration $\boldsymbol{V_A}$ $\boldsymbol{W_A}$ of the origin A , angular velocity $\boldsymbol{\omega}$, and acceleration $\boldsymbol{\varepsilon}$ of a movable frame of reference.



In the inertial frame of reference X, Y, Z, observer O will write down the basic law of the dynamics of particle M in the form:

$$m\boldsymbol{W_a} = \sum \boldsymbol{F_k}$$

Here $\boldsymbol{F_k}$ are physical forces acting from other particles.

According to the Coriolis theorem:

$$\boldsymbol{W_a} = \boldsymbol{W_e} + \boldsymbol{W_r} + \boldsymbol{W_c}$$

The Basic Law takes the form of:

$$m(\boldsymbol{W_e} + \boldsymbol{W_r} + \boldsymbol{W_c}) = \sum \boldsymbol{F_k}$$

Or:

$$m\boldsymbol{W_r} = \sum \boldsymbol{F_k} - m\boldsymbol{W_e} - m\boldsymbol{W_c}$$

The latter terms are called **the transport and Coriolis forces of inertia** , respectively:

$$\boldsymbol{\Phi_e} \equiv -m\boldsymbol{W_e}; \qquad \boldsymbol{\Phi_c} \equiv -m\boldsymbol{W_c}$$

$$\boldsymbol{\Phi_e} \equiv -m[\boldsymbol{W_A} + \boldsymbol{\varepsilon} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]; \qquad \boldsymbol{\Phi_c} \equiv -2m(\boldsymbol{\omega} \times \boldsymbol{V_r})$$

Expression

$$m\boldsymbol{W_r} = \sum \boldsymbol{F_k} + \boldsymbol{\Phi_e} + \boldsymbol{\Phi_c}$$

is called ***the basic equation of the dynamics of the relative motion of a particle***.

Thus

to construct the equation of motion of a particle with respect to a non-inertial frame of reference, it is necessary to add the forces of inertia $\boldsymbol{\Phi_e}, \boldsymbol{\Phi_c}$ to physical forces $\sum \boldsymbol{F_k}$

Inertial forces differ from physical forces in that they are not caused by the action of other particles, but are determined only by the movement of the mobile frame of reference. Therefore, it is said that the forces of inertia $\boldsymbol{\Phi_e}, \boldsymbol{\Phi_c}$ have ***kinematic*** character.

The dispute about the reality or imaginary nature of the forces of inertia $\boldsymbol{\Phi_e}, \boldsymbol{\Phi_c}$ seems to have a simple solution: they are real for the observer A of the mobile reference system and imaginary for the observer O of the absolute reference system.

For example, two observers will explain the break of the thread in different ways when the rotation velocity of the ball along a fixed plane increases.

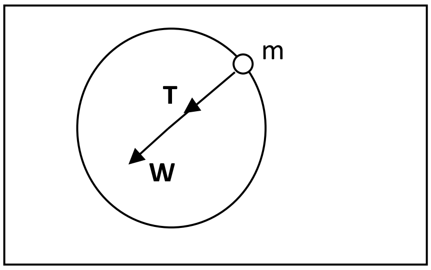


Fig.4

A stationary observer (Fig. 4) connected to a plane says: The tension ***T*** of the thread creates an axial acceleration ***W*** of the ball. The thread breaks because the force of its tension ***T*** reaches the limit value**.**

Movable Observer (Fig.5) connected to the thread says: the ball is stationary, and the tension force ***T*** balances the centrifugal force of inertia $\boldsymbol{\Phi_e}$.

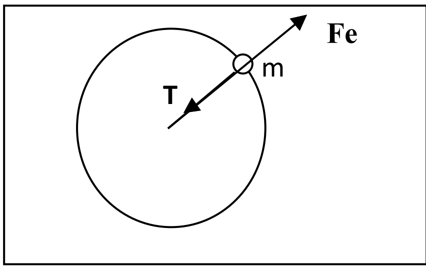


Fig.5

Failure to comply with Galileo's principle. Relative rest condition

Consider an isolated particle in a non-inertial frame of reference.

$$\sum \boldsymbol{F_k} = \boldsymbol{0}$$

Equation of Relative Motion of a Particle

$$m\boldsymbol{W_r} = \boldsymbol{\Phi_e} + \boldsymbol{\Phi_c} \neq \boldsymbol{0}$$

shows that the isolated particle does not conserve the velocity vector

$$\boldsymbol{V} \neq \boldsymbol{Const}$$

in a non-inertial frame of reference.

Let the particle now be in a state of relative rest. Then $\boldsymbol{\Phi_c} = \boldsymbol{0}$.

Let's release the particle. It will start moving, because

$$m\boldsymbol{W_r} = \boldsymbol{\Phi_e} \neq \boldsymbol{0}; \qquad \boldsymbol{V} \neq \boldsymbol{Const} = \boldsymbol{0}$$

Conclusion*: *Galileo's principle does not hold in a non-inertial frame of reference.

Let us stop the particle M, which is under the influence of physical forces

$$\sum \boldsymbol{F_k} \neq \boldsymbol{0}$$

relative to the movable frame of reference. Then

$$\boldsymbol{V_r} = \boldsymbol{0} \quad \text{и} \quad \boldsymbol{\Phi_c} \equiv -2m(\boldsymbol{\omega} \times \boldsymbol{V_r}) = \boldsymbol{0}$$

For a particle to remain at rest, there must be no relative acceleration of the particle. According to the basic equation of the dynamics of the relative motion of a particle

$$m\boldsymbol{W_r} = \sum \boldsymbol{F_k} + \boldsymbol{\Phi_e} = \boldsymbol{0}$$

Thus, ***the condition for relative rest*** is:

$$\sum \boldsymbol{F_k} + \boldsymbol{\Phi_e} = \boldsymbol{0}$$

It differs from the known rest condition of a particle in an inertial system

$$\sum \boldsymbol{F_k} = \boldsymbol{0}$$

Conditions of inertiality of a mobile system. Galileo's principle of relativity

Let us do away with the myth about the uniqueness of the heliocentric inertial frame of reference. Let us find out how the frame of reference should move in relation to the inertial frame of reference in order to also be inertial, i.e., so that Newton's laws are observed in it. Obviously, for this to happen, the following inertia forces must be absent in the mobile reference system:

$$\boldsymbol{\Phi_e} \equiv -m\boldsymbol{W_e} = \boldsymbol{0}; \qquad \boldsymbol{\Phi_c} \equiv -m\boldsymbol{W_c} = \boldsymbol{0}$$

To do this, the transport and Coriolis particle accelerations must turn to zero.

$$\boldsymbol{W_e} = \boldsymbol{W_A} + \boldsymbol{\varepsilon} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \boldsymbol{0} \qquad \boldsymbol{W_c} = 2(\boldsymbol{\omega} \times \boldsymbol{V_r}) = \boldsymbol{0}$$

Let the system move translatory, then

$$\boldsymbol{\omega} = \boldsymbol{0}; \quad \boldsymbol{\varepsilon} = \boldsymbol{0}$$

and

$$\boldsymbol{\Phi_c} = \boldsymbol{0}\,; \qquad \boldsymbol{\Phi_e} \equiv -m\boldsymbol{W_A} \neq \boldsymbol{0}$$

In order for both forces of inertia to turn to zero, it remains to demand:

$$\boldsymbol{W_A} = \boldsymbol{0}$$

Thus, ***the condition of inertiality of a moving frame of reference*** is its translational, rectilinear and uniform motion relative to the initial inertial system.

$$\boldsymbol{\omega} \equiv \boldsymbol{0}; \quad \boldsymbol{W_A} = \boldsymbol{0}$$

This leads to an optimistic conclusion:

If there is at least one inertial system, then there are countless of them, and all of them move translatory, in a straight line, and uniformly with respect to each other.

In all these inertial systems, Newton's laws hold in the same way.

Therefore, ***Galileo's principle of relativity is valid:***

No mechanical experiment can distinguish one inertial system from another.

Proof of the non-inertial nature of a geocentric system.

Since the Earth rotates with respect to an inertial heliocentric frame of reference, the space associated with the Earth cannot be inertial. This is evidenced by many phenomena on Earth.

1. ***The force of gravity is not equal to the force of attraction and depends on the latitude***.

Consider the particle of mass m , which is at rest near the Earth's surface at geocentric latitude φ. It is affected by the physical force of attraction of the Earth's mass M,

$$\boldsymbol{F} = m\boldsymbol{g_0};$$

directed towards the center of the Earth, and obeying the law of universal gravitation

$$F = \gamma \frac{mM}{R^2} = mg_0; \qquad g_0 = \gamma \frac{M}{R^2} = 9{,}832 \text{ м/с}^2$$

Here g_0*is* ***the acceleration of the force of attraction***.

In a non-inertial frame of reference, a transport force of inertia $\boldsymbol{\Phi_e}$ acts on a particle at rest

$$\Phi_e = m\omega^2 RCos\varphi$$

The force of gravity is the sum of

$$\boldsymbol{P} = \boldsymbol{F} + \boldsymbol{\Phi_e}$$

A particle cannot remain at rest under the influence of a single force. Let's hang the particle from the ceiling. Then the particle will find the long-awaited rest under the influence of two forces equal in modulus and opposite in direction:***P***

$$\boldsymbol{T} = -\boldsymbol{P}; \qquad T = P = mg$$

We see that the plumb line, like the force of gravity ***P***, is not directed to the center of the Earth. It forms an angle of ψ with the plane of the equator, called ***the geographical latitude.*** Due to the low angular velocity of the Earth's rotation, the maximum difference is about 4 degrees

$$\gamma = \psi - \varphi$$

$$Sin\,\gamma \approx 0; \quad Cos\,\gamma \approx 1 \qquad Cos\psi \approx Cos\varphi;$$

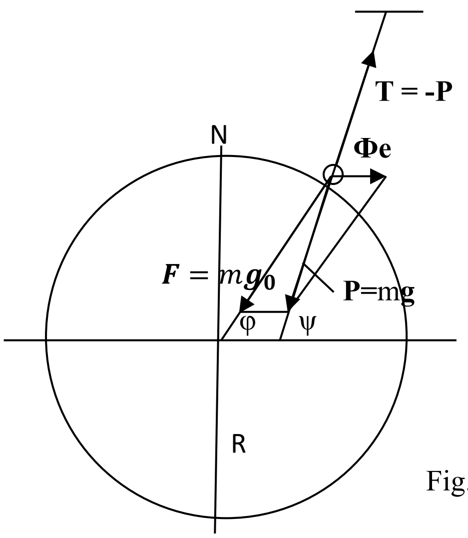


Fig.4

Let's project the equation of the relative rest of a particle

$$\boldsymbol{F} + \boldsymbol{T} + \boldsymbol{\Phi_e} = \boldsymbol{0}$$

on the direction of the plumb line

$$T - FCos\gamma + \Phi_e Cos\psi = 0; \qquad T = mg$$

Otherwise

$$mg = FCos\gamma - \Phi e Cos\psi \approx mg0 - \; m\omega^2 RCos2\,\varphi$$

From this we find the dependence of the acceleration of gravity from the latitude

$$g(\varphi) = g_0\left(1 - \frac{\omega^2}{g_0} RCos^2\varphi\right)$$

We see that at the poles $\varphi = \pm\frac{\pi}{2}$ the acceleration of the force of gravity is equal to the acceleration of the force of attraction

$$g = g_0 = 9{,}832\ \text{м/с}^2$$

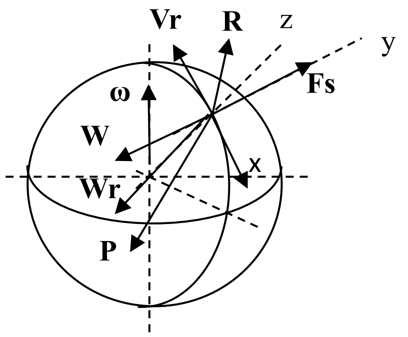
The acceleration of gravity reaches a minimum at the equator

$$g_{min} = g_0\left(1 - \frac{\omega^2}{g_0}R\right) = 9{,}780\ \text{м/с}^2$$

The action of centrifugal forces of inertia eventually turned the Earth into ***a geoid***, flattened at the poles.

2. *Erosion of river banks*

Let a river in the northern hemisphere flow from south to north at a relative velocity $\boldsymbol{V}_r$. Let us consider the volume of water of mass m between two river sections perpendicular to the riverbed. Let us put the origin of coordinates in the mass m, and direct the axes: x − to the south, y −to the east, z −vertically.



The mass m is affected by: the force of gravity $\boldsymbol{P}$ including the transport force of inertia $\boldsymbol{\Phi_e}$, it belongs to the plane $x\ z$,

The reaction $\boldsymbol{R}$ of the channel, the Coriolis force of inertia $\boldsymbol{\Phi}_c$ directed to the east, opposite of the Coriolis acceleration $\boldsymbol{W}_c$.

The relative acceleration $\boldsymbol{W}_r$is directed towards the center of the Earth, since the mass m moves uniformly along the meridian.

Equation of Relative Motion of Mass m

$$m\boldsymbol{W}_r = \boldsymbol{P} + \boldsymbol{R} + \boldsymbol{\Phi}_c$$

Let's project it on the axis of y:

$$0 = R_y + \Phi_c$$

Hence,

$$R_y = -\Phi_c < 0$$

This means that the reaction $\boldsymbol{R}$ of the bank is deflected to the West by the eastern (right) bank of the river. The pressure of the river of the same module erodes the right bank. If the river flows to the south, then the western (also right) bank is eroded. So, all rivers in the northern hemisphere are eroding their right banks. For the same reason, in the northern hemisphere, the right rails of railways wear out more.

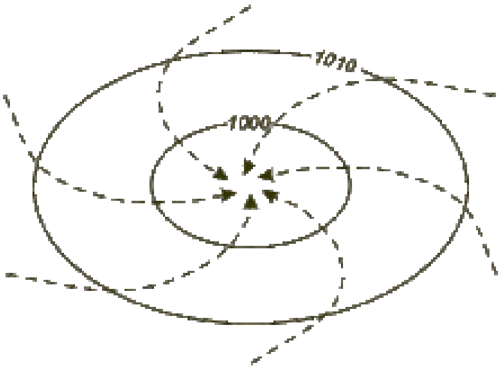
In the southern hemisphere, the left banks of rivers are eroded and the left rails wear out more.

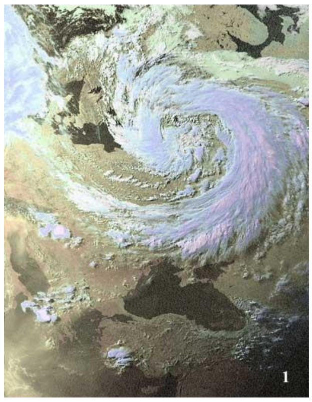
Thus, when a particle in the northern hemisphere moves, it always bends to the right. If this deviation is hindered by constraints (banks, rails), then the constraints experience additional pressure. If there are no constraints, then the motion acquires a specific character, which depends on the direction of the relative velocity.

For example, all ocean currents in the northern hemisphere move clockwise. In the southern hemisphere, the currents are counterclockwise.

In phenomena such as a whirlpool of water and an atmospheric cyclone, the particles move radially to the center of the whirlpool or cyclone, where the area of low pressure is located.

In the northern hemisphere, the radial relative velocity of particles causes them to deflect to the right, forming cyclones that rotate counterclockwise. In the southern hemisphere, cyclones rotate clockwise.



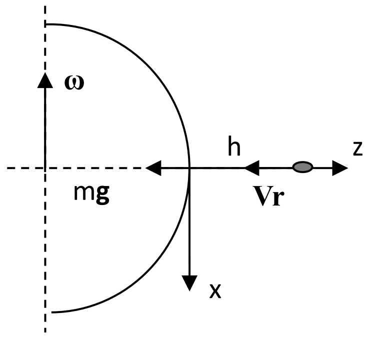


The direction of rotation of cyclones is easy to remember if you understand that they rotate in the direction of the Earth's rotation, as we see it from our pole: North, or South.

A terrible earthquake off the coast of Japan on 11.03.2011 caused a tsunami and a whirlpool that rotated counterclockwise.

3.Falling particle deviation

Consider a material particle falling from a height h to the earth from the state of relative rest. Let's suppose, for simplicity, it happens at the equator. Equation of Relative Motion:



$$m\boldsymbol{W}_r = m\boldsymbol{g} + \boldsymbol{\Phi}_c; \qquad \boldsymbol{\Phi}_c = -2m\boldsymbol{\omega}x\boldsymbol{V_r}$$

Force of gravity $m\boldsymbol{g}$ includes the transport force of inertia $\boldsymbol{\Phi_e}$.

The Coriolis acceleration $2\boldsymbol{\omega}x\boldsymbol{V_r}$ is directed westward (against the y-axis).

Projecting the equation on the moving axes: x —south, y —east, z —vertical

$$\ddot{x} = -2(\omega_y\dot{z} - \omega_z\dot{y}); \quad \ddot{y} = -2(\omega_z\dot{x} - \omega_x\dot{z})$$

$$\ddot{z} = -g - 2(\omega_x\dot{y} - \omega_y\dot{x})$$

Taking into account the fact that $\omega_x = -\omega; \quad \omega_y = \omega_z = 0$, we find

$$\ddot{x} = 0; \quad \ddot{y} = -2\omega\dot{z}; \quad \ddot{z} = -g + 2\omega\dot{y}$$

As the Earth rotates slowly

$$\omega < 0{,}0001\ c^{-1}$$

then it can be considered that

$$\ddot{z} \approx -g; \qquad z = h - gt^2/2$$

At the moment of falling T:

$$z = 0; \qquad T = \sqrt{\frac{2h}{g}}$$

Integrating the equation on y, we get:

$$\dot{y} = -2\omega z + \mathrm{C}$$

From the initial conditions

$$\mathrm{C} = 2\omega h$$

And now

$$\dot{y} = -2\omega(z - h) = \omega g t^2$$

$$y = \omega\frac{gt^3}{3}$$

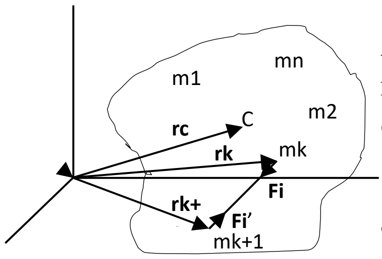
At the moment of falling, the deviation of the particle to the east reaches the value of

$$y(T) = \frac{\omega g}{3}\left(\frac{2h}{g}\right)^{\frac{3}{2}}$$

GENERAL THEOREMS OF SYSTEM DYNAMICS

Material system. Center of Mass and Center of Gravity.

Let us name ***Material system*** the set of interacting material particles $m_1, m_2, \dots . m_k \dots m_n$. Example: solar system.



A system of material particles, the interaction of which can be neglected in comparison with the interaction with the external environment, is not material. Example: a group of airplanes.

The mass of a system is an arithmetic quantity equal to the sum of the masses of the particles of the system

$$M = \sum m_k$$

The motion of the particles is considered in relation to the inertial frame of reference. ***A reference system*** is a three-dimensional space with which an observer who is able to measure distances and time is connected. The boundaries of the system are determined by the observer. A frame of reference is inertial if it holds Newton's laws.

The center of mass of the system is called the geometric point C, the radius-vector of which is equal to

$$\boldsymbol{r}_c = \frac{1}{M}\sum m_k \boldsymbol{r}_k$$

Its coordinates in Cartesian axes

$$x_c = \frac{1}{M}\sum m_k x_k; \quad y_c = \frac{1}{M}\sum m_k x_k; \quad z_c = \frac{1}{M}\sum m_k x_k;$$

A rigid body is a system of an infinite but countable number of mass particles, the distances between which are invariable in time. The elementary particle of the body of volume dV has a mass dm = γ(r)dV, where γ(r) is the density of the body depending on the radius-vector of the particle. A body is said to be homogeneous if the γ is independent of r.

The volume of a body is an integral in volume

$$V = \iiint dV$$

Body mass

$$M = \iiint \gamma(x, y, z) dxdydz$$

The center of mass of a body is determined by a vector

$$\boldsymbol{r}_c = \frac{1}{M}\iiint \boldsymbol{r}\gamma(r) dxdydz$$

For a homogeneous body

$$M = \gamma V \qquad \text{and} \qquad \boldsymbol{r}_c = \frac{1}{V}\iiint \boldsymbol{r} dxdydz$$

The gravitational field is determined by the gravitational acceleration vector **g (r).**

$$d\boldsymbol{P} = \boldsymbol{g}(\boldsymbol{r})dm = \boldsymbol{g}(\boldsymbol{r})\gamma(r)\, dV$$

The weight of the body is a vector

$$\boldsymbol{P} = \iiint \mathrm{d}\mathbf{P}$$

For a homogeneous body

$$d\boldsymbol{P} = \gamma\, \boldsymbol{g}\, (\boldsymbol{r}) dV$$

For small bodies near the Earth, the accelerations of gravitation can be considered parallel and dependent on the z coordinate of the vertical

$$\boldsymbol{g}\, (\boldsymbol{r}) = -g(z)\boldsymbol{k} \;\; (\boldsymbol{k} - \text{z ort})$$

Then, for a homogeneous body,

$$d\boldsymbol{P} = -\gamma g(z) dV\, \boldsymbol{k} = -dP\, \boldsymbol{k}$$

The center of gravity of a body is a point with a radius vector

$$\boldsymbol{r}_c = \frac{1}{P}\iiint \boldsymbol{r} dP$$

The field is homogeneous if **g** is the same for all particles. Then the gravity of the body is a vector

$$\boldsymbol{P} = M\boldsymbol{g}$$

and the center of gravity coincides with the center of mass of the body

The Earth's field is heterogeneous, so the center of gravity of television towers does not coincide with their center of mass. The question is: which is higher?

Classification of forces

The forces acting on the particles of the system are naturally divided into two classes.

Let us call the ***internal forces*** $\boldsymbol{F}_k^i$ the forces of interaction between the particles of the system, ***external forces*** $\boldsymbol{F}_k^e$ - the forces of interaction between the particles of the system and the particles outside the system. This division is conditional; it depends on the boundaries of the system chosen by us. For example, for chalk lying on the table, the force of interaction with the table can be internal, if the table is included in the system, and external, if the system is the chalk only.

Properties of internal forces

According to Newton's 3rd law, the internal forces are paired, which means that their principal vector and principal moment with respect to any point are equal to zero.

$$\boldsymbol{V}^i = \sum \boldsymbol{F}_k^i = \boldsymbol{0}, \quad \boldsymbol{M}_o^i = \sum \boldsymbol{m}_o\left(\boldsymbol{F}_k^i\right) = \boldsymbol{0}$$

Here $\boldsymbol{F}_k^i$ is the resultant (sum) of the internal forces applied to the particle m_k

Internal forces are balanced only for a solid. Thus, the solar system moves precisely under the influence of internal forces.

Differential equations of motion of a system

Newton's 2nd law for particles of a system

$$m_k\ddot{\boldsymbol{r}}_k = \boldsymbol{F}_k^e(\boldsymbol{r}_1 \ldots \boldsymbol{r}_n; \dot{\boldsymbol{r}}_1 \ldots \dot{\boldsymbol{r}}_n; t) + \boldsymbol{F}_k^i(\boldsymbol{r}_1 \ldots \boldsymbol{r}_n; \dot{\boldsymbol{r}}_1 \ldots \dot{\boldsymbol{r}}_n; t) \quad (k = 1,2, \ldots, n)$$

gives n ordinary vector differential equations of the 2nd order with respect to the laws of motion of particles $\boldsymbol{r}_k\,(t) \quad (k = 1,2, \ldots, n)$

To solve problems, scalar form of equations is required. In the Cartesian coordinate system, they are equivalent to 3n scalar equations.

$$m_k\ddot{x}_k = F_{kx}^e(x_1 \dots z_n; \dot{x}_1 \dots \dot{z}_n; t) + F_{kx}^i(x_1 \dots z_n; \dot{x}_1 \dots \dot{z}_n; t) \quad (k = 1,2, \dots, n)$$

$$m_k\ddot{y}_k = F_{kx}^e(x_1 \dots z_n; \dot{x}_1 \dots \dot{z}_n; t) + F_{kx}^i(x_1 \dots z_n; \dot{x}_1 \dots \dot{z}_n; t)$$

$$m_k\ddot{z}_k = F_{kx}^e(x_1 \dots z_n; \dot{x}_1 \dots \dot{z}_n; t) + F_{kx}^i(x_1 \dots z_n; \dot{x}_1 \dots \dot{z}_n; t)$$

In most cases, integrating these equations is analytically difficult because the internal forces are unknown functions. Even when they are known, for example, in the problem of three particles interacting according to the law of universal gravitation, there is no analytical solution. Numerically, they are solved without problems on the computer.

Sometimes it is enough to study the movement of the system "as a whole". This is especially true for a solid. For a solid, it is enough to learn how it’s center of mass moves and how the body rotates around the center of mass.

3 general theorems of system dynamics allow us to study the motion of the system as a whole:

- theorem of the change in momentum (theorem of the motion of the center of mass),

- Angular momentum change theorem,

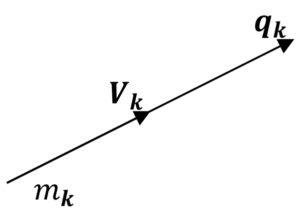
- Theorem of the change of kinetic energy.

THEOREM OF THE CHANGE IN MOMENTUM.

Theorem on the motion of the center of mass

The momentum of the particle m_k of the system is called the vector

$$\boldsymbol{q_k} = m_{\boldsymbol{k}}\boldsymbol{V_k}$$



where $\boldsymbol{V_k}$is the velocity of the particle at the moment.

Consider the system $m_1, m_2, \dots . m_k \dots m_n$

The momentum of the system is the principal vector of the momentums of all particles of the system

$$\boldsymbol{Q} = \sum \boldsymbol{q_k} = \sum m_{\boldsymbol{k}}\boldsymbol{V_k}$$

In projections on Cartesian axes

$$Q_x = \sum m_{\boldsymbol{k}}\dot{x}_k \quad Q_y = \sum m_{\boldsymbol{k}}\dot{y}_k \qquad Q_x = \sum m_{\boldsymbol{k}}\dot{x}_k$$

Since the masses of the particles are constant, Q can be expressed in terms of the velocity of the center of mass

$$\boldsymbol{Q} = \frac{d}{dt}\sum m_{\boldsymbol{k}}\boldsymbol{r}_k = \frac{d}{dt}M\boldsymbol{r}_c = M\boldsymbol{V_c}$$

$$Q_x = M\dot{x}_c, \qquad Q_y = M\dot{y}_c, \qquad Q_z = M\dot{z}_c$$

Examples.

a) If the center of mass of a rotating body lies on the axis of rotation, then $\boldsymbol{V_c} = \boldsymbol{0}$, and the momentum of the body is zero.

b) The momentum of the wheel depends only on the velocity of its center and does not depend at all on the velocity of its rotation. $\boldsymbol{V_c}$

Let us write Newton's 2nd law for a particle in the system in the formm_k

$$\dot{\boldsymbol{q}}_k = \boldsymbol{F}_k^i + \boldsymbol{F}_k^e$$

Here $\boldsymbol{F}_k^i$is the resultant of all internal forces, and $\boldsymbol{F}_k^e$ of all external forces applied to them_k particle. Summing up by k, we get

$$\dot{\boldsymbol{Q}} = \boldsymbol{V}^i + \boldsymbol{V}^e$$

The principal vector of internal forces $\boldsymbol{V}^i = \boldsymbol{0}$, which leads to the ***theorem of the change in the momentum*** of the system

$$\dot{\boldsymbol{Q}} = \boldsymbol{V}^e \qquad (4)$$

In projections on Cartesian axes

$$\dot{Q}_x = \sum F_{kx}^e, \qquad \dot{Q}_y = \sum F_{ky}^e, \qquad \dot{Q}_z = \sum F_{kz}^e$$

As

$$\dot{\boldsymbol{Q}} = M\dot{\boldsymbol{V}}_c = M\boldsymbol{W}_c$$

then this theorem can be written in the form ***of a theorem on the motion of the center of mass.***

$$M\boldsymbol{W}_c = \boldsymbol{V}^e \qquad (5)$$

It has the form of Newton's second law:

The center of mass of the system moves
as a particle with the mass of the system M, to which all the external forces are applied.

So, if we neglect air resistance, then after the explosion of a firework projectile, the center of mass of its parts continues to move along the same trajectory (parabola) as the unexploded projectile.

In projections on Cartesian axes

$$M\ddot{x}_c = \sum F_{kx}^e \qquad M\ddot{y}_c = \sum F_{ky}^e \qquad M\ddot{z}_c = \sum F_{kz}^e \qquad (6)$$

Consequences of theorems

1. Internal forces do not directly affect the momentum of the system $\boldsymbol{Q}$ and the velocity $\boldsymbol{V_c}$of the center of mass. However, they can cause external forces that can change the momentum.

   For example, internal forces in the car engine cause friction between the wheels and the road, which moves the car, changing the velocity of its center of mass.

   Another example explains the "miracle." In South America, there is a tree from which nuts fall in the fall. After a while, the hard nuts begin to jump, causing terror among the uninitiated - After all, a solid body cannot jump. An explanation was found, cracking a nut. There they found a bug that emerged from a larva that gnawed through a nut and ate its contents. In the resulting space, the bug begins to jump. A nut jumps with it. Thus, the internal forces of the bug cause an external reaction of the Earth, which sets in motion the center of mass of the system beetle - nut. In exactly the same way, you can stand on a chair, cover yourself with a box and jump with this hard shell.

2. If $\boldsymbol{V^e} = \boldsymbol{0}$, then $\boldsymbol{Q}$ and $\boldsymbol{V_c}$are conserved. Thus, the center of mass of the solar system moves uniformly and in a straight line in the universe.
3. If $V_x^e = 0$, then Q_x and $\dot{x}_c$ are conserved.

For example, when a car with a jet engine is moving, the center of mass of the vehicle-fuel system remains in place: the car and exhaust gases move in different directions.

Here we can also give an example of a well-known scam. In the 80s, "inertoids" were demonstrated on popular science TV shows. They allegedly proved the existence, in addition to the generally recognized support and reactive methods of propulsion, also of the "inertial" method.

A box on a cart with freely rotating wheels was demonstrated. The toggle switch turned on, and the mechanism began to buzz inside the box. The cart was placed on the floor and released without a push. The cart began to move, which, allegedly, proved the presence of an inertial method of movement.

This experiment does not prove anything new. He simply illustrates consequence 3.

If you open the box, you will find a motor with an unbalanced weight on its axe. At the moment of releasing the cart, external forces along the x-axis disappear, and then the center of mass retains the horizontal component $\boldsymbol{V_{Cx}}$of its velocity. At the same time, the cart itself does not move evenly, but jerkily.

THEOREM OF THE CHANGE IN ANGULAR MOMENTUM

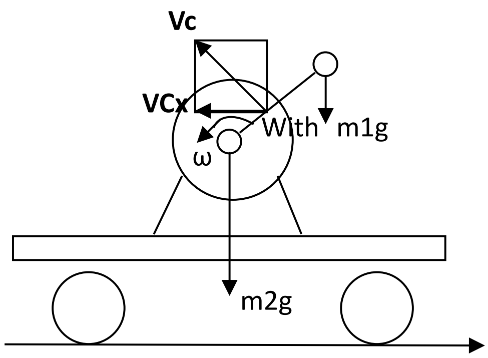
Angular momentum of particle and system with respect to a center and an axis

Let us consider a system of material particles with masses $m_1, m_2, \ldots . m_j \ldots m_n$ that have velocities $\boldsymbol{V_1}, \boldsymbol{V_2}, \ldots . \boldsymbol{V_n}$ relative to the inertial frame of reference at a given moment.

Angular momentum of particle m_j relative to the center O is a vector equal to the moment of its momentum relative to this center.

$$\boldsymbol{K_{oj}} = \boldsymbol{m_o}(\boldsymbol{q_j}) = \boldsymbol{r_j} \times \boldsymbol{m_j V_j} \qquad (j = 1,2 \ldots n)$$

It is known that vector product can be written in scalar form with the help of attached matrix *R* of the first factor radius-vector $\boldsymbol{r}$ **.**



Omitting the index j, we write the matrix expression in the axes x, y, z beginning at point O:

$$K_o = mRv$$

where *R is* the obliquely symmetric attached matrix of column *r*

$$\begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} = m\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = m\begin{pmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{pmatrix} \quad (7)$$

The projection of the Angular momentum on the axis is called ***the Angular momentum of the particle with respect to the axis***. It is calculated either analytically according to formulas (7) or as a moment of force relative to the axis. The moment is created only by the tangent component $\boldsymbol{q_\tau}$ of the vector $\boldsymbol{q}$ (Fig. 6).

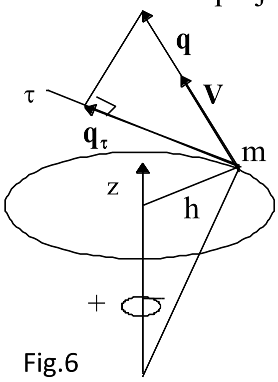


Fig.6

$$K_z = \mp q_\tau h$$

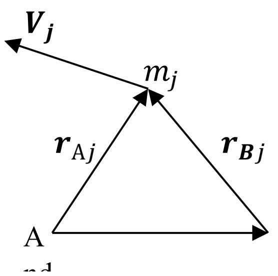
The angular momentum turns to zero if the momentum (the velocity of the particle) lies in the same plane with the axis (parallel or intersects the axis)

Angular momentum of the system relative to the center O is the main moment of momentums of the particles of the system relative to this center.

$$\boldsymbol{K_o} = \sum \boldsymbol{K_{oj}} = \sum m_j \boldsymbol{r_j} \times \boldsymbol{V_j}$$

Similarly to formula (7), the vector $\boldsymbol{K_o}$ projections form a column of angular momentums with respect to the coordinate axes

$$\begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} = \sum m_j \begin{pmatrix} y_j\dot{z}_j - z_j\dot{y}_j \\ z_j\dot{x}_j - x_j\dot{z}_j \\ x_j\dot{y}_j - y_j\dot{x}_j \end{pmatrix}$$



Let us find a connection between the angular momentums of the system with respect to the two fixed centers A and B. Let us denote the vectors of the particle m_j of the system relative to the centers A and B $\boldsymbol{r}_{Aj}$ and $\boldsymbol{r_{Bj}}$ respectively. It is obvious that ,

$$\boldsymbol{r}_{Aj} = \boldsymbol{AB} + \boldsymbol{r_{Bj}}$$

Then

$$\boldsymbol{K_A} = \sum m_j \boldsymbol{r_{Aj}} \times \boldsymbol{V_j} = \sum m_j(\boldsymbol{AB} + \boldsymbol{r_{Bj}}) \times \boldsymbol{V_j} = \boldsymbol{AB} \times \sum m_j \boldsymbol{V_j} + \sum m_j \boldsymbol{r_{Bj}} \times \boldsymbol{V_j}$$

Finally

$$\boldsymbol{K_A} = \boldsymbol{K}_B + \boldsymbol{AB} \times M\boldsymbol{V_c} \qquad \text{or} \qquad \boldsymbol{K_A} = \boldsymbol{K}_B + \boldsymbol{AB} \times \boldsymbol{Q} (8)$$

The formula resembles the dependence of the main moment of the system of forces on the center. We see that when the center of mass C of the body is stationary (for example, spherical motion around C or rotation of the body around the central axis), the Angular momentum does not depend on the center.

$$\boldsymbol{V_c} = \boldsymbol{0}: \ \boldsymbol{K_A} = \boldsymbol{K}_B = \boldsymbol{K}$$

Angular momentum of system in complex motion

In addition to the inertial reference frame with the axes x, y, and z, let us introduce translationally moving C-coordinates with the origin in the center of mass C. Now the movement of each particle can be represented as complex. The velocity of the particle will be the sum of the transport velocity $\boldsymbol{V_C}$, which is equal for all particles to the velocity of the center of mass C and the relative velocity $\boldsymbol{V_{jr}}$

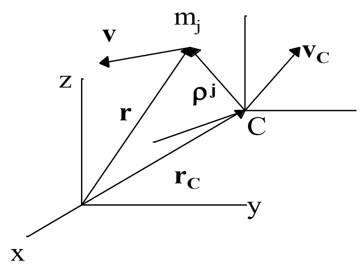


Fig.5

$$\boldsymbol{V_j} = \boldsymbol{V_C} + \boldsymbol{V_{jr}}$$

In addition, the figure shows that

$$\boldsymbol{r_j} = \boldsymbol{r_C} + \boldsymbol{\rho_j}$$

For now

$$\boldsymbol{K_o} = \sum m_j(\boldsymbol{r_C} + \boldsymbol{\rho_j}) \times (\boldsymbol{V_C} + \boldsymbol{V_{jr}}) = \boldsymbol{r_C} \times \boldsymbol{V_C} \sum m_j + \boldsymbol{r_C} \times \sum m_j \boldsymbol{V_{jr}} + \left(\sum m_j \boldsymbol{\rho_j}\right) \times \boldsymbol{V_C} +$$
$$+ \sum m_j \boldsymbol{\rho_j} \times \boldsymbol{V_{jr}}$$

Here, the second and third terms are equal to zero according to the definition of the center of mass

$$\sum m_j \boldsymbol{\rho_j} = M\boldsymbol{\rho_c} = \boldsymbol{0} \qquad \sum m_j \boldsymbol{V_{rj}} = \frac{d}{dt}\sum m_j \boldsymbol{\rho_j} = \boldsymbol{0}$$

The latter term can logically be called the ***relative Angular momentum*** of the system

$$\boldsymbol{K_C} = \sum m_j \boldsymbol{\rho_j} \times \boldsymbol{V_{rj}}$$

For now

$$\boldsymbol{K_O} = \boldsymbol{K_C} + \boldsymbol{r_C} \times M\boldsymbol{V_C} \qquad (9)$$

It should be noted that, in contrast to the similar formula relating the angular momentums with respect to the fixed centers, here C moves arbitrarily and the relative velocities of the particles are included in $\boldsymbol{K_C}$. The derivation of the formula shows that such a simple formula (9) is valid only for the center of mass, which emphasizes the importance of this center in dynamics.

Theorem of the change in the angular momentum of a system.

Differentiating

$$\boldsymbol{K_o} = \sum \boldsymbol{K_{oj}} = \sum m_j \boldsymbol{r_j} \times \boldsymbol{V_j}$$

by time, we find

$$d\boldsymbol{K_O}/dt = \sum m_j(\boldsymbol{V_j} \times \boldsymbol{V_j} + \boldsymbol{r_j} \times \boldsymbol{W_j}) = \sum \boldsymbol{r_j} \times m_j \boldsymbol{W_j} =$$

$$= \sum [\boldsymbol{r_j} \times (\boldsymbol{F_j^e} + \boldsymbol{F_j^i})] = \sum \boldsymbol{m_O}(\boldsymbol{F_j^e}) + \sum \boldsymbol{m_O}(\boldsymbol{F_j^i}) = \boldsymbol{M_O^e} + \boldsymbol{M_O^i} = \boldsymbol{M_O^e}$$

Here it is taken into account that the vector product of the vector by itself and the main moment of the internal forces are equal to zero. Thus, we come to ***the theorem of the change in kinetic momentum***

$$d\boldsymbol{K_O}/dt = \boldsymbol{M_O^e} \qquad (10)$$

In projections on Cartesian coordinate axes

$$\frac{dK_X}{dt} = \sum m_X(F_k^e), \qquad \frac{dK_Y}{dt} = \sum m_Y(F_k^e), \qquad \frac{dK_Z}{dt} = \sum m_Z(F_k^e),$$

Investigation

1. Internal forces do not change the Angular momentum directly. However, as in the theorem of the motion of the mass center, they can cause external forces that change the kinetic momentum.
2. If $\boldsymbol{M_O^e} = 0$, then vector $\boldsymbol{K_O} = \boldsymbol{Const}$. For example, for the Solar System, which can be considered isolated from the external influence of distant galaxies, the angular momentum vector retains its direction and modulus. The plane perpendicular to it, called the ***Laplace plane***, also retains its position in relation to the heliocentric inertial frame of reference.

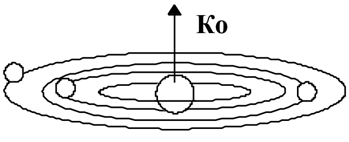


Рис.4

3. If $M_z = 0$, in the special case, only, then the corresponding projection $K_z = Const$ of the Angular momentum is preserved. Thus, the Angular momentum of a conical pendulum relative to the vertical axis will not change over time, since $M_z = 0$.

Let us substitute expression (9) into formula (10). After differentiation, we get

$$d\boldsymbol{K}_C /dt + \boldsymbol{V_C} \times M\boldsymbol{V_C} + \boldsymbol{r_C} \times M\boldsymbol{w_C} = \boldsymbol{M_O^e}$$

Taking into account the fact that $\boldsymbol{V_C} \times M\boldsymbol{V_C} = \boldsymbol{0}$, $M\boldsymbol{W_C} = \boldsymbol{Ve}$ and the theorem on the dependence of the principal moment on the center

$$\boldsymbol{M_O^e} - \boldsymbol{r_C} \times \boldsymbol{V^e} = \boldsymbol{M_c^e}$$

we arrive at ***the theorem of the change in relative angular momentum***

$$d\boldsymbol{K}_C /dt = \boldsymbol{M_C^e} \qquad (11)$$

It has the same form as a theorem in an inertial system.
In projections on the x, y, z axes with the beginning at the center of mass of the system

$$\frac{dK_x}{dt} = \sum m_x\,(F_k^e), \qquad \frac{dK_y}{dt} = \sum m_y\,(F_k^e), \qquad \frac{dK_z}{dt} = \sum m_z\,(F_k^e),$$

Angular momentum of a body in spherical motion. Inertia matrix

Let us consider a rigid body making a spherical motion around a fixed particle O. (Fig.7). The Angular momentum of the body should be calculated by the formula

$$\boldsymbol{K}_o = \iiint [\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})]dm = -\iiint [\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\omega})]dm \quad (12)$$

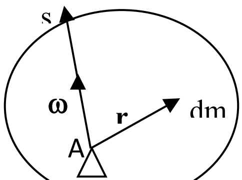


Fig.7

Here *dm* is the mass of the elementary particle, $\boldsymbol{\omega} \times \boldsymbol{r}$ – its velocity.
Let us represent formula (12) in matrix form by writing the vector product in terms of the attached obliquely symmetric matrix *R of* the radius - vector **r** in the Cartesian axes associated with the body.

$$\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\omega}) \;\; -> \;\; R\,(R\omega) = R^2\omega,$$

Where

$$R = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}, \qquad \omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Get

$$K_o = (-\iiint R^2 dm\,)\omega \qquad (13)$$

The value in parentheses at (13) is a 3x3 matrix, and is called ***the inertia matrix*** J_o at the center O and the x, y, and z axes.

$$Jo = -\iiint R^2 dm \qquad (14)$$

Axial and centrifugal moments of inertia

Let's calculate the inertia matrix according to formula (14).

$$-R^2 = -\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} = \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

The integral of a matrix is a matrix of the integrals of its elements, so

$$J_o = \begin{pmatrix} \iiint (y^2+z^2)dm & -\iiint xydm & -\iiint xzdm \\ -\iiint yxdm & \iiint (z^2+x^2)dm & -\iiint yzdm \\ -\iiint zxdm & -\iiint zy\,dm & \iiint (x^2+y^2)dm \end{pmatrix}$$

We see that the matrix *Jo* is symmetric ($\iiint yxdm = \iiint xydm$, etc.) and, therefore, has only six different elements. Diagonal elements are called ***axes moments of inertia*** of the body with respect to the x, y, and z.

$$J_x = \iiint (y^2+z^2)dm, \qquad J_y = \iiint (z^2+x^2)dm, \qquad J_z = \iiint (x^2+y^2)dm$$

The remaining three integrals are called- ***centrifugal moments of inertia***

$$J_{xy} = J_{yx} = \iiint xydm, \qquad J_{yz} = J_{zy} = \iiint yzdm, \qquad J_{zx} = J_{xz} = \iiint zxdm,$$

Dimension of all moments of inertia [J] = $kg\ m^2$.

In the accepted notations, the inertia matrix acquires the form

$$J_o = \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{pmatrix} \quad (15)$$

Now the Angular momentum of a rigid body in spherical motion takes the form

$$K_o = \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} = \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} J_x\omega_x - J_{xy}\omega_y - J_{xz}\omega_z \\ -J_{yx}\omega_x + J_y\omega_y - J_{yz}\omega_z \\ -J_{zx}\omega_x - J_{zy}\omega_y + J_z\omega_z \end{pmatrix} \quad (16)$$

Let us consider the main properties of moments of inertia (other properties will be considered in a special chapter).

Axial moments of inertia

Note that under the signs of the integral there are squares of the distances h from the particle dm to the corresponding axis. So $y^2 + z^2 = h_x^2$. Therefore, the moment of inertia of the body relative to the arbitrary axis L will be equal to

$$J_L = \iiint h_L^2\ dm$$

where h_L is the distance of the current particle to the axis.

We see that the axial moment cannot be negative or equal to zero, and characterizes the distance of the body masses from the axis. For example, the moment of inertia of the rod relative to the perpendicular axis will be greater than relative to the inclined axis (Fig.8) since $x > h$ for any particle of the member.

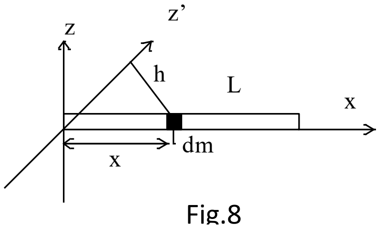


Fig.8

$$J_z > J_{z'}$$

Let us calculate the axial moment of inertia of the homogeneous rod of mass $M = \gamma L$ (γ is the linear density, L is the length of the rod) with respect to the z-axis

$$J_z = \int_0^L x^2 dm = \gamma \int_0^L x^2 dx = \gamma \frac{L^3}{3} = M \frac{L^2}{3} \text{ кг м}^2 \qquad (17)$$

Expressions of moments of inertia for the bodies of regular shape with respect to some axes can be found in reference books.

Centrifugal moments of inertia.

In contrast to axial moments of inertia, centrifugal moments of inertia can be negative or equal to zero. The axis is called ***the major axis of inertia at point O*** if both centrifugal moments with its index are zero. So, the z-axis will be the main axis in O if

$$J_{zx} = J_{yz} = 0$$

It can be shown that at any point in space for a given body there are three mutually perpendicular principal axes of inertia XYZ, in which the inertia matrix will be diagonal.

$$J_O = \begin{pmatrix} J_X & 0 & 0 \\ 0 & J_Y & 0 \\ 0 & 0 & J_Z \end{pmatrix}$$

Rotation of the body around a major or central axis

Let us align the z-axis with the axis of rotation and select the origin O on it (Fig.9). Then formula (16) will take the form $\omega_x = \omega_y = 0$ $\begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} = \begin{pmatrix} -J_{xz} \\ -J_{yz} \\ J_z \end{pmatrix} \omega_z$

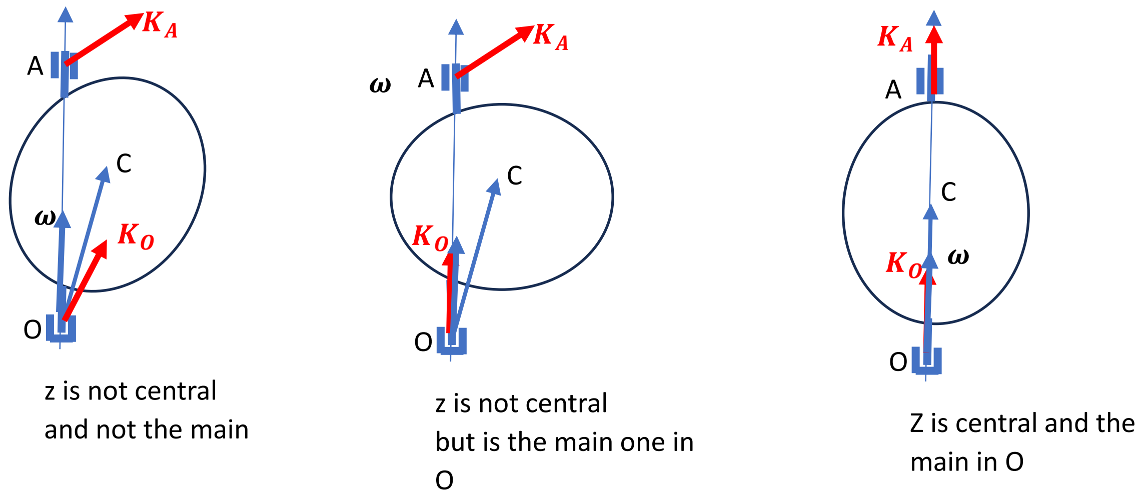


Fig.9

We see that in this case the vectors $\boldsymbol{K_O}$ and $\boldsymbol{\omega}$ are not collinear.

According to the formula

$$\boldsymbol{K_A} = \boldsymbol{K}_{\text{o}} + \boldsymbol{AO} \times \text{M}\boldsymbol{V_c} \neq \boldsymbol{K}_{\text{o}}$$

Let us now assume that the z-axis is not the central axis, but the main axis in O. Then $\boldsymbol{K}_{\text{o}}$ is directed along the axis of rotation (Fig. 9).

$$\begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ J_z \end{pmatrix} \omega_z$$

But still $\boldsymbol{K_A} \neq \boldsymbol{K_o}$, because the axis is not central.

If, finally, the z-axis is the main axis in O and the central axis, then the Angular momentum will not depend on the position of the point on the axis of rotation. This means that

$$\boldsymbol{K}_{\text{A}} = \boldsymbol{K}_{\text{O}} = \boldsymbol{K}_{\text{C}}$$

And all of them lie on the axis of rotation. It follows that

The main central axis is the main at any it's point.

Transformation of the inertia matrix. Steiner-Huygens formula

Let us consider a body in spherical motion around point O. The velocity of an arbitrary particle of the body, including the center of mass C, should be sought according to Euler's formula.

$$\boldsymbol{V_C} = \boldsymbol{\omega} \times \boldsymbol{r}_{\text{C}} = -\boldsymbol{r}_{\text{C}} \times \boldsymbol{\omega}$$

In matrix form

$$V_C = -R_C\omega$$

Here, R_C is the attached matrix of column r_C

By substituting this expression into the formula

$$\boldsymbol{K_o} = \boldsymbol{K_c} + \boldsymbol{r_C} \times \text{M}\boldsymbol{V_c}$$

get

$$J_o\omega = (J_c - MR_c^2)\omega$$

We come to ***the generalized Steiner-Huygens formula***

$$J_o = J_c + M(-R_c^2); \qquad -R_c^2 = \begin{pmatrix} Z_c^2 + Y_c^2 & -X_cY_c & -X_cZ_c \\ -Y_cX_c & Z_c^2 + X_c^2 & -Y_cZ_c \\ -Z_cX_c & -Z_cY_c & X_c^2 + Y_c^2 \end{pmatrix} \quad (18)$$

Formula (18) makes it possible to determine the components of the inertia matrix when the coordinate axes are transferred translatory.

Let us consider two parallel axes of coordinates, X, Y, Z with the origin in O and x, y, z in the center of mass C, respectively. Let us find out how the axial moment of inertia changes during the parallel transfer of the axis. Comparing the lower right elements of the matrix expression (18),

$$\begin{pmatrix} & & \\ & & J_Z \end{pmatrix} = \begin{pmatrix} & & \\ & & J_z \end{pmatrix} + M\begin{pmatrix} & & \\ & & X_c^2 + Y_c^2 \end{pmatrix}$$

Find

$$J_Z = J_z + M(X_c^2 + Y_c^2) \quad or \quad J_Z = J_z + Md^2 \qquad (19)$$

Here, d is the distance between the Z and z axes. This is the ***Steiner-Huygens formula,*** which expresses the moment of inertia of a body relative to an arbitrary axis through a moment of inertia relative to a central axis parallel to it.

Formula (19) shows that the moment of inertia with respect to the central axis is the minimum among all the axes parallel to it.

$$J_{zc} < J_Z$$

Comparing the non-diagonal elements of the matrix relation (18), we find the formula for transforming centrifugal moments of inertia when transferring the coordinate system. For example,

$$J_{XY} = J_{xy} - MX_C Y_C$$

DYNAMICS OF SOLIDS

The dynamics of a solid body is fully described by two general theorems that we have studied: the theorem on the motion of the center of mass and the theorem on the change of angular momentum.

General equations of motion of a rigid body. Dynamic equivalence of systems of forces

The main problem of the dynamics of a rigid body is to determine its motion under the action of given forces and constraints reactions. If the body is free (no constraints, Fig. 9), then you should find

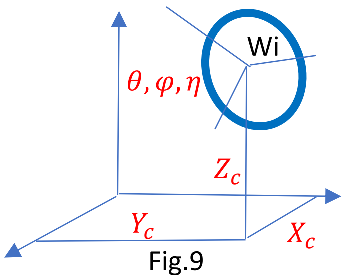


Fig.9

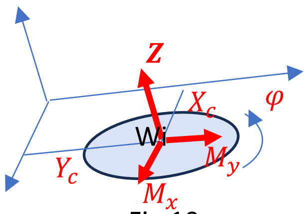


Fig.10

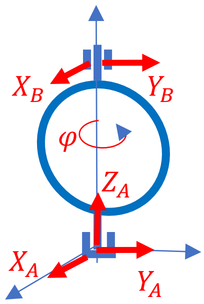


Fig.11

functions of the six functions of coordinates $x_c(t), y_c(t) z_c(t)$ и $\psi(t)$, $\theta(t)$, $\varphi(t)$.

If the body is not free, then, in addition to the law of motion, it is necessary to find reactions of constraints. Let us assume that there is no friction, and let us consider special cases of motion.

In plane motion (Fig.10) we look for three coordinate functions $x_c(t), y_c(t)\ \varphi\ (t)$ and three reactions of the plane along which the body moves: the normal reaction ***N*** and the moments relative to the axes x and y. Again, it turns out that there are six unknowns.

A rotating body (Fig.11) has one coordinate (angle of rotation φ) and five unknown reactions $\boldsymbol{X_A, Y_A, Z_A, X_B, Y_B}$. Here again six unknowns.

Thus, in any motion of a rigid body, it is necessary to have six scalar equations to determine the law of motion and constraint reactions. Let's call them ***general equations of motion of a body***.

The general equations of motion of a body are a consequence of two theorems: the motion of the center of mass and the change in relative angular momentum.

$$M\boldsymbol{w_C} = \boldsymbol{V^e}$$

$$d\boldsymbol{K}_\mathrm{C} /dt = \boldsymbol{M}_\mathrm{C}^{\boldsymbol{e}}$$

In matrix form

$$Mw_C = Ve \quad d\frac{K_\mathrm{C}}{dt} = M_\mathrm{C}^e \qquad (20)$$

However, it is difficult to use the second theorem (20) because the matrix of inertia of a rotating body in the moving axes is an unknown function of time $J_C\ (t)$, so it cannot be differentiated.

$$K_\mathrm{C}(t) = J_\mathrm{C}(t)\omega(t)$$

Therefore, it is necessary to move on to the frame of reference associated with the body. In it, the matrix of inertia will no longer depend on time.

$$K_\mathrm{C}(t) = J_\mathrm{C}\omega(t)$$

If the vector $\boldsymbol{K}_\mathrm{C}$ is given in a mobile reference system, then the derivative of it should be taken according to the theorem on the connection of derivatives (recall the complex motion of a particle).

$$\frac{d\boldsymbol{K}_\mathrm{C}}{dt} = \frac{d_r\boldsymbol{K}_\mathrm{C}}{dt} + \boldsymbol{\omega} \times \boldsymbol{K}_\mathrm{C}$$

In matrix form

$$\frac{dK_\mathrm{C}}{dt} = J_c\varepsilon + \Omega J_c\omega$$

We come to the ***general equations of motion*** of a body in the reference system associated with the body

$$Mw_C = V^a + V^R \quad (21)$$

$$J_c\varepsilon + \Omega J_c\omega = M_c^a + M_c^R$$

Here, the first equation is written in the fixed axes, the second in the axes associated with the body, and the external forces are divided into active and constraint reactions (index R)

In cases of spherical and rotational motions, C in the second formula can be replaced by a fixed particle O.

In their expanded form, general equations are a system of six scalar equations. These equations determine both the motion of the body according to the initial conditions and the reactions of the constraints.

We will call ***Equivalent*** systems of forces the systems that generate the same differential equations of the motion of a body and the reaction of constraints. There was no movement in Static, and we called ***statically Equivalent*** systems of forces that cause the same constraint reactions. It has been shown that the condition for the static equivalence of two systems of forces is the equality of their principal vectors and principal moments.

For a given body, equations (21) are the same for loads with the same principal vector and principal moment

Hence, the condition for the ***dynamic equivalence*** of two loads applied to a solid body is again the ***equality of their principal vectors and principal momentums***.

Equations of translatory motion of a body

Since the body does not rotate in translatory motion,

$$\omega \equiv 0 \ \ (\varepsilon \ = \ 0) \qquad\qquad W_c = \begin{pmatrix} \ddot{x}_c \\ \ddot{y}_c \\ \ddot{z}_c \end{pmatrix}$$

and the main vector of the reactions of the constraints is equal to zero, then equations (20) acquire the form

$$\begin{aligned} M\ddot{x}_c &= \sum F_{kx} & 0 &= \sum m_x(F_k) + M_x^R \\ M\ddot{y}_c &= \sum F_{ky} & 0 &= \sum m_y(F_k) + M_y^R \qquad (22) \\ M\ddot{z}_c &= \sum F_{kz} & 0 &= \sum m_z(F_k) + M_z^R \end{aligned}$$

Three differential equations define the law of motion of a body $x_c(t), y_c(t), z_c(t)$, and the remaining equations are used to find the main moments of the reactions of constraints with respect to the three axes.

Equations of rotational motion of a body.

Let the body rotate around the z-axis. In this case, the law of rotation φ (t) and five unknown reactions $\boldsymbol{X_A}, \boldsymbol{Y_A}, \boldsymbol{Z_A}, \boldsymbol{X_B}, \boldsymbol{Y_B}$ are determined by the load and initial conditions

$$\omega_x = \omega_y = 0, \qquad \omega_z = \dot{\varphi} \qquad \varepsilon_z = \ddot{\varphi} \qquad W_C = (E + \Omega^2)r$$

E and Ω are the attached angular acceleration and velocity matrices.

Now from (20) follow the complete equations of the rotational motion of the body

$$M\begin{pmatrix} -\omega_z^2 & -\varepsilon_z & 0 \\ \varepsilon_z & -\omega_z^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \begin{pmatrix} X_A + X_B + \sum F_{kx} \\ Y_A + Y_B + \sum F_{ky} \\ Z_A + \sum F_{kz} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -J_{xz}\omega_z \\ -J_{yz}\omega_z \\ J_z\omega_z \end{pmatrix} + \begin{pmatrix} -J_{xz}\varepsilon_z \\ -J_{yz}\varepsilon_z \\ J_z\varepsilon_z \end{pmatrix} = \begin{pmatrix} M_x^a + M_x^R \\ M_y^a + M_y^R \\ M_z^a + M_z^R \end{pmatrix}$$

Unfolded

$$-Mx_c\omega^2 - My_c\varepsilon_z = X_A + X_B + \sum F_{kx}$$

$$Mx_c\varepsilon_z - My_c\omega^2 = Y_A + Y_B + \sum F_{ky}$$

$$0 = Z_A + \sum F_{ky} \qquad (23)$$

$$J_{yz}\omega_z^2 - J_{xz}\varepsilon_z = \sum m_x(F_k) - Y_B AB$$

$$-J_{xz}\omega_z^2 - J_{yz}\varepsilon_z = \sum m_y(F_k) + X_B AB$$

$$J_z\varepsilon_z = \sum m_z(F_k)$$

There are six unknowns in these six equations: the law of rotation φ (t) and the five components of constraints reactions. Actually ***differential equation of rotation*** is the last equation

$$J_z\ddot{\varphi} = \sum m_z(F_k) \qquad (24)$$

It defines the law of rotation φ (t). The rest of the equations are used to determine the reactions of bearings according to the found law of rotation.

It should be reminded that the forces applied to the body can depend on the angle of rotation and the angular velocity of the body. With the help of equation (23) it is possible to solve direct and inverse problems of the dynamics of the rotation of a body.

It also follows from it force ***condition of uniformly accelerated rotation***. Obviously, in order to keep angular acceleration constant, it is necessary that the main moment of the applied forces be constant.

$$\sum m_z(F_k) = Const$$

In order for the body to rotate uniformly, this moment must be equal to zero

$$\sum m_z(F_k) = 0$$

Balance of a rotating body

After the law of motion is found from the differential equation of rotation, the reactions of the supports can be found from the other equations. Equations (23) show that a $\boldsymbol{Z_A}$ does not depend on the rotation of the body, and other reactions $\boldsymbol{X_A}, \boldsymbol{Y_A}, \boldsymbol{Z_A}, \boldsymbol{X_B}, \boldsymbol{Y_B}$ can depend.

Experience shows that at high angular velocity, these reactions can reach values that are dangerous for the destruction of bearings. Therefore, it is important to know the conditions that allow you to avoid such a danger.

The body is called ***dynamically balanced*** relative to the axis of rotation, if the reactions of the bearings do not depend on the rotational velocity of the body. To find the conditions of equilibrium, we investigate the equations for reactions that can depend on rotation. For simplicity, let's rotate the x,y axes so that $x_c = 0$.

$$-My_c\varepsilon_z = X_A + X_B + \sum F_{kx}$$

$$-My_c\omega^2 = Y_A + Y_B + \sum F_{ky} \qquad (25)$$

$$J_{yz}\omega_z^2 - J_{xz}\varepsilon_z = \sum m_x(F_k) - Y_B AB$$

$$-J_{xz}\omega_z^2 - J_{yz}\varepsilon_z = \sum m_y(F_k) + X_B AB$$

Obviously, if the left-hand sides of these equations could be equal to zero, then the reactions would not depend on rotation, but would be determined only by the active forces.

The first two equations (25) give a condition: the center of gravity should ly on the axis of rotation. In other words the axis should be ***Central***

$$x_c, y_c = 0$$

This condition ensures equilibrium, which is called ***static***, since it is easily verified by "static" experience. It is enough to place the axis of rotation of the body horizontally. If the body remains at rest at any angle of rotation, then the axis is central. For example, if you take a bicycle wheel by the axle, it is easy to determine that its axle is not central, because the tube nipple will force the wheel to turn to its lower position.

The second two equations (25) give a homogeneous system of linear equations with respect to the moments of inertia. Its determinant is non-zero

$$\begin{vmatrix} \omega_z^2 & -\varepsilon_z \\ \varepsilon_z & \omega_z^2 \end{vmatrix} = \omega_z^4 + {\varepsilon_z}^2 > 0$$

This means that the system will have only a zero solution if

$$J_{yz} = J_{xz} = 0$$

Thus, for that the body would be ***dynamically balanced*** it is necessary and sufficient that the axis of rotation is ***central and main*** axis of inertia.

$$x_c, y_c = 0, \qquad J_{yz} = J_{xz} = 0 \qquad (26)$$

The equilibrium conditions of a rotating body show how important is the problem of determining the principal axes of inertia in the body.

Equations of plane motion of a body

Let us consider the motion of a plane figure obtained by a cross-section of a body through the center of mass C parallelly to the plane of motion of the body.

Let's align the plane of the axes x, y with the plane figure. Then $\boldsymbol{W_C}$ and the both main moments of the reactions of the smooth plane will lie in the plane x, y, and $\boldsymbol{\omega}, \boldsymbol{\varepsilon}$, the normal reaction of the plane are directed along the Z axis. Equations of motion

$$M\begin{pmatrix} \ddot{x}_c \\ \ddot{y}_c \\ 0 \end{pmatrix} = \begin{pmatrix} \sum F_{kx} \\ \sum F_{ky} \\ \sum F_{kz} + R_z \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -J_{xz}\omega_z \\ -J_{yz}\omega_z \\ J_z\omega_z \end{pmatrix} + \begin{pmatrix} -J_{xz}\varepsilon_z \\ -J_{yz}\varepsilon_z \\ J_z\varepsilon_z \end{pmatrix} = \begin{pmatrix} M_x^a + M_x^R \\ M_y^a + M_y^R \\ M_z^a \end{pmatrix}$$

Expanded

$$M\ddot{x}_c = \sum F_{kx}, \quad M\ddot{y}_c = \sum F_{ky}, \quad 0 = R_z + \sum F_{kz}$$

$$J_{yz}\omega_z^2 - J_{xz}\varepsilon_z = \sum m_x(F_k) + M_x^R \qquad (27)$$

$$-J_{xz}\omega_z^2 - J_{yz}\varepsilon_z = \sum m_y(F_k) + M_y^R$$

$$J_z\ddot{\varphi} = \sum m_z(F_k)$$

First, second, and last equations

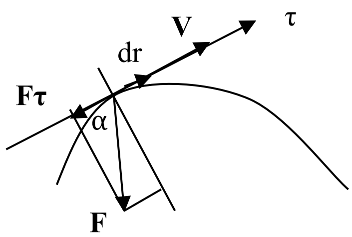
$$M\ddot{x}_c = \sum F_{kx}\,, \quad M\ddot{y}_c = \sum F_{ky}\,, \qquad J_z\ddot{\varphi} = \sum m_z(F_k) \qquad (28)$$

are ***differential equations of plane motion***. After determining $x(t), y(t), \varphi(t)$ and the accelerations, it is possible to find the projections of the main reaction vector R_z and the main moments of the reactions $M_x^R M_y^R$ from the remaining three equations. Note that the plane's reactions will not be affected by motion if the z_C is the main axis.

THEOREM OF THE CHANGE IN KINETIC ENERGY

Elementary work and power of force.

Newton's second law for a free particle



$$m\boldsymbol{W} = \boldsymbol{F}$$

relates the acceleration $\boldsymbol{W}$ of a particle to force $\boldsymbol{F}$.

As is known, the motion of a particle (its velocity and trajectory) is determined not only by the force, but also by the initial conditions. By setting the position and velocity of the particle arbitrarily, it is possible to find the initial conditions corresponding to them. The set of initial conditions corresponds to a set of ***possible velocities V*** at a given particle position. The ***actual velocity V*** at each point of the trajectory corresponds to the specific initial conditions.

Let us multiply Newton's law scalarly by the actual velocity $\boldsymbol{V}$ of the particle

$$m\boldsymbol{W}\cdot\boldsymbol{V} = \boldsymbol{F}\cdot\boldsymbol{V} \qquad (29)$$

The left side of the expression can be represented as

$$m\boldsymbol{W}\cdot\boldsymbol{V} = m\boldsymbol{V}\cdot\dot{\boldsymbol{V}} = \frac{d}{dt}\left(\frac{m\boldsymbol{V}\cdot\boldsymbol{V}}{2}\right) = \frac{d}{dt}\left(\frac{mV^2}{2}\right) = \frac{dT}{dt} = \dot{T}$$

Positive value

$$T = \frac{mV^2}{2} > 0$$

is called the ***kinetic energy*** of the particle.

Right side of (29)

$$N(F) = \boldsymbol{F}\cdot\boldsymbol{V} \qquad (30)$$

is called ***the power*** of force **F**

We arrive at ***the theorem of the change in kinetic energy***

$$\dot{T} = N \qquad (31)$$

The speed of change of the kinetic energy of a particle

is equal to the power of the force.

The theorem holds true in both possible and actual motion. The theorem shows that the speed of change of kinetic energy is maximum if the force is collinear with the velocity, and it is equal to zero when they are mutually perpendicular.

It follows, for example, that the frictional force of the wheel grip does not develop power in the absence of slippage. Also, the driving force or torque applied to the wheel has zero power at the moment of start.

Theorem (31) can be written as

$$dT = \boldsymbol{F} \cdot d\boldsymbol{r} = d'A \qquad d\boldsymbol{r} = \boldsymbol{v}dt \qquad (32)$$

Value

$$d'A = \boldsymbol{F} \cdot d\boldsymbol{r} \qquad (33)$$

is called ***the elementary work*** of force ***F***. The stroke in the notation is intended to emphasize that, in the general case, elementary work is not a differential of some function A. We will see that it is only for "potential" forces. Reveal the dot product $d'A$

$$d'A = \boldsymbol{F} \circ d\boldsymbol{r} = FdrCos\alpha = F_\tau dr_\tau = F_x dx + F_y dy + F_z dz$$

It follows:

1. The sign of work is determined by the sign of Cos: the work is positive if the directions of the force and displacement coincide with an accuracy of π/2.
2. Only the tangent component of the force does the work.
3. The work is zero if the force is perpendicular to the displacement.

Let us consider the motion of a system of material particles $\{ m_1, m_2, \dots, m_k, \dots m_n\}$ in an inertial frame of reference. ***The kinetic energy of a system*** is a positive value

$$T = \frac{1}{2}\sum m_k V_k^2 > 0 \qquad (34)$$

The resultant external and internal forces acting on the particle m_k are denoted by $\boldsymbol{F}_k^e$ и $\boldsymbol{F}_k^i$. The theorem of the change in the kinetic energy of a system can be written in the form

$$m_k \boldsymbol{w}_k \cdot \boldsymbol{V}_k = \left(\boldsymbol{F}_k^e + \boldsymbol{F}_k^i\right) \cdot \boldsymbol{V}_k$$

(A repeating index k indicates the summation by the index: from 1 to n).

This means that the derivative of the kinetic energy of the system is equal to the sum of the powers of external and internal forces.

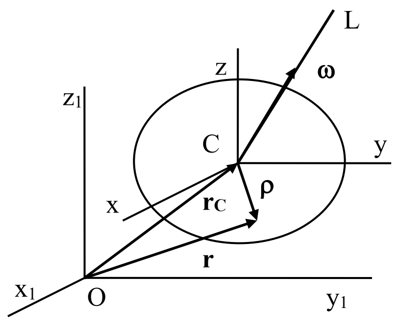
$$\dot{T} = N^e + N^i$$

Koenig's theorem.

The center of mass of the system $\{m_1, m_2, \dots, m_k, \dots m_n\}$ has a radius vector

$$\boldsymbol{r}_c = \frac{1}{M}\sum m_k \boldsymbol{r}_k$$

At the center of mass C, let us choose the origin of the axes x, y, z of a movable reference system moving translatory. Let us call it the ***C-system.*** The radius vector of the particle of the system with respect to the center of mass is denoted by $\boldsymbol{\rho}$**.** Now the absolute velocity of the particle $m_k$is represented as



$$\boldsymbol{V}_k = \boldsymbol{V}_k^{\boldsymbol{e}} + \boldsymbol{V}_k^{\boldsymbol{r}}$$

The transport velocity $\boldsymbol{V}_k^{\boldsymbol{e}}$is the same for all particles in the system

$$\boldsymbol{V}_k^{\boldsymbol{e}} = \boldsymbol{V}_{\boldsymbol{c}} \qquad \boldsymbol{V}_k = \boldsymbol{V}_{\boldsymbol{c}} + \boldsymbol{V}_k^{\boldsymbol{r}}$$

Substituting the formula into the kinetic energy

$$T = \frac{1}{2}\sum m_k\,(\boldsymbol{V}_{\boldsymbol{c}} + \boldsymbol{V}_k^{\boldsymbol{r}})^2 =$$

$$= \frac{1}{2}V_c^2\sum m_k + \boldsymbol{V}_c\sum m_k\boldsymbol{V}_k^{\boldsymbol{r}} + \frac{1}{2}\sum m_k\boldsymbol{V}_k^{\boldsymbol{r2}}$$

$$\sum m_k\boldsymbol{V}_k^{\boldsymbol{r}} = \frac{d}{dt}\sum m_k\rho_k = \frac{d}{dt}M\rho_c = 0$$

Arriving at ***König's theorem***

$$T = \frac{1}{2}MV_c^2 + T^r \qquad\qquad T^r = \frac{1}{2}\sum m_kV_k^{\boldsymbol{r2}} \qquad (35)$$

The kinetic energy of the system consists of the energy of the translatory motion with the center of mass and the energy $\boldsymbol{T^r}$ of its motion relative to the C-system.

Kinetic energy of a solid.

Consider the movement of a free solids relative to the inertial reference system. In the Koenig formula

$$T = \frac{1}{2}MV_c^2 + \frac{1}{2}\sum m_kV_k^{\boldsymbol{r2}}$$

for a solid, the sum becomes integral, the mass dm is elementary,

$$T^r = \frac{1}{2}\iiint V_r^2 dm \qquad (36)$$

and the relative velocity V_r of the particle in spherical motion around the center of mass C must be found by Euler's formula.

$$\boldsymbol{V_r} = \boldsymbol{\omega} \times \boldsymbol{\rho} = -\,\boldsymbol{\rho} \times \boldsymbol{\omega}$$

In the matrix form

$$V_r = \Omega\varrho = -P\omega$$

Here P*is* the attached obliquely symmetric ($P^{\mathrm{T}} = -P$*)* radius matrix of the vector $\boldsymbol{\rho}$

Let us calculate the square of the relative velocity of a particle

$$V_r^2 = V_r^TV_r = (-\mathrm{P}\omega)^T(-\mathrm{P}\omega) = \omega^T\mathrm{P}^T\mathrm{P}\omega = -\omega^T\mathrm{P}^2\omega$$

Substituting this expression in formula (36), we get

$$T^r = \frac{1}{2}\omega^T[-\iiint \mathrm{P}^2 dm]\omega$$

In square brackets, we recognize the expression of the inertia matrix with respect to the center of mass C.

$$J_C = -\iiint \mathrm{P}^2 dm$$

Now the formula of the kinetic energy of a body in voluntary motion takes the form

$$T = \frac{1}{2}\left(MV_c^2 + \omega^T J_C \omega\right) \quad (37)$$

Translatory motion

In this case, the body does not rotate ($\boldsymbol{\omega} \equiv 0$), the velocities of all particles are similar and therefore

$$T = \frac{1}{2} MV^2 \quad (38)$$

Spherical motion around the center O

Repeating the calculations for T^r, but for the center O, we get a similar formula

$$1) \qquad T = \omega^T J_o \omega \quad (39)$$

On the other hand, we know that the velocity of a particle of a body in spherical motion can be found in terms of the distance $h_L$from the instantaneous axis L

$$V = \omega h_L$$

Then

$$\mathrm{T} = \frac{1}{2}\omega^2 \iiint h_L^2 dm$$

Integral gives the moment of inertia with respect to the instantaneous axis

$$J_L = \iiint h_L^2 dm$$

And we come to the second formula

$$2) \ \ \mathrm{T} = \frac{1}{2} J_L \omega^2 \quad (40)$$

Rotary motion

It is a special case of spherical motion, when the instantaneous axis coincides with the axis of rotation z:

$$\mathrm{T} = \frac{1}{2} J_z \omega^2 \qquad (41)$$

Plane motion in the x, y plane

The first formula is derived from Koenig's theorem

$$1) \qquad T = \frac{1}{2}\left(MV_c^2 + J_{z_c}\omega^2\right) \quad (42)$$

Another formula is obtained by introducing the velocity center P into consideration. Then the velocity of any particle is expressed in terms of its distance h_P from P.

$$V = \omega h_P$$

So there is a second formula, through the instantaneous center:

2) $\mathrm{T} = \frac{1}{2} J_{z_P}\omega^2$ (43)

The power of the force applied to a solid.

Free movement. Let the motion of the body be characterized by the velocity $\boldsymbol{V_A}$ of the pole A and the angular velocity $\boldsymbol{\omega}$. Let us find the power of the force $\boldsymbol{F}$ applied at some point M of the body.

$$N(\boldsymbol{F}) = \boldsymbol{F} \cdot \boldsymbol{V} = \boldsymbol{F} \cdot (\boldsymbol{V_A} + \boldsymbol{\omega} \times \boldsymbol{\rho})$$

The velocity distribution theorem is taken into account here

$$\boldsymbol{V} = \boldsymbol{V_A} + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

Let's make a circular permutation in the mixed product

$$\boldsymbol{F} \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \boldsymbol{\omega} \cdot (\boldsymbol{\rho} \times \boldsymbol{F})$$

In parentheses, we recognize the expression of the moment of force **F** with respect to the pole A. Since **ω** is directed along the instantaneous axis S, then

$$\boldsymbol{\omega} \cdot (\boldsymbol{\rho} \times \boldsymbol{F}) = \boldsymbol{\omega} \cdot \boldsymbol{m_A}(\boldsymbol{F}) = \omega_S m_S(F)$$

where ω_S– is the projection of angular velocity on S, and the moment $m_S(F)$– is relative to this axis. We come to the expression of the power of force:

$$N(\boldsymbol{F}) = \boldsymbol{F} \cdot \boldsymbol{V}_A + m_S(F)\omega_S \qquad (44)$$

Note that the sign of the second term is easier to determine by comparing the directions of momentum and rotation. Therefore, almost often the sign is determined separately and the work is calculated according to the formula

$$N(\boldsymbol{F}) = \boldsymbol{F} \cdot \boldsymbol{V}_A \pm |m_S(F)||\omega_S|$$

Note that, unlike Koenig's formula for kinetic energy, here pole A– is an arbitrary particle of the body, not necessarily the center of mass.

If a system of forces $\{\boldsymbol{F}\} = \{\boldsymbol{F_1}, \boldsymbol{F_2}, \dots, \boldsymbol{F_k}, \dots, \boldsymbol{F_n}\}$ is applied to the body, then after summing by k, we get

$$N\{\boldsymbol{F}\} = \boldsymbol{V}\{\boldsymbol{F}\} \cdot \boldsymbol{V}_A \pm |M_S(F)||\omega_S| \qquad (45)$$

Here $\boldsymbol{V}\{\boldsymbol{F}\}$ is – the main vector of the system of external forces, and the M_S– main moment relative to the S axis

Using the general formula, we obtain the expressions of work for the simplest movements of the body.

Translatory motion

A body in translational motion does not rotate ($\boldsymbol{\omega} = 0$) and all its particles have the similar velocity $\boldsymbol{V}$

$$N\{\boldsymbol{F}\} = \boldsymbol{V}\{\boldsymbol{F}\} \cdot \boldsymbol{V} \quad (46)$$

Rotary motion

Here it makes sense to choose the pole A on the axis of rotation z, which is the axis S. Then:

$$\boldsymbol{V_A} = 0; \qquad N\{\boldsymbol{F}\} = \pm|M_S(F)||\omega_S| \quad (47)$$

A plus sign if the moment is directed as the angular velocity.

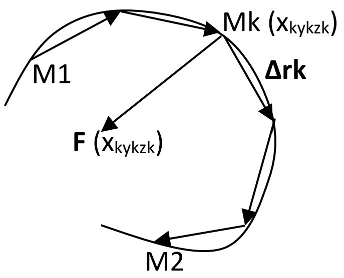
Plane Motion

Let us remind that in this motion the instantaneous axis passes through the instantaneous center of velocities perpendicular to the plane of motion. The formula takes the form of

$$N\{\boldsymbol{F}\} = M_{z_P}\omega_z \quad (48)$$

The finite work of force.

Let us consider the motion of the particle m under the influence of the force $\boldsymbol{F}$ along the trajectory from the position M1 to the position M2. Let us divide the curve M_1 M2 into n parts. Let's draw vectors of movements from node to node and indicate the work on these movements through



$$\Delta A_k = \boldsymbol{F}\,(x_k y_k z_k) \circ \boldsymbol{\Delta r_k}$$

The ***final work*** of the force $\boldsymbol{F}$ on the way from position M1 to position M2 is called a scalar quantity equal to the limit

$$A_{12} = \lim_{n\to\infty,\Delta r_k\to 0} \sum \Delta A_k$$

This limit is a curvilinear integral of the 2nd kind

$$A_{12} = \int_{1-2} (\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \circ \, d\mathbf{r})$$

What do we need to know to calculate this integral?

1. If the force depends on all parameters, then you need to know the law of motion of the particle $\boldsymbol{r}(t)$

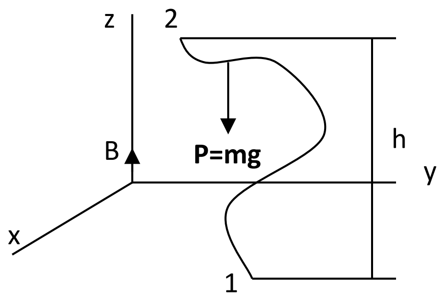
$$A_{12} = \int_{t_1}^{t_2} (\mathbf{F}(t) \circ V(t)\, dt)$$

2. In the case of a ***force field*** – a space at each point of it a force function $\boldsymbol{F}(\boldsymbol{r})$ is given, you need to know the trajectory of the particle:

$$A_{12} = \int_{r_1}^{r_2} (\mathbf{F}(r) \circ d\mathbf{r})$$

3. There are force fields, called ***potential fields***, in which only the initial and final positions of the particle need to be known in order to calculate the final work. Such fields will be considered in detail below. Here are examples of gravitational and elastic force fields.

Gravity force work



$$\boldsymbol{P} = m\boldsymbol{g} = -mg\boldsymbol{k} \quad P_z = -mg$$

$$d'A = P_z dz = -mgdz,$$

$$A_{12} = -mg \int_{z_1}^{z_2} dz = mg(z_1 - z_2)$$

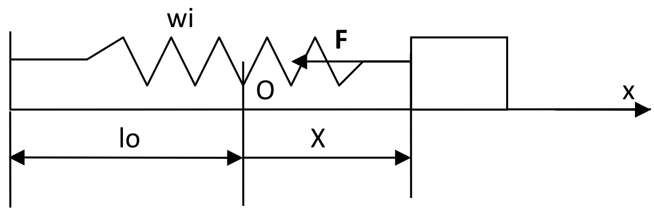
Usually, this formula is written in the form

$$A_{12}(mg) = \pm mgh \quad (49)$$

The work of gravity is positive if $(z_1 - z_2) > 0$, i.e. the particle goes down.

The work of the elastic forces of a linear spring:

The change in length l_o of a spring is called the deformation Δ. ***The stiffness*** of the spring "c" is the force required to elongate it per unit of length. The deformation Δ causes the elastic force ***F***. A spring is linearly elastic if the elastic force is linearly dependent on the deformation $F = c\,\Delta$. The force is directed to the origin O of the x coordinate chosen in the equilibrium position of the load. Therefore, the force is called ***restoring*** (equilibrium position), $|x| = \Delta$ and



$$F_x = -c\,x$$

When moving the end of the spring from the position with the coordinate x_1 to the position x_2, the elastic force performs elementary work

$$d'A = -cx\,dx$$

and the final work

$$A_{12} = -c \int_{x_1}^{x_2} xdx = \frac{1}{2} c(x_1^2 - x_2^2)$$

Replace coordinate squares with deformation squares

$$A_{12} = \frac{1}{2} c\,(\Delta_1^2 - \Delta_2^2) \qquad (50)$$

The sign of operation is determined by the ratio of the initial and final deformations of the spring.

The work of Elastic Moment of Spiral Spring

Consider a rod rotating around a vertical axis under the action of a coil spring. The stiffness of such a spring is equal to the torque that twists the spring by one radian. Its deformation is measured by the angle of torsion in radians. Deformation causes the moment of elasticity $c'\Delta' = \varphi M_z = -c'\varphi$.

Elementary Work

$$d'A = -c'\,\varphi\,d\,\varphi$$

When the rod is rotated from φ_1 position to φ_2 position the elastic moment performs the final work

$$A_{12} = -c' \int_{\varphi_1}^{\varphi_2} \varphi d\varphi = \frac{1}{2} c'(\varphi_1^2 - \varphi_2^2)$$

Replace coordinate squares with deformation squares

$$A_{12} = \frac{1}{2} c(\varDelta'^2_1 - \varDelta'^2_2)$$

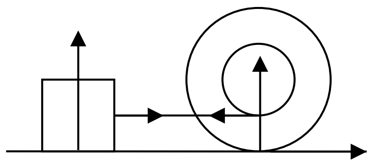
The sign of operation is determined by the ratio of the initial and final deformations of the spring.

LAGRANGE'S ANALYTICAL MECHANICS

Newtonian mechanics provides a complete system of equations for the solution of the basic problem of mechanics: the determination of the law of motion of the system and the reactions of constraints determined by load and initial conditions. As we have seen in the example of a rigid body, a part of these equations equal to the number of degrees of freedom of the body are differential equations of motion, the rest determine the reactions of constraints.

The Lagrange method makes it possible to find directly only the differential equations of motion of a system.

Consider the two-body system. The non-stretch thread and the absence of roller slippage leave the system with one degree of freedom. Three external reactions of the plane and one internal reaction of the thread will be included in Newton's 5 equations: 2 for the body and 3 for the roller.



Of these, only one will be the differential equation of motion of the system, which is most often the goal.

The Lagrange method makes it possible to compose one differential equation at once. In this case, the reactions of ideal bonds will initially be excluded from consideration. The Lagrange method is based on the concept of ***possible displacement***.

Classification of constraints.

Let us consider the motion of a system of n particles in an inertial frame of reference with coordinates x, y, z. The position of the system is determined by the values of 3n coordinates $x_1 \dots z_n$ and 3n velocities $\dot{x}_1 \dots \dot{z}_n$

Constraint is a condition imposed on the coordinates and velocities of particles. In general, the equations of the s constraints can be written in the form:

$$\Phi_i\ (x_1 \dots z_n;\ \dot{x}_1 \dots \dot{z}_n;\ t) \geq 0\ \ (i = 1, 2, \dots, s)$$

Other parameters, including angular ones, can also act as coordinates.

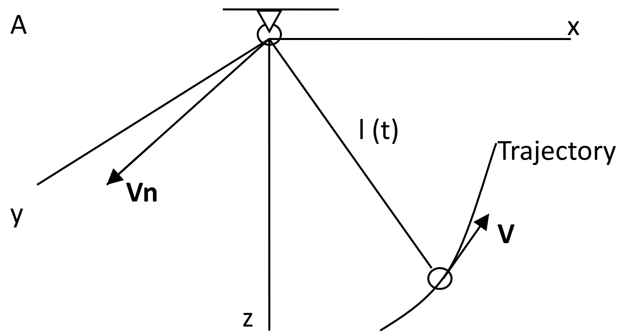
Constraints are divided into:

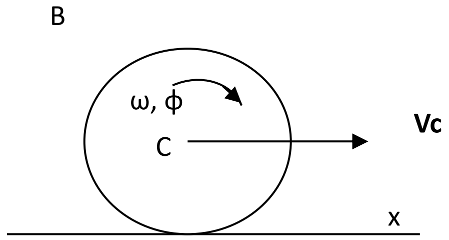
1. ***Geometric and kinematic***

The equations of geometric constraints do not include velocities. Consider 2 examples: a pendulum A of variable length and a wheel B that rolls without slippage.

The distance from the pendulum A to the origin cannot be greater than the variable length of the thread, so the equation of the constraint is as follows: $x^2 + y^2 + z^2 \leq l^2\ (t)$

The constraint is geometrical, since there are no derived coordinates in its equation.





When wheel B is rolling without slipping, the velocity of its center and angular velocity are related by the ratio

$$\dot{x} = \text{r}\ \dot{\varphi}$$

It is kinematic constraint

2. ***Stationary and nonstationary***
   In the equations of stationary constraints, time t is not included.
   Examples: A) – nonstationary, because time is included in equation, B) – stationary
3. ***Retentional (bilateral) and non-retentional (one-sided***)
   Equations of retentional constraints are written through equality, non-retention - through inequality.
   Examples: A) – non-retentional constraint. The name of one-sided follows from the fact that the thread does not stretch, but can crumple. Thus, the thread acts in one direction, from the center. C) – retentional
4. ***Holonomic and nonholonomic***:
   Holonomic constraints are all geometric constraints, as well as those kinematic constraints that can be integrated into geometric ones. The wheel coupling equation can be integrated and reduced to the form of a geometric

$$\text{x} - \text{r}\ \varphi = 0$$

The study of systems with nonholonomic constraints is a complex section of analytical mechanics and is beyond the scope of the course. Therefore, we will consider only systems with holonomic constraints.

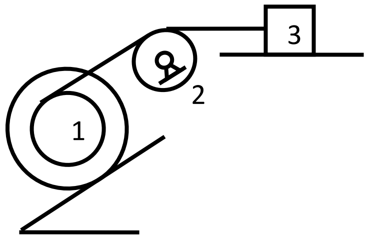
Generalized coordinates. The number of degrees of freedom of a system.

The position of the system $\{m_1, m_2, \dots, m_k, \dots m_n\}$ in the inertial frame of reference is determined by their coordinates. These 3n coordinates obey s equations of holonomic constraints:

$$\Phi_i\ (x_1 \dots z_n; t) \geq 0\ \ (i = 1, 2, \dots, s)$$

Thus, $l = 3n - s$ of the $3n$ coordinates are only independent. The rest s are expressed through them using constraints equations.

For holonomic systems, the number *l* is called ***the number of degrees of freedom***. The number *l* can be defined as the number of coordinates that must be fixed in order for the system to stop.



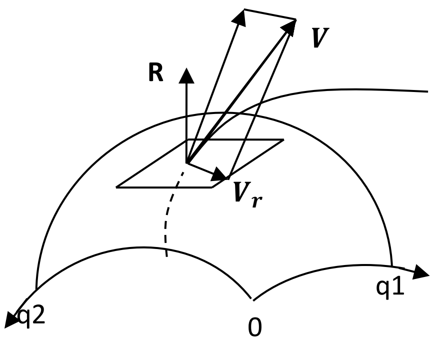
Cartesian coordinates are not always convenient. In addition, angular coordinates and their combinations with linear coordinates are used. ***Generalized coordinates*** q_i are parameters of any dimension that determine the position of the system.

Thus, the generalized coordinates of the three-body system above can be: the coordinates of the center and the angle of rotation of the roller 1, the angle of rotation of block 2, the coordinates of the body 3. All of them are interconnected due to the absence of slippage of the roller, the non-extensibility of the thread, the presence of guides. Only one of them is independent, since if any of the listed parameters is fixed, the system will lose mobility. This means that the system has one degree of freedom.

In the future, we will agree that generalized coordinates are understood only as independent coordinates $q_i \ (i = 1,2 \ldots l)$.

The possible, actual and virtual velocities of the system particle.

Let us consider the motion of a non-free particle under the influence of a force ***F*** and a nonstationary geometric constraint



$$\Phi(x, y, z; t) = 0 \qquad (51)$$

Force and constraint allow many ***possible movements*** of a particle, differing in initial conditions. A possible movement that actually takes place and meets specific initial conditions is ***actual***.

The equation of constraint (51) is convenient to interpret as the equation of the moving surface on which the particle moves. The figure shows a photograph of the surface at moment *t*.

At a given moment *t*, the particle can be at an arbitrary point on the constraint surface and have an arbitrarily directed ***possible velocity V*** that corresponds to arbitrary initial conditions.

Any possible velocity $\boldsymbol{V}$ is the sum of the transport velocity $\boldsymbol{V_e}$ together with the surface and the relative velocity $\boldsymbol{V_r}$ tangent to the constraint surface.

In the case of nonstationary constraint, the relative velocity $\boldsymbol{V_r}$ cannot coincide with the actual velocity. Therefore, it is called imaginary or ***virtual***.

If the bond surface is smooth, then its reaction $\boldsymbol{N}$ is normal to the surface. The Lagrange method is based on the fact that a normal reaction $\boldsymbol{N}$ does not produce power at virtual velocity $\boldsymbol{V_r}$.

$$\boldsymbol{N}\cdot\boldsymbol{V_r}=0 \qquad but \qquad \boldsymbol{N}\cdot\boldsymbol{V}\neq 0$$

Excluding such reactions from consideration, we find the equations of motion of the particle along the surface of the bond, that is, the differential equations of motion we are looking for.

In stationary communication, a particle moves along a stationary surface, its transport velocity is zero, possible and virtual velocities coincide and are tangent to the constraint surface.

Generalized forces and reactions. Perfect constraints.

Let's consider a particle m_k of the system. Let us denote the resultant active forces $\mathbf{F}_k$ and reactions$\mathbf{N}_k$ of the constraints acting on the particle. All possible laws of motion, including the actual law of motion of a particle, are functions of independent generalized coordinates and time и

$$\boldsymbol{r}_k(q_1\ q_2\ \dots q_l;t)$$

They satisfy the constraints equations and Newton's equations

$$m_k\boldsymbol{w_k}=\boldsymbol{F_k}+\boldsymbol{N_k}$$

The possible velocity $\boldsymbol{V}_k$of a particle is made up of the transport $\boldsymbol{V}_{ke}$and virtual $\boldsymbol{V}_{kr}$ (relative) velocities

$$\boldsymbol{V}_k=\boldsymbol{V}_{ke}+\boldsymbol{V}_{kr}, \quad \boldsymbol{V}_{ke}=\frac{\partial\mathbf{r}_k}{\partial t}, \qquad \boldsymbol{V}_{kr}=\frac{\partial\mathbf{r}_k}{\partial q_i}\dot{q}_i$$

Hereinafter, the repeating index indicates the summation of the index: k from 1 to n, i from 1 to l.

Let's calculate the power of all forces at virtual velocities $\boldsymbol{V}_{kr}$

$$\sum_i\left(\sum_k(\boldsymbol{F}_k+\boldsymbol{N}_k)\cdot\frac{\partial\boldsymbol{r}_k}{\partial q_i}\right)\dot{q}_i=(Q_i+R_i)\,\dot{q}_i$$

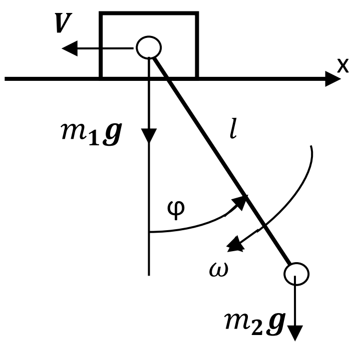
It is logical to call Q_i − ***generalized forces*** and R_i − ***generalized reactions***,

$$Q_i=\boldsymbol{F}_k\cdot\frac{\partial\boldsymbol{r}_k}{\partial q_i} \qquad R_i=\boldsymbol{N}_k\cdot\frac{\partial\boldsymbol{r}_k}{\partial q_i}$$

since they are coefficients at generalized velocities in the expression of power. Usually, generalized forces Q_i are found as a coefficient in the expression of the virtual power of active forces

$$\boldsymbol{F}_k\boldsymbol{V}_{kr}=Q_i\dot{q}_i$$

Let us see how this is done using the example *of an* ***elliptical pendulum***. A pendulum consists a body of mass m_1, sliding translatory without friction along the axis x , and a mathematical pendulum of length l and mass $m_2$hinged to it. Constraints are stationary, so virtual velocities are possible velocities.



While calculating the possible power of the active forces we take advantage of the independence and arbitrariness of the possible velocities of the system. Their independence means that we can give them independently. The constraints are perfect and allow each of the possible velocities $V_x\omega_z$ in two directions. Let's try to give negative velocities.

First, for the calculation of Q_x, we give the negative velocity $V_x < 0$, putting $\omega_z = 0$. The whole system moves progressively to the left at a velocity V_x. In horizontal motion, vertical gravitational forces have no power, so

$$N_\text{x} = 0 \quad \text{and} \quad Q_x = 0$$

To calculate the generalized force Q_φ, we give the system the possible velocities $V_\text{x} = 0,\ \omega_z < 0$. The body $m_1$remains motionless; the pendulum rotates clockwise. Only moment of $m_2\boldsymbol{g}$ creates power at ω_z

$$N_\varphi = m_2 g\, l\, Sin\varphi\, |\omega_z| = -\, m_2 g\, l\, Sin\varphi\, \omega_z = Q_\varphi \omega_z$$

Thus

$$Q_\varphi = -\, m_2 g\, l\, Sin\varphi$$

In order not to make a mistake in the sign of power, it is convenient to always give positive generalized possible movements.

Constraints are called ***ideal*** if all their generalized reactions are zero.

$$R_i = 0 \quad (i = 1,2, \dots, l)$$

Smooth surfaces, frictionless hinges, non-stretchable threads, etc. are ideal.

Lagrange showed that the reactions of ideal bonds do not affect the motion of the system along nonstationary bonds.

Static principle of possible velocities.

Consider a system with ideal stationary constraints at rest. Since the constraints are stationary, there are no trasport velocities and virtual velocities are possible.

The principle

For the system to remain at rest in the equilibrium position It is necessary and sufficient the equality to zero of all generalized forces.

$$Q_i = 0 \qquad (52)$$

Necessity. If the system is at rest, then the velocities of its particles, and hence the power of all forces, are equal to zero.

$$(Q_i + R_i)\, \dot{q}_i = 0$$

In view of the independence and arbitrariness of generalized velocities $\dot{q}_i$

$$Q_i + R_i = 0$$

Since the bonds are ideal, all their generalized reactions are equal to zero and the generalized forces are equal to zeroR_i

$$Q_i = 0$$

Sufficiency: Let $Q_i = 0$. Let's show that the system remains at rest. Let's assume the opposite – the system has begun to move. Then the kinetic energy of the system begins to increase, hence

$$\dot{T} = (Q_i + R_i)\dot{q}_i = Q_i\dot{q}_i > 0$$

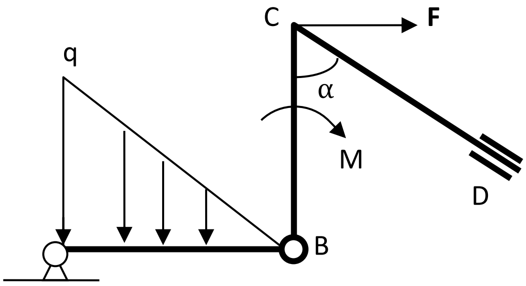
From where

$$Q_i > 0$$

which contradicts the original assumption. This means that the system will remain at rest, which was to be proved.

Sets of control problems on the principle of possible velocities can be downloaded from: https://disk.yandex.ru/d/itjDbi4afDNMGg

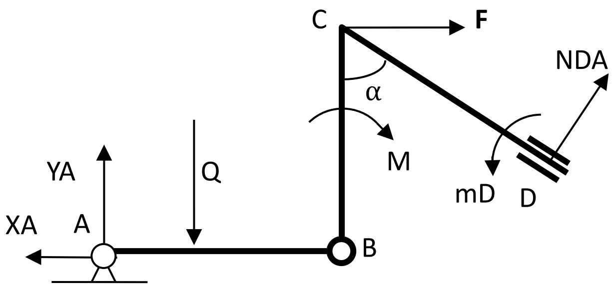
An example of solving a problem on the principle of possible velocities



A system of 2 bodies is affected by force, torque and distributed load.

1. For each external reaction, transform the external support. Write down the principle of possible velocities and velocity ratios.
2. The question is: how should the straight rod be directed so that the problem becomes statically indefinable and the constraints insufficient?

1. Solution



The supports are sufficient, i.e. they ensure the rest of the two-body system under any plane load.

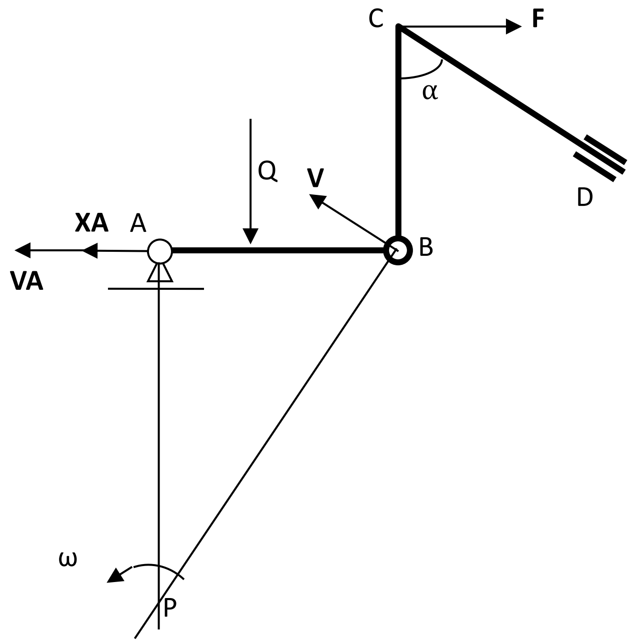
The constraints are definable, since the reaction in the sliding embedding, being an arbitrary force perpendicular to the direction of slipping, cannot be on the AB line of the other two reactions.

In order to apply the principle of possible velocities, it is necessary to transform the support for each unknown reaction so that the system becomes a mechanism with one degree of freedom along the desired reaction.

We will depict only those forces and moments that create power in the possible motion of the transformed system.

X_A reaction

We replace the hinge A with a roller along the x-axis. We get a mechanism with one degree of freedom. We perceive the desired XA reaction as an active force that balances the other forces.



The BCD rod can move progressively along the guide. Let us give it a possible velocity V, depicting it at a common point B.

In this case, point A of the rod AB will acquire velocity $\boldsymbol{V_A}$ along the reaction $\boldsymbol{X_A}$.

The rod AB makes a plane motion, rotating at a given moment around the instantaneous center of velocities (ICV) P with a possible angular velocity ω.

Since the mechanism is at rest, the principle of possible velocities is necessarily fulfilled.

The possible power of the forces applied to the rod AB in plane motion is calculated as the product of the angular moment relative to P and the possible angular velocity of the rod ω

$$X_A AP\omega - Q\frac{AB}{3}\omega - FV \ \ Sin \propto = 0$$

The velocity ratio is always found through a common particle (B in this problem)):

$$V \ = \omega \text{BP}, \qquad BP = \frac{AB}{Cos \propto}, \qquad AP = ABtg \propto, \ \ Q = \frac{1}{2}qAB$$

By substituting the ratio of velocities in principle, it is possible to find the reaction X_A

***Y_A* reaction**

We replace the hinge A with a roller along the axis y. The possible velocity of the joint A is directed along the axis y. The velocity of the joint B is parallel to CD. Therefore, the ICV of the AB rod is located at the point B. This means that the rod BCD is stationary, and the rod AB rotates around B with a possible angular velocity ω**.**

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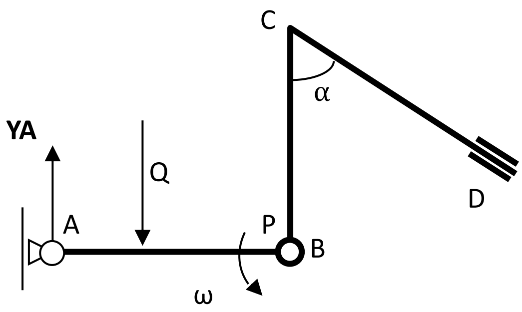
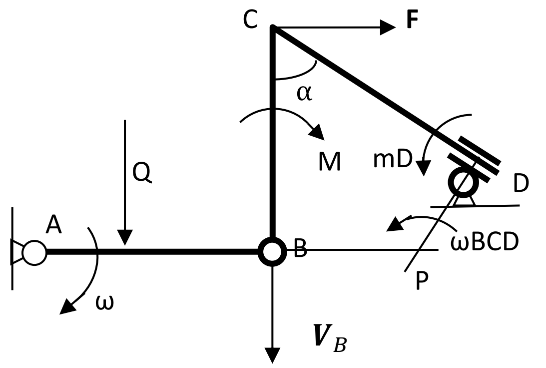


We equate to zero the possible power of all the forces applied to the rod AB, calculated as the product of the moment of forces relative to B and the angular velocity of the rod ω.

$$-Y_A AB\omega + Q\frac{2AB}{3}\omega = 0$$

This equation determines the reaction Y_A.

Moment m_D



Put the sliding support D on the hinge, allowing it to rotate around D.

Give the rod AB the possible angular velocity ω. The common particle B acquires the possible velocity **V**.

The velocity of the point D is directed along CD, so the rod in plane motion rotates around the ICV P with a possible angular velocity ω_{BCD}. Its direction is determined by velocity **V**.

The power of the forces is calculated as the product of the moment relative to their velocity centers A and P, respectively, by the angular velocities of the rods.

$$Q\frac{AB}{3}\omega \quad + (m_D - FCB - M)\omega_{BCD} = 0$$

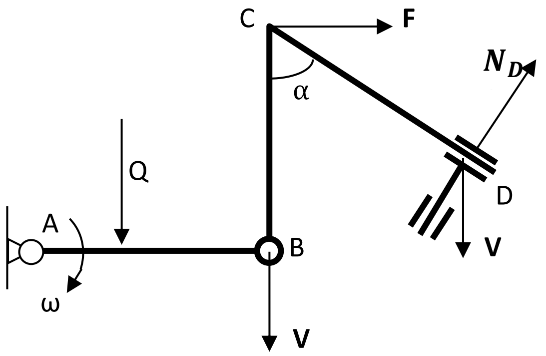
The ratio of velocities ω and ω_{BCD} is found with the velocity $\boldsymbol{V_B}$ of the common point B:

$$V_B = AB\omega \quad = BP\omega_{BCD}, \qquad BP = CDSin \propto -(BC - CDCos \propto)Ctg \propto,$$

Substituting the velocity ratio into the equation of the principle makes it possible to find the reaction moment m_D

$\boldsymbol{N_D}$ reaction

We replace the sliding seal D with a double sliding seal, which allows the BCD rod to move freely, but only translatory. The direction of this translatory motion determines the velocity of the common joint B.



Give the rod AB the possible angular velocity ω around the hinge A. The possible velocities of all particles of the BCD rod are similar and equal **V**.

The power of the forces applied to the rod AB in its rotation around A is calculated as the product of the moment and the angular velocity ω of the rod.

$$-N_D V \;\; Sin \propto + Q\frac{AB}{3}\omega = 0$$

Kinematic ratio:

$$V \;\; = AB\omega$$

Substituting the velocity relationship into the power equation allows you to find the reaction N_D

2. **Answer to the question**

Perpendicular to CD.

Here we must remind the fact that with the right number of unknowns, as soon as the constraints become redundant in one direction, they become insufficient in the other direction.

If the AB rod is perpendicular to the CD, then there is freedom to move the BCD rod along the CD. At the same time, constraints will be redundant in the direction of AB.

A COMPLETE SYSTEM OF EQUATIONS FOR A SYSTEM WITH IDEAL HOLONOMIC NON-STATIONARY CONSTRAINTS

As is known, the theorem of the change in the kinetic energy of a particle is a projection of Newton's second law on the direction of movement (velocity) of the particle, and leads to the differential equation of its motion. Lagrange used the same technique to derive the differential equations of a system with ideal holonomic non-stationary constraints. We will additionally project Newton's law onto the direction of motion of a nonstationary bond, which will allow us to find the reactions of ideal bonds.

Lagrange identities

Let us consider a system of particles $\{m_1, m_2, \dots, m_k, \dots m_n\}$ with ideal holonomic nonstationary constraints.

$$f_j(\boldsymbol{r}_k; t) = 0 \quad (j = 1,2, \dots, s) \qquad (53)$$

All possible laws, including the actual law of motion of a particle m_k, are functions of independent generalized coordinates and time

$$\boldsymbol{r}_{\mathrm{k}}(q_1\, q_2 \dots q_l; t)$$

Let us calculate the velocity of the k-th particle:

$$\mathbf{V}_{\mathrm{k}} = \dot{\boldsymbol{r}}_k = \frac{\partial \mathbf{r}_{\mathrm{k}}}{\partial \mathrm{q}_{\mathrm{i}}}\dot{q}_i + \frac{\partial \mathbf{r}_{\mathrm{k}}}{\partial \mathrm{t}} \qquad (*)$$

A repeating index indicates the summation of the index: k from 1 to n, i from 1 to l.
Herewith

$$\frac{\partial \boldsymbol{r}_k}{\partial q_i}(q_1 \dots q_l\,;\, t) \quad (**)$$

Let's prove the ***first Lagrange identity*** L_1

$$\frac{\partial \boldsymbol{V}_k}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{r}_k}{\partial q_i} \qquad (\mathrm{L}_1)$$

Since (*) is a linear function $\dot{q}_i$ with coefficients $\partial \boldsymbol{r}_k/\partial q_i$, the identity L_1 is true.

Lagrange's Second Identity L_2

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_k}{\partial q_j}\right) = \frac{\partial \boldsymbol{V}_k}{\partial q_j} \qquad (\mathrm{L}_2)$$

is proved by direct calculation of the right and left parts of the identity.

Differentiating (**) by time, we get

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_k}{\partial q_j}\right) = \sum \frac{\partial^2 \mathbf{r}_{\mathrm{k}}}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 \mathbf{r}_{\mathrm{k}}}{\partial q_j\, \partial \mathrm{t}}$$

Differentiating (*) by , we get the same expressionq_j

$$\frac{\partial \boldsymbol{V}_k}{\partial q_j} = \sum \frac{\partial^2 \boldsymbol{r}_k}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 \boldsymbol{r}_k}{\partial q_j \partial t}$$

Identity L2 is proved.

Newton's equations in projections on generalized coordinates and on the velocity of the constraint.

According to Newton's second law, the resultant of the active forces $\boldsymbol{F}_k$ and the reactions $\boldsymbol{N}_k$ of ideal constraints acting on a particle m_k of the system determine its absolute acceleration $\boldsymbol{w}_k$ in the inertial frame of reference **:**

$$m_k\boldsymbol{w}_k = \boldsymbol{F}_k + \boldsymbol{N}_k \qquad (54)$$

The problem of using Newton's equations (54) to derive the differential equations of motion of the system consists in their vector form and in the dependence of unknown reactions $\boldsymbol{N}_k$, and hence accelerations $\boldsymbol{w}_k$, on the velocities of particles and the motion of nonstationary constraints.

Equations (53, 54) correspond to the set of ***possible*** positions, laws of motion $\boldsymbol{r}_k(t)$, and velocities $\boldsymbol{V}_k$ generated by a set of initial conditions.

The possible positions of the system on the constraints can be specified by independent generalized coordinates $\{q_1, q_2, \dots, q_i, \dots q_l\}$, where l - the number of degrees of freedom of the system. Then, the possible law of motion of the particle m_k will turn out to be a function of generalized coordinates and time $\boldsymbol{r}_k(q_i, t)$.

Let us represent the motion of the system from an arbitrary position as the sum of two motions: the transport motion with the constraints and the relative motion along the constraints. The possible velocity of a particle m_k is the sum of the transport and relative velocities:

$$\boldsymbol{V}_k = \frac{d\boldsymbol{r}_k}{dt} = \frac{\partial \boldsymbol{r}_k}{\partial \mathrm{t}} + \frac{\partial \boldsymbol{r}_k}{\partial q_i}\dot{q}_i = \boldsymbol{V}_{ke} + \boldsymbol{V}_{kr} \qquad (55)$$

$$\boldsymbol{V}_{ke} = \frac{\partial \boldsymbol{r}_k}{\partial t}, \qquad (56)$$

$$\boldsymbol{V}_{kr} = \frac{\partial \boldsymbol{r}_k}{\partial q_i}\dot{q}_i \qquad (57)$$

Hereinafter, a repeating index speaks of summation by index.

In the arbitrary position of the system on the constraints, the transport velocities $\boldsymbol{V}_{ke}$ have a single value determined by the position of the particle at a given moment of time on the constraint and the equations of the constraint (53).

The constraints (53) pictured at a given moment of time correspond to a set of relative velocities $\boldsymbol{V}_{kr}$ generated by a set of initial conditions. All relative velocities are directed arbitrarily in the tangent plane to the bracing surface, and have an arbitrary modulus. It is the relative velocities that create the many possible velocities of the system. The relative velocities $\boldsymbol{V}_{kr}$ of particles are usually called ***virtual*** velocities of the system.

It follows from expression (55) that in stationary constraints the sets of possible and virtual velocities coincide.

In the given position of the system, the derivatives $\partial\boldsymbol{r}_k/\partial q_i$ in formula (55) have a single meaning. Arbitrary in formula (55) are only generalized velocities $\dot{q}_i$, which we will call ***virtual generalized velocities***.

Multiplying Newton's law for each particle by its possible velocity, after summing by k, we arrive at the theorem of the change in the kinetic energy of the system for possible velocities:

$$\dot{T} = m_k\boldsymbol{w}_k \cdot \boldsymbol{V}_k = (\boldsymbol{F}_k + \boldsymbol{N}_k) \cdot \boldsymbol{V}_k\,, \qquad T = \frac{m_k {V_k}^{\mathbf{2}}}{2}$$

$$m_k\boldsymbol{w}_k \cdot (\boldsymbol{V}_{ke} + \boldsymbol{V}_{kr}) = (\boldsymbol{F}_k + \boldsymbol{N}_k)(\boldsymbol{V}_{ke} + \boldsymbol{V}_{kr})$$

Since the constraints are ideal, the sum of the power of their reactions on any virtual velocity $\boldsymbol{V}_{kr}$in an arbitrary position of the system is equal to zero

$$\boldsymbol{N}_k \cdot \boldsymbol{V}_{kr} = 0$$

We come to two ratios

$$m_k \boldsymbol{w}_k \cdot (\boldsymbol{V}_{ke} + \boldsymbol{V}_{kr}) = (\boldsymbol{F}_k + \boldsymbol{N}_k)\boldsymbol{V}_{ke} + \boldsymbol{F}_k \boldsymbol{V}_{kr}$$

$$m_k \boldsymbol{w}_k \cdot \boldsymbol{V}_{kr} = \boldsymbol{F}_k \cdot \boldsymbol{V}_{kr} \quad (58)$$

$$m_k \boldsymbol{w}_k \cdot \boldsymbol{V}_{ke} = (\boldsymbol{F}_k + \boldsymbol{N}_k) \cdot \boldsymbol{V}_{ke} \quad (59)$$

Lagrange equations

Let us show that from the relations (58) follow the Lagrange equations, and the relations (59) make it possible to find the reactions of the constraints.

Substitute in (58) the expression of relative velocities (57), and sum $\boldsymbol{V}_{kr}$

$$\sum_i \left[\sum_k m_k \boldsymbol{w}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} \right] \dot{q}_i = \sum_i \left[\sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} \right] \dot{q}_i$$

Dot product

$$m_k \boldsymbol{w}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i}$$

is a projection of the acceleration of a particle $\boldsymbol{w}_k$on a tangent $\frac{\partial r_k}{\partial q_i}$ of the constraint surface

The acceleration component tangent to the constraint surface does not depend on the tangent velocities $\dot{q}_i$ (only the normal reactions of ideal constraints depend on them), and is equal to the projection of the resultant external forces $\boldsymbol{F}_k$ on the same direction.

Indeed, due to the independence of the generalized velocities, the coefficients in parentheses by them are equal:

$$\sum_k m_k \boldsymbol{w}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} = \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} \quad (60)$$

Let us show that these relations in generalized coordinates lead to Lagrange equations.

Here, the right-hand sides are projections of forces on generalized coordinates, so the sums of

$$Q_i = \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} \equiv \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} \quad (61)$$

naturally to call ***generalized forces***.

Let us express the left sums in terms of generalized coordinates.

$$m_k \boldsymbol{w}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} = m_k \dot{\boldsymbol{V}}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} = \frac{d}{dt}\left(m_k \boldsymbol{V}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} \right) - m_k \boldsymbol{V}_k \cdot \frac{d}{dt}\frac{\partial \boldsymbol{r}_k}{\partial q_i} =$$

$$= \frac{d}{dt}\left(m_k \boldsymbol{V}_k \cdot \frac{\partial \boldsymbol{V}_k}{\partial \dot{q}_i} \right) - m_k \boldsymbol{V}_k \cdot \frac{\partial \boldsymbol{V}_k}{\partial q_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\frac{m_k {V_k}^2}{2} - \frac{\partial}{\partial q_i}\frac{m_k {V_k}^2}{2} =$$

$$= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}$$

Lagrange identities are used here

$$\frac{\partial \boldsymbol{r}_k}{\partial q_i} = \frac{\partial \boldsymbol{v}_k}{\partial \dot{q}_i}, \qquad \frac{d}{dt}\frac{\partial \boldsymbol{r}_k}{\partial q_i} = \frac{\partial \boldsymbol{v}_k}{\partial q_i}$$

Thus, expressions (60) bring us to Lagrange equations.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \qquad (62)$$

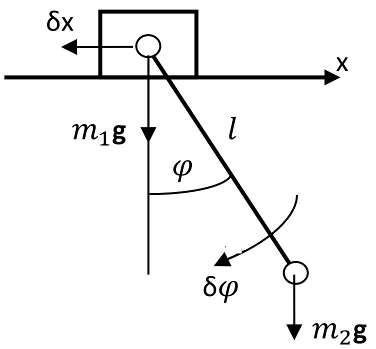
Lagrange's equations lead to the differential equations of motion of the system along the constraints. After their integration, the question of the reactions of ideal constraints remains open. Usually, vector equations of relative motion dynamics are used to determine reactions.

The Lagrange equation is the most universal way to derive differential equations of motion of a holonomic system with ideal constraints, including non-stationary ones.

Advantages and disadvantages of the Lagrange method compared to the Newton method:

1) The formalism of the Lagrange method, which consists in the fact that the problem is reduced to the differentiation of the function T, is convenient, but it does not allow us to see physical laws, as in Newton's method.
2) The Lagrange method to initially exclude the reaction of ideal constraints from the consideration, makes possible to quickly obtain differential equations of motion of the system. To determine these reactions after integrating the equations, however, we will have to return to Newton's method.

Example of solving the problem



In order to obtain the differential equations of motion of an elliptic pendulum by Newton***'s method***, it would be necessary:

- take into account the reaction of the ideal constraint in the form of tension of the thread,
- make one equation of the translatory motion of the body m_1, and two equations of the plane motion of the particle m_2.
- Of the three equations, two will be differential and one will serve to determine the tension of the thread.

Let's find differential equations using the ***Lagrange method:***

The system has two degrees of freedom, corresponding to the generalized coordinates x, φ and Lagrange equations

$$\frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx} = Q_x \qquad\qquad \frac{d}{dt}\frac{dT}{d\dot{\varphi}} - \frac{dT}{d\varphi} = Q_\varphi$$

We found generalized forces earlier

$$Q_x = 0 \qquad Q_\varphi = -m_2 g l Sin\varphi$$

The kinetic energy of the system T is sought at the moment when the system passes the equilibrium position

$$T = \frac{1}{2}\left[m_1\dot{x}^2 + m_2(\dot{x} + l\dot{\varphi})^2\right]$$

T is independent of x: $\frac{dT}{dx} = 0$, and $Q_x = 0$

$$\frac{dT}{d\dot{x}} = (m_1 + m_2)\dot{x} + m_2 l\dot{\varphi}Cos\varphi = Const$$

Note that this integral expresses the expected fact of conservation of the system momentum along the x-axis.

The first differential equation of motion of the system is obtained after differentiation

$$(m_1 + m_2)\ddot{x} + m_2 l(\ddot{\varphi}Cos\varphi - \dot{\varphi}^2 Sin\varphi) = 0$$

To obtain the second equation, let's find the corresponding derivatives.

$$\frac{dT}{d\dot{\varphi}} = m_2 l(l\dot{\varphi} + \dot{x}Cos\varphi)$$

$$\frac{d}{dt}\frac{dT}{d\dot{\varphi}} = m_2 l(l\ddot{\varphi} + \ddot{x}Cos\varphi - \dot{x}\dot{\varphi}Sin\varphi)$$

$$\frac{dT}{d\varphi} = -m_2 l\dot{x}\dot{\varphi}Sin\varphi$$

When substituted into the second Lagrange equation, last expressions are reduced, and we find the second differential equation of motion of the system

$$l\ddot{\varphi} + \ddot{x}Cos\varphi = -gSin\varphi$$

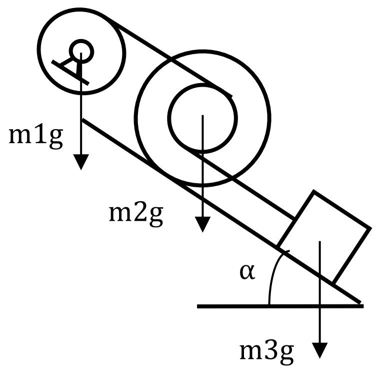
When fixing the body m_1, we obtain the known equation of oscillations of a mathematical pendulum m_2

$$l\ddot{\varphi} = -gSin\varphi$$

Set of control problems, to solve by the of Newton and Lagrange methods, can be found here: https://disk.yandex.ru/d/MgeI-bf-5ddnOg

Example of deriving the differential equation of motion of a system with the 1st degree of freedom by Newton and Lagrange methods

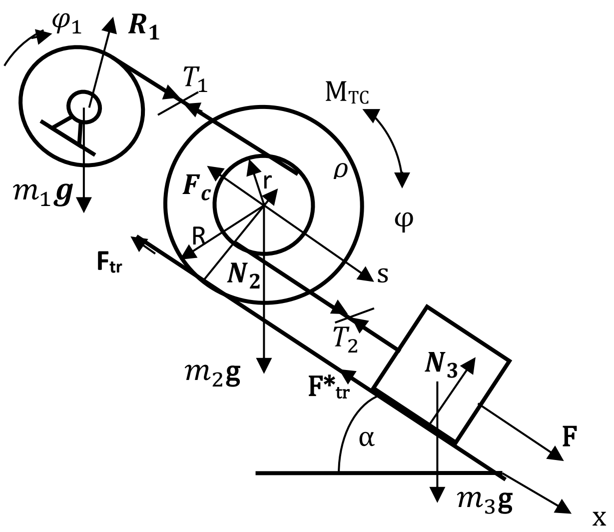
The three bodies are connected by a thread. The roller of mass m_1 moves without slippage, but with rolling resistance (coefficient k), and road resistance (k_1). Radius of inertia of the roller relative to the central axis ρ. Weight m_1 slips with friction (coefficient f)



1. Apply a force or moment to one of the bodies so that the threads are stretched and the system moves from a state of rest in the direction of the force (moment).
2. Deduce the differential equation of motion of the system using the Lagrange method.
3. Write the differential equations of motion of each of the bodies using Newton's method and the ratio of the accelerations of the bodies.
4. Check for yourself that it gives the same differential equation of motion of the system as by the Lagrange method.

1. A counterclockwise moment can be applied to block 1. Nothing can be applied to body 2, because one of the threads may weaken. Let's apply to the body 3 a force **F**, directed downwards along an inclined plane. Let us assume that the system moves from a state of rest in the direction of the force F

2. Lagrange



2.1 The position of the system can be defined by several generalized coordinates: the rotation angles φ and φ_1, the s coordinate of the center of the roller 2, the x coordinate of the body 3. The system will stop if any of the listed coordinates is fixed with the threads stretched. This means that the system has one degree of freedom, and only one of the generalized coordinates is independent. Let's choose the angle of rotation φ roller 2 for an independent coordinate, and agree on the directions of all coordinates so that they increase simultaneously.

Let's write down a single Lagrange equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} - \frac{\partial T}{\partial \varphi} = Q_\varphi$$

2.2 Let us find the kinetic energy of the system as the sum of the energies in the actual motion of the system corresponding to the type of motion of the bodies: rotational for body 1, plane for body 2, and translatory for body 3.

$$T = \frac{J_1}{2}\dot{\varphi}_1{}^2 + \frac{m_2}{2}\dot{s}^2 + \frac{J_2}{2}\dot{\varphi}^2 + \frac{m_3}{2}\dot{x}^2$$

Moments of inertia of bodies 1 (let it be a solid disk) and 2:

$$J_1 = \frac{m_1}{2}r_1{}^2; \qquad J_2 = m_2\rho^2$$

Since the system has one degree of freedom, we express all velocities in terms of generalized velocity $\dot{\varphi}$. In the roller, the velocities are linearly dependent on the distance to the ICV (R).

$$\dot{s} = R\dot{\varphi}; \qquad \dot{x} = (R - r)\dot{\varphi};$$

Top thread velocity

$$v = (R + r)\dot{\varphi}$$

Angular velocity of block 1:

$$\dot{\varphi}_1 = \frac{v}{r_1} = \frac{R + r}{r_1}\dot{\varphi}$$

By substituting the formulas of kinematic constraints, we obtain the expression of kinetic energy in terms of the generalized velocity $\dot{\varphi}$

$$T = \frac{m_1}{4}r_1{}^2\left(\frac{R + r}{r_1}\right)^2\dot{\varphi}^2 + \frac{m_2}{2}R^2\,\dot{\varphi}^2 + \frac{m_2\rho^2}{2}\dot{\varphi}^2 + \frac{m_3}{2}(R - r)^2\dot{\varphi}^2 =$$

$$= \frac{1}{2}\left[\frac{m_1}{2}(R + r)^2 + m_2(R^2 + \rho^2) + m_3(R - r)^2\right]\dot{\varphi}^2$$

$$T = \frac{J}{2}\dot{\varphi}^2$$

The constant value in square brackets can be called the moment of inertia J of the system reduced to the axis of the roller.

The left side of the Lagrange equation takes the form:

$$\frac{\partial T}{\partial \varphi} = 0; \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} = J\ddot{\varphi}$$

2.3 Let us find the generalized force $Q_\varphi$as a coefficient at the generalized velocity $\dot{\varphi}$ in the expression of the possible power of the active forces. Reactions of non-ideal constraints can be conventionally considered as unknown active forces. Let us depict all the external forces of the system. The reaction of the "soft" road to the "soft" wheel is reduced to the normal reaction N_2, the frictional force F_{tr}, the rolling friction torque M_{tc},

$$M_{тк} = N_2 k = k m_2 g Cos \propto;$$

directed against the rotation of the wheel, and the drag force **Fc**

$$F_c = N_2 \frac{k_1}{R} = \frac{k_1}{R} m_2 g Cos \propto$$

directed against the movement of its center.

The sliding friction force F^*_{tr} is related with the normal reaction N_3 by Coulomb's law

$$F^*_{тр} = f N_3 = f m_3 g Cos \propto$$

The power of forces in rotational and plane motion will be calculated as the product of the moment of forces relative to the center of velocities and the angular velocity of the body. The power is positive when the directions of the multipliers coincide.

Let's list the forces that do not have power

$N(R_1, m_2 g) = 0$ the point of application is stationary

$N(F_{тр}, N_2) = 0$ there is no moment relative to ICV

$N(N_3) = 0$ perpendicular to the velocity

$N(T_1, T_2) = 0$ the thread is inextensible

Give the system a possible generalized velocity. It is determined only by constraints, has nothing to do with actual motion, and can have an arbitrary direction and module. In order not to make a mistake in the sign, we always give a positive possible velocity $\omega_z > 0$.

Since real velocities belong to a set of possible velocities, it is convenient to use the found ratios of real velocities, considering them positive.

Let's calculate the power of all forces at positive velocities:

$$N = m_2 g \dot{s} Sin \propto + m_3 g \dot{x} Sin \propto + F\dot{x} - M_{тк}\dot{\varphi} - F_c \dot{s} - F^*_{тр}\dot{x}$$

Substituting here the velocity ratios, we find

$$N = [g\{(m_2 R + m_3(R - r)\} Sin \propto + F(R - r) - g(k + k_1) m_2 Cos \propto - g f m_3 Cos \propto (R - r)]\dot{\varphi} = Q_\varphi \dot{\varphi}$$

A constant quantity in square brackets is a generalized force Q_φ that has the dimension of a moment.

2.4 From the Lagrange equation:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} - \frac{\partial T}{\partial \varphi} = Q_\varphi$$

we obtain the differential equation of uniformly accelerated motion of the system:

$$J\ddot{\varphi} = Q_\varphi$$

3. Newton

Let us make differential equations of motion for each of the bodies of the system. To do this, mentally cut the threads and introduce their tensions T_1 and T_2.

Block m_1

Rotates. Differential equation of rotation

$$J_1 \ddot{\varphi}_1 = T_1 r_1$$

Rink m_2

Plane motion. Making three equations

$$m_2 \ddot{s} = m_2 g Sin\alpha + T_2 - T_1 - F_c - F_{тр}$$

$$0 = N_2 - m_2 g Cos\alpha$$

$$J_2 \ddot{\varphi} = F_{тр} R - (T_1 + T_2) r - M_{тк}$$

(moments relative to the center of the rink!)

Body m_3

Translatory motion. Newton's equations:

$$m_3\ddot{x} = F + m_3 g Sin\alpha - T_2 - F^*_{\text{тр}}$$
$$0 = N_3 - m_3 g Cos\alpha$$

There are 9 unknowns $\ddot{\varphi}_1 T_1 \ddot{s} T_2 N_2 \ddot{\varphi} F_{\text{тр}} \ddot{x} N_3$ in the resulting 6 equations:

The missing 3 equations are found by integrating the equations of kinematic relations

$$\ddot{s} = R\ddot{\varphi}; \quad \ddot{x} = (R - r)\ddot{\varphi}; \quad \ddot{\varphi}_1 = \frac{R + r}{r}\ddot{\varphi}$$

Analytical equations for determining the reactions of ideal constraints

Let us show that the relations (59) make it possible to find the reactions of constraints analytically, with the function of the kinetic energy T. Let us write (59) in the form:

$$m_k \boldsymbol{w}_k \cdot (\boldsymbol{V_k} - \boldsymbol{V}_{kr}) = (\boldsymbol{F}_k + \boldsymbol{N}_k) \cdot \boldsymbol{V}_{ke}$$

Taking into account the Lagrange equations, and the fact that $m_k \boldsymbol{w}_k \cdot \boldsymbol{V}_k = \dot{T}$, we get

$$m_k \boldsymbol{w}_k \cdot (\boldsymbol{V_k} - \boldsymbol{V}_{kr}) = \dot{T} - \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right)\dot{q}_i = \dot{T} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial T}{\partial q_i}\dot{q}_i =$$
$$= \dot{T} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial q_i}\dot{q}_i = 2\dot{T} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) - \frac{\partial T}{\partial t} =$$
$$= 2\dot{T} - 2\dot{T}_2 - \dot{T}_1 - \frac{\partial T}{\partial t} = 2\dot{T}_2 + 2\dot{T}_1 + 2\dot{T}_0 - 2\dot{T}_2 - \dot{T}_1 - \frac{\partial T}{\partial t} = \dot{T}_1 + 2\dot{T}_0 - \frac{\partial T}{\partial t}$$

Since

$$\frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial q_i}\dot{q}_i = \dot{T} - \frac{\partial T}{\partial t}$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) = \frac{d}{dt}\left(\frac{\partial T_2}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial T_1}{\partial \dot{q}_i}\dot{q}_i\right) = 2\dot{T}_2 + \dot{T}_1$$

Here it is taken into account that in nonstationary relations, kinetic energy is the sum of quadratic, linear and zero forms of generalized velocities.

$$T = \sum (T_{k2} + T_{k1} + T_{k0})$$

Thus,

for each of the particles of the system

$$(\boldsymbol{F}_k + \boldsymbol{N}_k) \cdot \boldsymbol{V}_{ke} = \dot{T}_{k1} + 2\dot{T}_{k0} - \frac{\partial T_k}{\partial t} \qquad (63)$$

From these ratios, it is possible to determine the reactions$\boldsymbol{N}_k$ of external and internal ideal constraints.

1. Examples

1.1. Non-stationary constraints of general type.

The particle of mass m moves in the plane x,y according to a law in which the generalized coordinate q along a moving plane curve is distinguished

$$x = x(q,t), \quad y = y(q,t)$$

For example, a particle moves along an ellipse, the semi-axes of which change in time

$$x = a(t) Sin\frac{q}{a(t)}, \quad y = b(t) Cos\frac{q}{b(t)}$$

The kinetic energy of a particle is the sum of three forms of generalized velocity

$$T(\dot{q}, q, t) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = T_2 + T_1 + T_0$$
$$T_2 = A_2(q,t)\dot{q}^2, \quad T_1 = A_1(q,t)\dot{q}, \quad T_0 = A_0(q,t)$$

Derived forms:

$$\dot T_1 = A_1\ddot q + \frac{\partial A_1}{\partial q}\dot q + \frac{\partial A_1}{\partial t}, \qquad \dot T_0 = \frac{\partial A_0}{\partial q}\dot q + \frac{\partial A_0}{\partial t}, \qquad \frac{\partial T}{\partial t} = \frac{\partial A_2}{\partial t}\dot q^2 + \frac{\partial A_1}{\partial t}\dot q + \frac{\partial A_0}{\partial t}$$

From the power equation

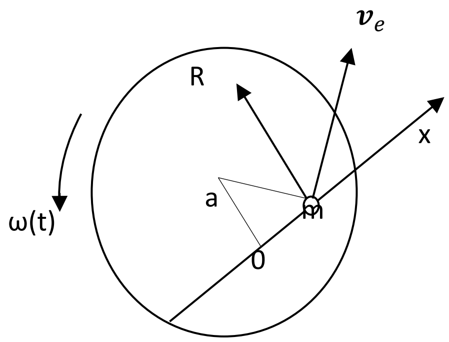
$$\dot T_1 + 2\dot T_0 - \frac{\partial T}{\partial t} = A_1\ddot q - \frac{\partial A_2}{\partial t}\dot q^2 + 2\frac{\partial A_0}{\partial q}\dot q + \frac{\partial (A_1 + A_0)}{\partial t} = \boldsymbol{N}\cdot\frac{\partial \boldsymbol{r}}{\partial t}$$

Finding the Connection Reaction

$$N_x\frac{\partial x}{\partial t} + N_y\frac{\partial y}{\partial t} = A_1\ddot q - \frac{\partial A_2}{\partial t}\dot q^2 + 2\frac{\partial A_0}{\partial q}\dot q + \frac{\partial (A_1 + A_0)}{\partial t}$$

3.2 The particle of mass m travels frictionlessly along the chord of a disk rotating in the horizontal plane with a variable angular velocity ω (t). Find the horizontal reaction ***R*** of the guide.

3.2.1 General Formula



$$\boldsymbol{R}\cdot\boldsymbol{v}_e = \dot T_1 + 2\dot T_0 - \frac{\partial T}{\partial t}$$

$$v_e = \omega\sqrt{a^2 + x^2} \qquad \boldsymbol{R}\cdot\boldsymbol{v}_e = R\omega x$$

$$T = \frac{m}{2}[\dot x^2 + \omega^2(a^2 + x^2) + 2\omega\dot x a]$$

$$T_1 = m\omega\dot x a, \qquad T_0 = \frac{m}{2}\omega^2(a^2 + x^2),$$

$$\dot T_1 = ma(\omega\ddot x + \dot\omega\dot x) \qquad \dot T_0 = mx\dot x\omega^2 + m\omega\dot\omega(a^2 + x^2)$$

$$\frac{\partial T}{\partial t} = m\omega\dot\omega(a^2 + x^2) + m\dot\omega\dot x a$$

For now

$$Rx = 2mx\dot x\omega \quad + m\dot\omega(a^2 + x^2) + ma\ddot x$$

Taking into account the differential equation of motion

$$\ddot x + \dot\omega a - \omega^2 x = 0$$

Get

$$R = m[2\omega\dot x + \dot\omega x + a\omega^2]$$

3.2.2 . Let's free the coordinate φ

$$T = \frac{m}{2}[\dot x^2 + \dot\varphi^2(a^2 + x^2) + 2\dot\varphi\dot x a]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\varphi}\right) = m[\ddot\varphi(a^2 + x^2) + 2\dot\varphi\dot x x + a\ddot x], \qquad \frac{\partial T}{\partial \varphi} = 0$$

Generalized reaction force (moment) by φ:

$$Q_\varphi = Rx$$

We get the same result

$$Rx = 2mx\dot x\dot\varphi \quad + m\ddot\varphi(a^2 + x^2) + ma\ddot x$$

$$R = m[2\dot\varphi\dot x + \ddot\varphi x + a\dot\varphi^2]$$

After integrating the equation of relative motion, we find the reaction R from the functions ω (t) and x(t).

A similar result can be obtained using the basic equation of the dynamics of the relative motion of a particle.

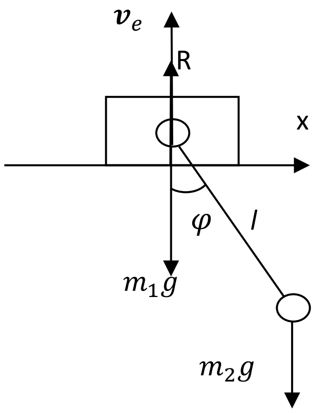
3.3 *Internal* reactions.

For a part of the system, the rest of it is a non-stationary connection. Therefore, the reaction of the discarded part can be found in the following ways:

Find the tension of the thread S connecting the bodies in an elliptical pendulum.

3.3.1 General formula.

Kinetic Energy of Mass m_2



$$T' = \frac{m_2}{2}(\dot{x}^2 + l^2\dot{\varphi}^2 + 2l\dot{x}\dot{\varphi}Cos\varphi)$$

Here it is considered as a given function of the velocity of nonstationary communication, so$\dot{x}$

$$T_o' = \frac{m_2}{2}\dot{x}^2, \quad T_1' = m_2 l\dot{\varphi}\dot{x}\ Cos\varphi$$

$$\dot{T}_o' = m_2\dot{x}\ddot{x}, \quad \dot{T}_1' = m_2 l[(\dot{\varphi}\ddot{x} + \dot{x}\ddot{\varphi})Cos\varphi - \dot{\varphi}^2\dot{x}Sin\varphi],$$

$$\frac{\partial T}{\partial t} = m_2(\dot{x}\ddot{x} + l\dot{\varphi}\ddot{x}Cos\varphi)$$

$$2\dot{T}_o' + \dot{T}_1' - \frac{\partial T}{\partial t} = -S\dot{x}Sin\varphi$$

$$SSin\varphi = m_2[l(\dot{\varphi}^2 Sin\varphi - \ddot{\varphi}Cos\varphi) - \ddot{x}]$$

3.3.2 Free the x-coordinate The reaction of the rod S is considered to be an unknown active force. Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m_2[\ddot{x} + l(\ddot{\varphi}Cos\varphi - \dot{\varphi}^2 Sin\varphi)], \quad \frac{\partial T}{\partial x} = 0, \quad Q_x = -SSin\varphi$$

leads to the same result:

$$SSin\varphi = m_2[l(\dot{\varphi}^2 Sin\varphi - \ddot{\varphi}Cos\varphi) - \ddot{x}]$$

The obvious term enters the right side of the solution through accelerations. This expression gives the solution for $m_2 g$. To find the tension of the filament at $\varphi \neq 0$, the system must be given a constant vertical velocity, as is done in the following example. $\varphi = 0$

3.4 ***Reactions of stationary constraints***.

With stationary bonds, the reactions of ideal bonds do not create power. To find reactions, it is possible to artificially make the bonds nonstationary by giving them a constant velocity in the direction of the bond. At the same time, the frame of reference remains inertial, and the reactions of the bonds do not change.

Let us find the normal reaction R acting on the body of the elliptulum pendulum.m_1
Let's give the base a vertical constant velocity. Kinetic energy of the system$\boldsymbol{v}_e$

$$T = \frac{m_1}{2}(\dot{x}^2 + {v_e}^2) + \frac{m_2}{2}[(\dot{x} + l\dot{\varphi}Cos\varphi)^2 + (v_e + l\dot{\varphi}Sin\varphi)^2]$$

3.4.1 General Formula

$$T_1 = m_2 v_e l\dot{\varphi}Sin\varphi, \qquad T_o = \frac{1}{2}(m_1 + m_2){v_e}^2 = Const$$

$$\dot{T}_1 = m_2 l v_e(\ddot{\varphi}Sin\varphi + \dot{\varphi}^2 Cos\varphi) = [R - (m_1 + m_2)g]v_e$$

$$R = (m_1 + m_2)g + m_2 l(\ddot{\varphi}Sin\varphi + \dot{\varphi}^2 Cos\varphi)$$

3.4.2 Release the coordinate y (.$\dot{y} = v_e, \ \ddot{y} = 0$)

$$T = \frac{m_1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m_2}{2}[(\dot{x} + l\dot{\varphi}Cos\varphi)^2 + (\dot{y} + l\dot{\varphi}Sin\varphi)^2]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = (m_1 + m_2)\ddot{y} + m_2 l(\ddot{\varphi}Sin\varphi + \dot{\varphi}^2 Cos\varphi), \quad \frac{\partial T}{\partial y} = 0, \quad \ddot{y} = 0$$

Generalized strength according to y: $Q_y = R - (m_1 + m_2)g$

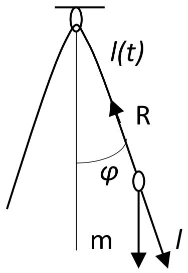
We come to the same result:

$$R = (m_1 + m_2)g + m_2 l(\ddot{\varphi}Sin\varphi + \dot{\varphi}^2 Cos\varphi)$$

This result can be obtained from the theorem on the motion of the center of mass.

3.5 Let us find the reaction of the filament R of the pendulum of mass *m* and the variable length *l(t)*.

3.5.1 General Formula



$$T = \frac{m}{2}(\dot{l}^2 + l^2\dot{\varphi}^2)$$

$$T_1 = 0, \quad T_o = \frac{m}{2}\dot{l}^2, \quad \dot{T}_o = m\dot{l}\ddot{l}, \quad \frac{\partial T}{\partial t} = m\dot{l}\ddot{l} + ml\dot{l}\dot{\varphi}^2$$

$$2\dot{T}_0 + \dot{T}_1 - \frac{\partial T}{\partial t} = (\boldsymbol{R} + m\boldsymbol{g}) \cdot \boldsymbol{v}_e$$

$$2m\dot{l}\ddot{l} - m\dot{l}\ddot{l} - ml\dot{l}\dot{\varphi}^2 = R_l\dot{l} + mg\dot{l}Cos\varphi$$

$$R_l = m(\ddot{l} - l\dot{\varphi}^2 - g\ Cos\varphi)$$

3.5.2 Free the coordinate l

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{l}}\right) = m\ddot{l}, \qquad \frac{\partial T}{\partial l} = mx\dot{\varphi}^2 \qquad Q_l = R_l + mgCos\varphi$$

We come to the same result

$$R_l = m(\ddot{l} - l\dot{\varphi}^2 - g\ Cos\varphi)$$

This result can also be obtained with the help of the basic equation of the dynamics of the relative motion of a particle.

Note that freeing the coordinates leads to a result faster than the general formula. But this method is applicable only to constraints with a finite number of degrees of freedom.

CONSERVATIVE SYSTEMS

Definition and properties of a potential force field.

A ***force field*** is a three-dimensional space, at each point of which a force function $\boldsymbol{F}(\boldsymbol{r}; t)$ is given.

Consider a stationary force field given in Cartesian coordinates by x, y, z functions:

$$F_x(x, y, z); \quad F_y(x, y, z); \quad F_z(x, y, z) \qquad (67)$$

As has been shown; in order to calculate the final work of the field force, it is necessary to know the trajectory of the particle. Among field forces, there is a class of ***potential force fields*** for which the final work of the force is determined only by the initial and final position of the particle and does not depend on the trajectory.

A force field (67) is said to be ***potential*** if there exists a ***potential energy*** function $\Pi(x, y, z)$,such that

$$F_x = -\frac{\partial \Pi}{\partial \mathrm{x}}; \quad F_y = -\frac{\partial \Pi}{\partial y}; \quad F_z = -\frac{\partial \Pi}{\partial z}$$

Let the field (67) be given. How to check if it is potential? We believe that the potential energy Π is a continuous, doubly differentiable function of coordinates. Then you can use the property: the order in which the mixed derivative is taken does not affect the result:

$$\frac{\partial^2 \Pi}{\partial x \partial y} = \frac{\partial^2 \Pi}{\partial y \partial x}, \qquad \frac{\partial^2 \Pi}{\partial z \partial y} = \frac{\partial^2 \Pi}{\partial y \partial z}, \qquad \frac{\partial^2 \Pi}{\partial x \partial z} = \frac{\partial^2 \Pi}{\partial z \partial x}$$

Hence the ***criteria for the potentiality of a silent field***

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

Properties of the work of potential forces.

1) The elementary work of the potential force is equal to minus the potential energy differential. Really

$$d'A = \boldsymbol{F} \bullet d\boldsymbol{r} = F_x dx + F_y dy + F_z dz = -\left(\frac{\partial \Pi}{\partial x} dx + \frac{\partial \Pi}{\partial y} dy + \frac{\partial \Pi}{\partial z} dz\right) = -d\Pi$$

The following properties follow from this.

2) The final work of the potential force depends only on the initial and final position of the particle

$$A_{12} = \int_{1-2} d'A = -\int_1^2 d\Pi = \Pi_1 - \Pi_2$$

3) Work in a vicious circle is equal to zero:

$\Pi_1 = \Pi_2$, therefore Ao=0

Calculation of potential energy. The law of conservation of total mechanical energy.

The surface on which Π retains the value is called ***equipotential***:

$$\Pi\,(x, y, z) = C_1 = const$$

Let us find out the direction of ***F*** in relation to the equipotential surface. Let the particle travel along the equipotential surface. According to the property of the work, the potential force ***F*** does not perform work since $\Pi = C_1$

$$d'A = \boldsymbol{F} \bullet d\boldsymbol{r} = 0$$

Since $d\boldsymbol{r}$ is directed arbitrarily in the tangent plane to the surface, the force is directed perpendicular to the equipotential surfaces $\Pi = C_1$

On the other hand

$$\boldsymbol{F} = -\left(\frac{\partial \Pi}{\partial x}\boldsymbol{i} + \frac{\partial \Pi}{\partial y}\boldsymbol{j} + \frac{\partial \Pi}{\partial z}\boldsymbol{k}\right) = -\boldsymbol{grad}\,\Pi$$

This means that the force is directed in the direction of decreasing Π.

According to the properties of differentiation, both functions $\Pi(x, y, z)$ *and* $\Pi(x, y, z) + C$, where C is an arbitrary additive constant, determine the same force field. The potential energy is said to be determined to ***the accuracy of the additive constant.***

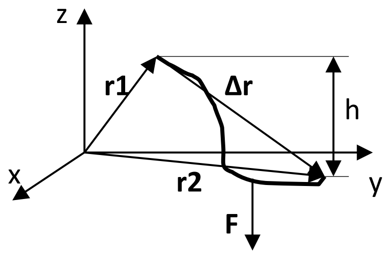
Let's choose a zero level of potential energy. Then move a particle from an arbitrary position $M(x, y, z)$ to any point of the zero level and calculate the work of the force:

$$A_{MMo} = \Pi(x, y, z)$$

Hence the ***rule for calculating potential energy functions***:

The function Π (x,y,z) is computed as
the work of the potential force to move from an arbitrary point M(x,y,z) to the zero level

Examples:



3) ***Constant force F*** $= \boldsymbol{const}$:

$$A_{12} = \boldsymbol{F} \bullet (\boldsymbol{r_2} - \boldsymbol{r_1}) = \boldsymbol{F} \bullet \boldsymbol{\Delta r}$$

4) ***Gravity***. This is a particular example of constant force:

A field is homogeneous if

$$\boldsymbol{F} = m\boldsymbol{g}, \qquad \boldsymbol{g} = \boldsymbol{const}$$

Direct the axis z vertically upwards, then

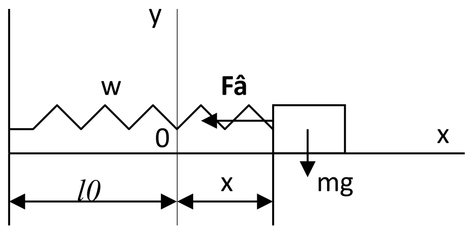
$$F_x = F_y = 0 \qquad\qquad F_z = -mg$$

All surfaces $z = const$ are equipotential. Therefore

$$A_{12} = F_z\,(z_1 - z_2) = \pm\, mgh$$

The work is positive if the particle drops.

5) ***Straight Linear Spring***:



Natural length of undeformed spring is l_0 . When the length is changed on what is called spring deformation $\Delta = l - l_0$, elastic force $\boldsymbol{F}_{\text{в}}$ is generated. It always tends to restore the undeformed state of the spring, so it is called ***the restoring force.,.***

A spring ***is linear*** if the force $\boldsymbol{F}_{\text{в}}$is linearly dependent on the deformation:

$$F_{\text{в}} = c\,\Delta$$

The coefficient c (n/m) is called the spring stiffness. If the beginning of the x-axis is chosen in the position where $\Delta = 0$, then

$$F_{\text{вх}} = -c\,x$$

Elementary work of $\boldsymbol{F}_{\text{в}}$

$$d'A = F_{\text{в}x}\,dx = -cx\,dx$$

The final work of $\boldsymbol{F}_{\text{в}}$

$$A_{12} = -c\int_{x_1}^{x_2} x dx = \frac{1}{2}c(x_1^2 - x_2^2)$$

Coordinate can be replaced by their deformation:

$$A_{12} = \frac{1}{2}c(\Delta_1^2 - \Delta_2^2)$$

A system is called ***conservative*** if all the forces acting on it are potential.
Theorem on the change of kinetic energy for a conservative system in integral form:

$$\text{T}_2 - \text{T}_1 = \text{A}_{12} = \Pi_1 - \Pi_2 \quad \text{или} \quad \text{T}_2 + \Pi_2 = \text{T}_1 + \Pi_1$$

The total mechanical energy of a system is the sum of its kinetic and potential energies:

$$E = T + \Pi$$

As we can see that the total mechanical energy of the conservative system is conserved

$$E = const$$

Suppose that in addition to potential forces, there are non-potential forces acting on the system, then:

$$dT = d'A_{\text{пот}} + d'A_{\text{не пот}} = -d\Pi + d'A_{\text{не пот}}$$

Dividing by dt, we find that

the speed of change of the total mechanical energy is equal to the power of the non-potential forces.

$$dE/dt = N_{\text{не пот}}$$

For example, in the presence of ***a viscous drag force***

$$\boldsymbol{F}_{\text{сопр}} = -\beta \boldsymbol{V} \qquad \beta = Const$$

Total mechanical energy decreases at a speed

$$dE \,/\, dt = -\beta \boldsymbol{V} \bullet \boldsymbol{V} = -\beta V^2$$

Generalized forces.

Static principle of possible velocities for a conservative system.

Consider a conservative non free system with potential energy Π (x, y, z) and generalized coordinates $q_1 \ldots . q_l$. Let's find the generalized forces of the system by definition ,

$$\mathrm{Q_i} = \sum F_\mathrm{k} \cdot \frac{\partial \boldsymbol{r}_\mathrm{k}}{\partial q_\mathrm{i}} = -\sum \left(\frac{\partial \Pi}{\partial \mathrm{x}} \frac{\partial x}{\partial q_\mathrm{i}} + \frac{\partial \Pi}{\partial \mathrm{y}} \frac{\partial y}{\partial q_\mathrm{i}} + \frac{\partial \Pi}{\partial \mathrm{z}} \frac{\partial z}{\partial q_\mathrm{i}} \right) = -\frac{\partial \Pi}{\partial q_\mathrm{i}}$$

Example: elliptical pendulum

Let's take the position $x = 0, \varphi = 0$ as the zero level of potential energy and calculate the work when the system returns to the origin

$$\Pi = m_2 g l \,(1 - Cos\,\varphi)$$

Π does not depend on x, so $Q_x = 0$

$$Q_\varphi = -\partial \Pi / \partial \varphi = -m_2 g l \,Sin\,\varphi$$

Static principle of possible velocties:

$$\delta A = \sum Q_i \delta q_i = 0$$

Since the generalized possible displacements δq_i are independent, the principle can be read as follows:

In the equilibrium position, all generalized forces turn to zero.

$$Q_i = 0 \quad (i = 1,2, \ldots, l)$$

This means that

In the equilibrium position, the potential energy of a conserved system has an extremum

$$\partial \Pi / \partial q_i = 0 (i = 1,2, \ldots, l)$$

Consequently, finding the equilibrium positions of a conservative system is reduced to finding extremes of the function Π.

Lagrange equation for conservative systems. Cyclic coordinates and integrals.

Consider a conservative non free system with *l* degrees of freedom. Potential energy $\Pi(q_1 \ldots q_l)$ determines generalized forces

$$Q_i = -\frac{\partial \Pi}{\partial q_i} \quad (i = 1,2, \ldots, l)$$

Lagrange equations take the form

$$\frac{d}{dt} \frac{\partial (T - \Pi)}{\partial \dot{q}_i} - \frac{\partial (T - \Pi)}{\partial q_i} = 0 \qquad (i = 1,2, .., l)$$

Here it is taken into account that the potential energy does not depend on generalized velocities

$$\frac{\partial \Pi}{\partial \dot{\mathrm{q}}_\mathrm{i}} = 0 \quad (i = 1,2, .., l)$$

Let's write the Lagrange equations using the ***Lagrange function***

$$L = T - \Pi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1,2,..,l)$$

The coordinate q_σ is called cyclic if the Lagrange function does not depend on it

$$\partial L/\partial q_\sigma = 0$$

The Lagrange equation with the number σ acquires the form

$$\frac{d}{dt}\frac{dL}{d\dot{q}_\sigma} = 0$$

and has ***a cyclic integral***

$$\frac{dL}{d\dot{q}_\sigma} = Const$$

Often this integral describes the case of conservation of momentum or angular momentum.

Example: elliptical pendulum

Π and T do not depend on x, so x is a cyclic coordinate, and there is an integral

$$\frac{dT}{d\dot{x}} = (m_1 + m_2)\dot{x} + m_2 l\dot{\varphi}Cos\varphi = Const$$

We have already noted that this integral expresses the expected conservation of the momentum of the system along the x-axis.