Министерство науки и высшего образования Российской Федерации

САНКТ-ПЕТЕРБУРГСКИЙ

ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ ПЕТРА ВЕЛИКОГО

А.В. Костарев   Т.А. Костарева

ТЕОРЕТИЧЕСКАЯ МЕХАНИКА

КОЛЕБАНИЯ

ГИРОСКОПЫ

ТЕОРИЯ И ПРИМЕРЫ РЕШЕНИЯ ЗАДАЧ

Учебное пособие на английском языке

Санкт-Петербург

2025

CONTENTS

INTRODUCTION TO THE THEORY OF OSCILLATIONS

ELEMENTARY GYROSCOPE THEORY

INTRODUCTION TO THE THEORY OF OSCILLATIONS

Everything around us, even seemingly at rest, is in a periodic motion called oscillation. Characteristic conditions for the occurrence of oscillation are the presence of:

1. The equilibrium position (state or process) around which the oscillations occur.
2. Forces that tend to return the system to an equilibrium position and are therefore called ***restoring forces.***

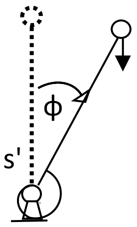
Determination of the equilibrium position of the system.

A system with ideal holonomic stationary constraints is considered. Let the number of degrees of freedom of the system be $l = 1$, and the system be conservative with the potential $\Pi(q)$.

To determine whether the system has equilibrium positions, we will use the principle of possible velocities, which states that if there is such a position, then the potential energy in it has an extremum.

$$\partial\Pi/\partial q = 0$$

Thus we have obtained an equation for finding the equilibrium position. If it has solutions, then the system has equilibrium positions.



Example: A reversed mathematical pendulum.

This is the name of the mathematical pendulum of length l and mass m, held in a vertical equilibrium position by a spiral spring of stiffness c'. Let us choose the equilibrium position for the zero level of potential energy: $\Pi(0) = 0$. The function $\Pi(\varphi)$ is calculated as the work of gravity and spring when the pendulum returns to the equilibrium position.

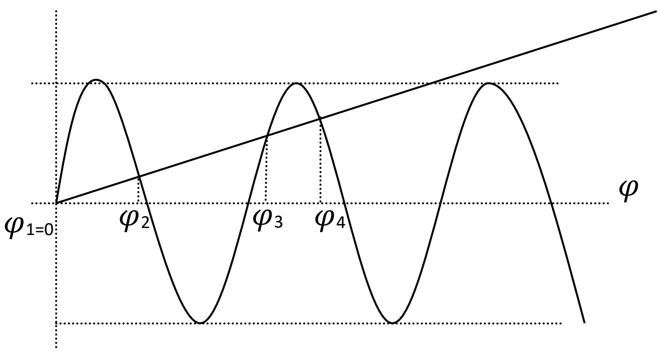
$$\Pi(\varphi) = -mgl(1 - Cos\varphi) + \frac{1}{2}c'\varphi^2$$

The static principle of possible displacements gives the equilibrium condition:

$$\Pi' = 0 \qquad \text{or} \qquad c'\varphi = mglSin\varphi$$

The solutions to this equation are found at the points of intersection of the line $y = c'\varphi$ and the sine wave $y = mglSin\varphi$

The lower the spring stiffness c', the more equilibrium positions the system will have. The graph shows that the system has 4 equilibrium positions. In the absence of a spring, there are countless of them, but physically these are two vertical positions.

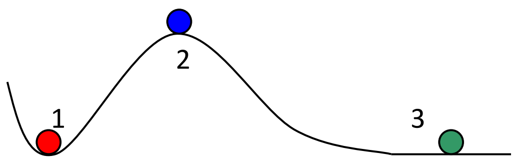


If the spring is rigid

$$c' > mgl$$

then the pendulum has only one equilibrium position $\varphi = 0$.

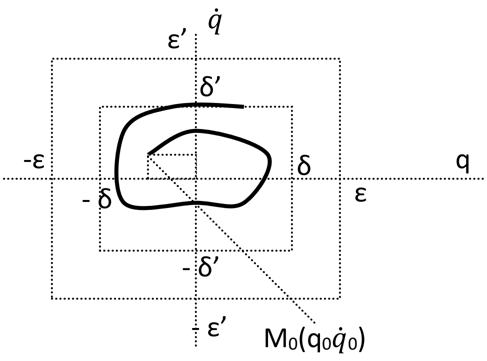
In general, there are three types of equilibrium position: stable, unstable, indifferent. For the ball, these are positions (1), (2) and (3). When deviating from a stable position, the ball tends to return to it. At the slightest deviation from the unstable position, the ball will never return there. The positions of indifferent equilibrium make up a continuum - next to any of them there is the similar.



Experience shows that oscillations occur only near a stable equilibrium position.

Stability of the equilibrium position according to Lyapunov.

Let us consider a system with one degree of freedom and an equilibrium position in which we choose the origin of the generalized coordinate q. The ***state of the system*** is determined by the values of its coordinate $q(t)$ and velocity $\dot{q}$ (t). We will take these parameters as the coordinates of the ***phase plane*** $q\ \dot{q}$. The origin of the phase coordinates corresponds to the equilibrium position of the system.



Let's see how the system will move if it is brought out of its equilibrium state. At $t = 0$, let's give the system ***a perturbation***: $q_o \dot{q}_0$. . Then the system will make ***a perturbed motion,*** and the depicting point will describe ***the phase trajectory***.

The considered equilibrium position is called ***stable according to Lyapunov*** if for any two arbitrarily small numbers ε, ε' it is possible to specify such two arbitrarily small numbers δ, δ' that the phase trajectory with the beginning in the δ region will never leave the ε region.

Linear and nonlinear systems. Linearization.

Consider a conservative system with potential energy $\Pi(q)$and an equilibrium position in which we choose the beginning of q and the zero level of potential energy:

$$\Pi(0) = 0 \qquad\qquad \Pi'(0) = 0 \text{ – equilibrium condition}$$

Let's decompose $\Pi(q)$ into a McLaren series, taking into account the condition of equilibrium:

$$\Pi(q) = \Pi(0) + \Pi'(0)q + \frac{1}{2}\Pi''(0)q^2 + \cdots . = \frac{1}{2}\Pi''(0)q^2 + \cdots .$$

The first remaining term in the series is called the ***quadratic form*** because it contains the square of q.

A system is called Π ***linear with respect to*** q if it is a quadratic form, i.e. all following terms of the expansion of Π are equal to zero.

Kinetic energy of the system.

$$T = \frac{1}{2}\sum m_k V_k^2 \qquad\qquad \boldsymbol{V}_\mathrm{k} = \frac{\partial \mathbf{r}_\mathrm{k}}{\partial q}\dot{q}$$

$$T = \frac{1}{2}\left[\sum \left(\frac{\partial \mathbf{r}_\mathrm{k}}{\partial q}\right)^2\right]\dot{q}^2 = \frac{1}{2}a(q)\dot{q}^2$$

Let 's decompose the function $a(q)$into a McLaren series.

$$a(q) = a(0) + a'(0)q + \ldots$$

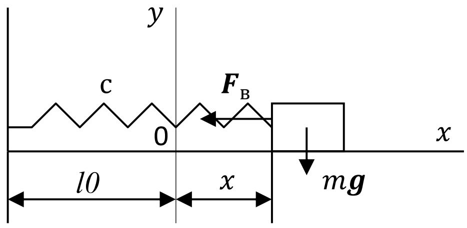
A system is said to be ***linear with respect to T*** if T is a quadratic form of $\dot{q}^2$ with a constant coefficient a, i.e. $a(q) = Const.$ A system ***is linear*** if it is linear with respect to both T and Π.

If the system is not linear, then we have to linearize it. ***Linearization*** of system is the introduction of constraints that allow the system to be considered almost linear. If we consider the small motions of the system $q, \dot{q} << 1$, then only the first term will remain in the decomposition of the function $\Pi(q)$.

$$\Pi \approx \frac{1}{2} cq^2, \qquad c = \Pi''(0)$$ – ***rigidity of the system***

After linearization in $a(q)$ McLaren series, we have only $a(0) \equiv a$

$$T = \frac{1}{2} a\dot{q}^2$$



Corollary: it is easier to obtain a quadratic form for T by calculating T while it is passing by the equilibrium position of the system.

Examples:

a) Linear spring: $T = \frac{1}{2}m\dot{x}^2$; $\Pi = \frac{1}{2} cx^2$ - linear both in T and Π

b) Reversed pendulum: $T = \frac{1}{2}ml^2\dot{\varphi}^2$; $\Pi = -mgl(1 - Cos\varphi) + \frac{1}{2} c'\varphi^2$

– nonlinear in Π, but linear in T.

Lagrange–Dirichlet theorem (without proof). Sylvester's criterion.

Theorem: In order that a given position of equilibrium be stable according to Lyapunov, it is necessary (but not sufficient) that the function Π has a minimum in this position.

Let's choose the origin and the zero level of potential energy in the equilibrium position. After linearization (if required), we get:

For a system with ***one degree of freedom***

$$\Pi = \frac{1}{2} cq^2, \qquad c = \Pi''(0) > 0$$ – the condition of minimum and stability

For a system with ***l degrees of freedom***:

$$\Pi = \Pi_0 + \sum \left(\frac{\partial \Pi}{\partial q_i}\right)_0 q_i + \frac{1}{2}\sum \left(\frac{\partial^2 \Pi}{\partial q_i \partial q_j}\right)_0 q_i q_j + \cdots \approx \frac{1}{2}\sum c_{ij} q_i q_j$$

System Stiffness Factors

$$c_{ij} = \left(\frac{\partial^2 \Pi}{\partial q_i \partial q_j}\right)_0 \qquad i,j = 1,2\ldots l$$

form the stiffness matrix of the system

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1l} \\ \vdots & \ddots & \vdots \\ c_{l1} & \cdots & c_{ll} \end{pmatrix}$$

According to the Lagrange-Dirichlet theorem, for the stability of the equilibrium position, it is necessary that the function Π has a minimum at the origin. Since Π is equal to zero there, it should be required that in the vicinity of zero, the function Π be positively defined.

It is known from mathematics that the condition for the positive definiteness of a quadratic form at zero is the ***Sylvester criterion***:

the positivity of all the main diagonal minors of the stiffness matrix.

$$\Delta_1 = c_{11} > 0$$

$$\Delta_2 = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}c_{22} - {c_{12}}^2 > 0$$

……….

$$\Delta_l = |C| > 0$$

If it is fulfilled, then this equilibrium position is stable according to Lyapunov. If the criterion is not met, then more subtle methods of stability research are required.

SYSTEM WITH ONE DEGREE OF FREEDOM

Free oscillations without resistance.

The motion of a conservative system with one degree of freedom is considered near a stable equilibrium position, where the origin of the q coordinate and the zero level of potential energy are chosen. After linearization (if needed), the kinetic and potential energies of the system will take the form of quadratic forms with constant coefficients.

$$T = \frac{1}{2}a\dot{q}^2, \qquad \Pi = \frac{1}{2}cq^2$$

$a > 0$ due to the positivity of kinetic energy, $c > 0$ due to the stability of the equilibrium position

Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q}$$

leads to a ***differential equation of free oscillations without resistance***

$$a\ddot{q} = -cq \quad \text{or} \quad \ddot{q} + k^2 q = 0 \qquad (k^2 = \frac{c}{a}\ s^{-1})$$

Let's try to find a solution of this equation in the form of an exponent. Substituting

$$q = e^{\lambda t}$$

into the equation, after the reduction of $e^{\lambda t}$, we obtain a ***characteristic equation*** for determining the unknown parameter λ

$$\lambda^2 + k^2 = 0$$

The equation has two imaginary roots

$$\lambda = \pm ki$$

This means that the equation has two independent solutions. The general solution (second integral) of the equation

$$q = C_1 Coskt + C_2 Sinkt$$

contains two arbitrary constant integrations C_1 and C_2, which can be found from initial conditions:

$$t = 0, \qquad q = q_0, \qquad \dot{q} = \dot{q}_0$$

To use them, we find the law of velocity (the first integral of the equation)

$$\dot{q} = -C_1 k Sinkt + C_2 k Coskt$$

Substituting the initial conditions at $t = 0$, we find that

$$q_0 = C_1, \quad \dot{q}_0 = C_2\, k \quad hence$$

$$C_1 = q_0, \quad C_2 = \frac{\dot{q}_0}{k}$$

Finally

$$q = q_0 Coskt + \frac{\dot{q}_0}{k} Sinkt$$

That means that if position of equilibrium is stable

$$c > 0$$

then the system performs periodic movement at *a **natural frequency***

$$k = \sqrt{\frac{c}{a}} \text{ сек}^{-1}$$

It is more convenient to represent the law of motion as a single function of the sine. To do this, let's introduce the new constants A $and\ \alpha$ so that we get the sine of the sum

$$C_1 = ASin\ \alpha, \quad C_2 = ACos\ \alpha$$

We get

$$q = Asin(kt + \alpha)$$

Here A is the amplitude, $(kt + \alpha)$is the phase, α is the initial phase of oscillations. After a time period of oscillation T, the sine phase will change by 2π radians:

$$k(t + T) + \alpha = kt + \alpha + 2\pi$$

consequently, the period of oscillations is

$$T = 2\pi/k \text{ sec}$$

Relay dissipative function

Almost any system oscillates in some medium. When the system moves, the resistance forces of the medium arise. For example, the ***viscous damp forces*** proportional to the first power of the velocity of the points of the system:

$$\boldsymbol{F}_k = -\beta_k \boldsymbol{v}_k (k = 1,2,\ldots,n)$$

Let's find a generalized resistance force, taking into account the Lagrange identity:

$$Q_{res} = \sum \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r_k}}{\partial q} = -\sum \beta_k\, \boldsymbol{v}_k \cdot \frac{\partial \boldsymbol{v_k}}{\partial \dot{q}} = -\frac{\partial \Phi}{\partial \dot{q}}$$

The dissipative ***Relay*** function of the viscous damp forces is introduced here:

$$\Phi = \frac{1}{2}\sum_{k=1}^{n} \beta_k \boldsymbol{v}_k{}^{\boldsymbol{2}}$$

We see that the expression Φ has the form of kinetic energy where instead of the masses are replaced the coefficients of resistance. To find Q_{res} we need to write the ***Relay*** function in generalized coordinates:

$$\boldsymbol{v}_k = \frac{\partial \boldsymbol{r_k}}{\partial q}\dot{q}, \qquad \Phi = \frac{1}{2}\left[\sum \left(\frac{\partial \boldsymbol{r_k}}{\partial q}\right)^2\right]\dot{q}^2 = \frac{1}{2}b(q)\dot{q}^2$$

The system is linear with respect to Φ, if $b(q) = Const$ (analogy with T).
If not, then small movements are considered: $q << 1$ – the system is linearized: $b(q) \approx b(0)$. So, like T, the Relay function should be calculated at the moment when the system passes the equilibrium position $q = 0$, which always simplifies the calculations.

Viscous resistance arises in linear and angular dampers. The Relay function is calculated in common case using the formula

$$\Phi = \frac{1}{2}\sum \alpha_i\, V_i{}^2 + \frac{1}{2}\sum \beta_j\, \omega_j{}^2$$

where α_i are the coefficients of resistance of linear dampers (shock absorbers), V_i - the velocities of their pistons, β_j —the coefficients of rotational resistance, ω_j — the angular velocities of rotating bodies.

Φ and total mechanical energy.

Consider a system with one degree of freedom and viscous resistance.
Potential and kinetic energy, Relay function for a nonlinear system

$$\Pi(q) = \frac{1}{2}cq^2 \quad (c = Const > 0), \quad T = \frac{1}{2}a(q)\dot{q}^2, \quad \Phi = \frac{1}{2}b(q)\dot{q}^2$$

They have the properties of

$$\frac{\partial T}{\partial \dot{q}} = 2T, \quad \frac{\partial \Phi}{\partial \dot{q}} = 2\Phi, \quad \dot{\Pi} = \frac{\partial \Pi}{\partial q}\dot{q}, \quad \dot{T} = \frac{\partial T}{\partial \dot{q}}\ddot{q} + \frac{\partial T}{\partial q}\dot{q}$$

Let us multiply the Lagrange equation for this system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} - \frac{\partial \Phi}{\partial \dot{q}}$$

by $\dot{q}$

$$\dot{q}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}\dot{q} = -\frac{\partial \Pi}{\partial q}\dot{q} - \frac{\partial \Phi}{\partial \dot{q}}\dot{q}$$

Using the formula

$$\dot{q}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\dot{q}\right) - \frac{\partial T}{\partial \dot{q}}\ddot{q} = 2\dot{T} - \frac{\partial T}{\partial \dot{q}}\ddot{q}$$

Taking into account the properties of the functions T, Φ, Π we get

$$2\dot{T} - \frac{\partial T}{\partial \dot{q}}\ddot{q} - \frac{\partial T}{\partial q}\dot{q} = -\dot{\Pi} - 2\Phi \qquad \text{or} \qquad \dot{T} + \dot{\Pi} = -2\Phi$$

$$\dot{E} = -2\Phi$$

This result can be formulated as follows:

Total mechanical energy $E = T + \Pi$ ***decreases at a rate of*** 2Φ

Influence of viscous resistance on the motion of the system.

The differential equation of a system with one degree of freedom and viscous resistance is obtained from the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} - \frac{\partial \Phi}{\partial \dot{q}}$$

After linearization (if required), we get quadratic forms

$$\Pi(q) = \frac{1}{2}cq^2, \quad T = \frac{1}{2}a\dot{q}^2, \quad \Phi = \frac{1}{2}b\dot{q}^2$$

After substituting it into the Lagrange equation, we obtain ***the differential equation of damped oscillations***

$$a\ddot{q} = -cq - b\dot{q} \qquad \text{or}$$

$$\ddot{q} + 2n\dot{q} + k^2 q = 0$$

Here the notation is entered: coefficient of resistance $2n = b/a$ and the natural frequency square $k^2 = c/a$

Let's find the solution to the equation in the form of an exponent:

$$q = e^{\lambda t}$$

By substituting this solution into the equation, after reducing by $e^{\lambda t}$, we obtain ***the characteristic equation***

$$\lambda^2 + 2n\lambda + k^2 = 0$$

This equation has 2 roots

$$\lambda = -n \pm \sqrt{n^2 - k^2}$$

which correspond to 2 independent solutions, as many as should have the equation of the second order.

The type of solution depends on the sign of the radical expression

1. ***Low Resistance Case*** $\mathbf{n} < \boldsymbol{k}$

In this case, the roots are complex and the solution looks like

$$q = e^{-nt}\left(C_1 Cos\tilde{k}t + C_2 Sin\tilde{k}t\right), \qquad \tilde{k} = \sqrt{k^2 - n^2} < k$$

This is the "second integral" integral of the differential equation under consideration

The first integral is a generalized velocity

$$\dot{q} = -ne^{-nt}\left(C_1 Cos\tilde{k}t + C_2 Sin\tilde{k}t\right) + e^{-nt}\left(-C_1\tilde{k}Sin\tilde{k}t + C_2\tilde{k}Cos\tilde{k}t\right)$$

As always, constants C_1, C_2 are found from the initial conditions:

$$t = 0:\; q = q_0, \quad \dot{q} = \dot{q}_0$$

From where

$$C_1 = q_0 \qquad\qquad C_2 = \frac{1}{\tilde{k}}\left(\dot{q}_0 + nq_0\right)$$

Let's explore this solution by moving on to the new permanent integrations

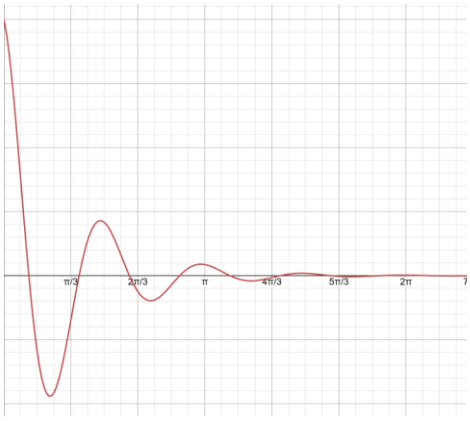
$$C_1 = ASin\alpha, \quad C_2 = ACos\alpha$$

For now $$q = Ae^{-nt}Sin\tilde{k}t$$

Denote

$$\tilde{A} = Ae^{-nt}$$

– amplitude, which decreases over time.



We see that the system makes damped oscillations. They are quasi-periodic, since only the equilibrium position of the system is passed through equal intervals of time. The quasi-period is calculated as for oscillations without resistance

$$\tilde{T} = \frac{2\pi}{\tilde{k}} > T = \frac{2\pi}{k}$$

We see that with an increase in resistance n, the quasi period increases and becomes infinite with

$$n \to k$$

i.e. when $n = k$ the oscillations stop altogether.

The rate of attenuation of oscillations is characterized by the ratio of adjacent swings (maximum deviations from the equilibrium position)

$$\mu = \frac{a_i}{a_{i+1}} = \frac{e^{-nt}}{e^{-n(t+\tilde{T}/2)}} = e^{n\tilde{T}/2}$$

called ***decrement*** (attenuation). Logarithmic decrement is often used.

$$\gamma = ln\mu = n\tilde{T}/2$$

By measuring two adjacent swings and time $\tilde{T}/2$, it is possible to calculate the resistance coefficient n

$$n = \frac{2}{\tilde{T}} ln \frac{a_i}{a_{i+1}}$$

2. Case of large resistance n > k

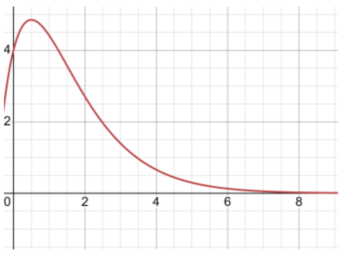
In this case, the roots of the characteristic equation are real numbers,

$$\lambda_{1,2} = -n \pm \sqrt{n^2 - k^2}$$

therefore

$$q = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

C_1 и C_2 are found from the initial conditions.



We see that the motion is not oscillatory (aperiodic). Let the initial deviation be positive. The traffic schedule can have one of three types.

a) $\dot{q}_0 > 0$ After deflection, the system asymptotically returns to the equilibrium position.

b) $\dot{q}_0 < 0, \quad |\dot{q}_0| < q_0(n + \sqrt{n^2 - k^2})$ The system immediately asymptotically returns to the equilibrium position.

$c)\ \ \dot{q}_0 < 0, |\dot{q}_0| > q_0(n + \sqrt{n^2 - k^2})$ The system will pass through the equilibrium position once and return to the equilibrium position from the other side.

3. Case $\boldsymbol{n = k}$

Practically, an unlikely coincidence. The aperiodic solution takes the form.

$$q = e^{-nt}(C_1 + C_2 t)$$

The movements are similar to the case $n > k$

Forced oscillations without resistance.

As we have found out, a conservative system without resistance retains full energy and makes unquenchable oscillations. If we take into account the influence of the medium viscous resistance, then the oscillations are either absent or attenuated, and the total energy of the system decreases, passing into the medium.

Energy can also enter the system from the medium. Let the action of the medium on the system be expressed by a periodic generalized force. As we know, any periodic function can be decomposed into a Fourier series:

$$Q = \sum H_i Sin(p_i t + \delta_i)$$

Here H_i the amplitude of the i-th harmonic, with the frequency of this harmonic$-p_i$, δ_i is the initial phase of this harmonic.

The Lagrange equation of the system:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} + Q$$

By substituting the quadratic forms T and Π

$$T = \frac{1}{2}a\dot{q}^2, \qquad \Pi = \frac{1}{2}cq^2,$$

we obtain an inhomogeneous differential equation

$$a\ddot{q} + cq = \sum H_i Sin(p_i t + \delta_i)$$

Its solution consists of a general solution of a homogeneous equation and a partial solution.

The partial solution will have the form of the right part, i.e. it will be a sum of solutions of the same kind (harmonics). Therefore, it is enough for us to consider the generalized force in the form of only one of the harmonics.

$$Q = HSin(pt + \delta)$$

We obtain the differential ***equation of forced oscillations without resistance***

$$\ddot{q} + k^2 q = hSin(pt + \delta), \quad h = \frac{H}{a}$$

The solution consists of the general solution of a homogeneous equation

$$q_{oo} = C_1 Coskt + C_2 Sinkt$$

and a particular solution, which we will look for in the form of the right side

$$q_ч = ASin(pt + \delta)$$

A is the amplitude of the partial solution, Let's find A .

$$\ddot{q}_ч = -p^2 ASin(pt + \delta)$$

Substituting in the equation, after the abbreviation by Sin, we get

$$(k^2 - p^2)A = h \qquad A = \frac{h}{k^2 - p^2}$$

A partial solution has the form of

$$q_ч = \frac{h}{k^2 - p^2} Sin(pt + \delta)$$

Now the complete solution takes the form of

$$q = C_1 Coskt + C_2 Sinkt + \frac{h}{k^2 - p^2} Sin(pt + \delta)$$

$$\dot{q} = -kC_1 Sinkt + kC_2 Coskt + \frac{ph}{k^2 - p^2} Cos(pt + \delta)$$

Let's find C1, C2 from the initial conditions:

$$t = 0: \qquad q = q_0; \qquad \dot{q} = \dot{q}_0$$

$$q_0 = C_1 + \frac{h}{k^2 - p^2} Sin\delta \qquad \dot{q}_0 = kC_2 + \frac{ph}{k^2 - p^2} Cos\delta$$

From where

$$C_1 = q_0 - \frac{h}{k^2 - p^2} Sin\delta \qquad C_2 = \frac{\dot{q}_0}{k} + \frac{p}{k} \frac{h}{k^2 - p^2} Cos\delta$$

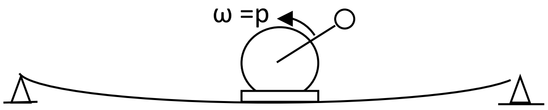
By substituting C_1 and C_2 in the solution, we will find the law of motion

$$q = (q_0 Coskt + \frac{\dot{q}_0}{k} Sinkt) - \frac{h}{k^2 - p^2}\left(Sin\delta\ Coskt + \frac{p}{k}\ Cos\delta\ Sinkt\right) + \frac{h}{k^2 - p^2}\ Sin(pt + \delta)$$

We see that the motion of the system consists of three oscillations. The first is an oscillation with a natural frequency k and amplitude that depends on initial conditions, the second is an oscillation with a natural frequency k and amplitude that does not depend on initial conditions, and the third is forced oscillations with a frequency of the forcing force p and amplitude that does not depend on initial conditions.

Beats and resonance in the absence of resistance.

How does a disturbing force arise? It can be created by placing an electric motor with an unbalanced mass on an elastic beam. The forcing frequency will be the angular speed of rotation of the electric motor $p = \omega$. When the motor does not rotate $\omega = 0$ it oscillates on the beam with a natural frequency k. If you turn on the motor, then at $\omega \rightarrow k$ the amplitude A increases, tending to infinity.



Beats

Let's find out how the system behaves in this case when $p/k \sim 1$. For simplicity, let's put the initial conditions as zero. Then the solution will take the form:

$$q_{ч} = \frac{h}{k^2 - p^2}[Sin(pt + \delta) - Sin(kt + \delta)] =$$

$$= \frac{2h}{k^2 - p^2} Sin\frac{(p - k)t}{2}\ Cos(pt + \delta) = A(t)\ Cos(pt + \delta)$$

We can see that when $p \rightarrow k$ the amplitude of forced oscillations $A(t)$ becomes a periodic function of low frequency $\frac{(p-k)}{2}$ we have the ***beats***. Beats can be heard in a motor aircraft when the engine rotation speed approaches the natural frequency of some part of the fuselage.

Resonance

The previously found partial solution

$$A = \frac{h}{k^2 - p^2}$$

loses its meaning at $p = k$, since its amplitude strives for infinity. The phenomenon of increasing the amplitude of forced oscillations A at certain values of the forcing frequency p is called ***resonance.*** Let us find out how the amplitude changes in time with $p = k$.

Let's try to find a private solution in the form of

$$q_{ч} = BtCos(pt + \delta)$$

$$\dot{q}_{ч} = BCos(pt + \delta) - BptSin(pt + \delta)$$

$$\ddot{q}_{ч} = -BpSin(pt + \delta) - BpSin(pt + \delta) - Bp^2tCos(pt + \delta)$$

Substituting these expressions into the differential equation, taking into account $p = k$

$$B = -\frac{h}{2p}$$

and partial solution

$$q_{ч} = -\frac{h}{2p}tCos(pt + \delta) = A(t)\ Cos(pt + \delta)$$

So, if the engine on the beam (see above) is immediately started with angular velocity $\omega = p = k$, then the amplitude of forced oscillations (and the deformation of the beam) will increase linearly in time. When the limit deformations of the beam are reached, the beam will break.

Dependencies of the coefficient of dynamism and phase shift

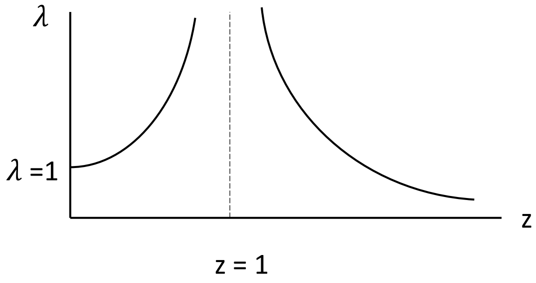
Coefficient of dynamism

Let us find the dependence of the amplitude A of the proper forced oscillations on the forcing frequency p. To construct qualitative dependencies, it is customary to switch to dimensionless quantities. Instead of amplitude A, consider its relation to "static deformation"

$$A_{ст} = \frac{H}{c} = \frac{H}{a}\frac{a}{c} = \frac{h}{k^2},$$

which is called ***the coefficient of dynamism.***

$$\lambda = \frac{A}{A_{ст}} = \frac{1}{1 - z^2}$$



In here

$$z = \frac{p}{k}$$

is a dimensionless forcing frequency called ***the tuning coefficient*** .

At $z = 0$ $\lambda = 1$, at $z \to \infty$ $\lambda \to 0$. The graph takes the form of $\lambda(z)$

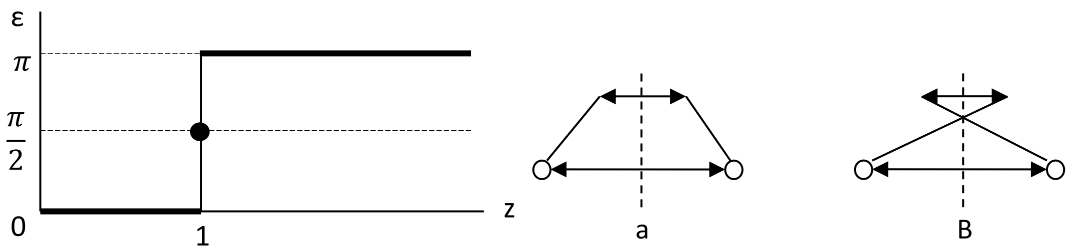
To avoid the danger of system failure, do not work near resonance z = 1

Phase shift dependence ε (z)

Phase shift ε is the difference between the phase of the forcing force (pt+δ) and the phase of the partial solution. Let's find ε at different values of z.

	Partial Solution	Phase Shift
At $z < 1\ (p < k)$:	$q_ч = \frac{h}{\lvert k^2 - p^2 \rvert} Sin(pt + \delta)$	ε = 0
At $z = 1\ (p = k)$:	$q_ч = -\frac{h}{2p} tCos(pt + \delta) = \frac{h}{2p} tSin\left(pt + \delta - \frac{\pi}{2}\right)$	$\varepsilon = \frac{\pi}{2}$
At $z > 1\ (p > k)$:	$q_ч = -\frac{h}{\lvert k^2 - p^2 \rvert} Sin(pt + \delta) = \frac{h}{\lvert k^2 - p^2 \rvert} Sin(pt + \delta - \pi)$	$\varepsilon = \pi$

Now we can draw a graph of the dependence ε (z).



The phase shift can be observed by swinging the "scatter" - ball on an elastic band. If the frequency of hand movements is less than the natural frequency of oscillations, then the ball moves in the same phase (inphase) with the hand (a). With a high frequency of hand movements, the ball moves "in antiphase" with the hand (b).

Forced oscillations with viscous resistance. The law of motion.

The same system is considered, which, along with potential forces, is affected by the damping forces and disturbing forces.

Potential forces are determined by the function of potential energy $\Pi(q)$– the zero level is chosen in the stable equilibrium position, where: $\Pi(0) = 0$ and $\Pi'(0) = 0, \Pi''(0) = c > 0$.

The viscous damping forces are characterized by the Relay function Ф,. After linearization, we have quadratic forms:

$$\Pi = \frac{1}{2}cq^2 \ (c > 0) \qquad T = \frac{1}{2}a\dot{q}^2 \ \ (a > 0) \qquad \Phi = \frac{1}{2}b\dot{q}^2 \ \ (b > 0)$$

The forcing forces are represented by the generalized force Q

$$Q = HSin(pt + \delta)$$

Let's write down the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} - \frac{\partial \Phi}{\partial \dot{q}} + Q$$

Substituting expressions for T, П, Ф, Q we get an equation

$$\ddot{q} + 2n\dot{q} + k^2 q = hSin(pt + \delta)$$

$$2n = \frac{b}{a}; \quad k^2 = \frac{c}{a}; \quad h = \frac{H}{a}$$

The solution of this inhomogeneous equation consists of the general solution q_{oo} of the homogeneous equation and the partial solution $q_ч$ of the inhomogeneous equation.

Low resistance $n < k$ solution q_{oo} attenuates over time

$$q_{oo} = e^{-nt}\left(C_1 Cos\tilde{k}t + C_2 Sin\tilde{k}t\right), \qquad \tilde{k} = \sqrt{k^2 - n^2} < k$$

We are looking for a partial solution in the form of:

$q_ч = ASin(pt + \delta - \varepsilon)$, A is the amplitude, ε –phase shift.

$$\dot{q}_ч = Ap\, Cos(pt + \delta - \varepsilon) \qquad \ddot{q}_ч = -Ap^2 Sin(pt + \delta - \varepsilon)$$

The right side of the equation is represented as

$$hSin(pt + \delta) = hSin[(pt + \delta - \varepsilon) + \varepsilon] = h\, Sin\varepsilon\, Cos(pt + \delta - \varepsilon) + h\, Cos\varepsilon\, Sin(pt + \delta - \varepsilon)$$

After substituting it into the equation, we find

$$A(k^2 - p^2)Sin(pt + \delta - \varepsilon) + 2n\, Ap\, Cos(pt + \delta - \varepsilon) =$$

$$= h\, Sin\varepsilon\, Cos(pt + \delta - \varepsilon) + h\, Cos\varepsilon\, Sin(pt + \delta - \varepsilon)$$

Collecting the coefficients for $Sin(pt + \delta - \varepsilon)$ and $Cos(pt + \delta - \varepsilon)$ we have

$Sin(pt + \delta - \varepsilon)$: $\quad A(k^2 - p^2) = h\, Cos\varepsilon$

$Cos(pt + \delta - \varepsilon)$: $\quad 2n\, Ap = h\, Sin\varepsilon$

Squaring and adding, we find the amplitude of forced oscillations:

$$A = \frac{h}{\sqrt{(k^2 - p^2)^2 + 4n^2p^2}}$$

Dividing the second by the first, we find the phase displacement tangent:

$$tg\varepsilon = \frac{2np}{k^2 - p^2}$$

Final partial solution

$$q_ч = \frac{h}{\sqrt{(k^2 - p^2)^2 + 4n^2p^2}} Sin(pt + \delta - \varepsilon)$$

General solution of the differential equation of oscillations ($n < k$):

$$q = e^{-nt}\left(C_1 Cos\tilde{k}t + C_2 Sin\tilde{k}t\right) + \frac{h}{\sqrt{(k^2 - p^2)^2 + 4n^2p^2}} Sin(pt + \delta - \varepsilon)$$

$$\dot{q} = -ne^{-nt}\left(C_1 Cos\tilde{k} + C_2 Sin\tilde{k}t\right) + e^{-nt}\left(-C_1\tilde{k}Sin\tilde{k}t + C_2\tilde{k}Cos\tilde{k}t\right) + \frac{hp}{\sqrt{(k^2 - p^2)^2 + 4n^2p^2}} Cos(pt + \delta - \varepsilon)$$

As always, the constant integrations of C_1 and C_2 are found from the initial conditions

$$t = 0: \quad q = q_0; \quad \dot{q} = \dot{q}_0$$

$$q_0 = C_1 + ASin(\delta - \varepsilon) \qquad \dot{q}_0 = -nC_1 + C_2\tilde{k} + ACos(\delta - \varepsilon)$$

From where

$$C_1 = q_0 - ASin(\delta - \varepsilon) \qquad C_2 = \frac{1}{\tilde{k}}(\dot{q}_0 + nC_1 - ACos(\delta - \varepsilon))$$

We see that C_1 and C_2 consist of initial conditions and components that depend on the disturbing force. Substituting C_1 and C_2 in the solution, we see that, as it was in forced oscillations without resistance, the motion of the system consists of three oscillations ($n < k$):

1. Oscillations with a quasi-frequency $\tilde{k}$ and amplitude depending on the initial conditions,
2. Oscillations with a quasi-frequency $\tilde{k}$ and amplitude independent of initial conditions
3. Proper forced oscillations with a frequency of p.

Regardless of the value of the resistance n, the first two oscillations disappear over time and the proper forced oscillations (the partial solution) remain. Therefore, it is of particular interest.

Characteristics $\lambda(z)$ and $\varepsilon(z)$

We will construct qualitative characteristics in dimensionless values of the coefficients of dynamism λ and adjustment z

$$\lambda = \frac{A}{A_{ст}} = \frac{1}{\sqrt{(1 - z^2)^2 + 4\nu^2z^2}} \qquad tg\varepsilon = \frac{2\nu z}{1 - z^2}$$

Where $\nu = \frac{n}{k}$ —is the dimensionless coefficient of resistance.

Let's explore the dependence $\lambda(z)$on extremes.

Obviously, $\lambda = 1$ with $z = 0$, and $\lambda \to 0$ with $z \to \infty$, Let's consider the radical expression

$$y = (1 - z^2)^2 + 4\nu^2 z^2$$

Let's find the extremum suspicious points.

$$y' = -4z(1 - z^2) + 8\nu^2 z = 0$$

The root $z_1 = 0$ exists at any resistance ν

The second root will be found from

$$1 - z^2 - 2\nu^2 = 0 \qquad z_2 = \sqrt{1 - 2\nu^2} < 1$$

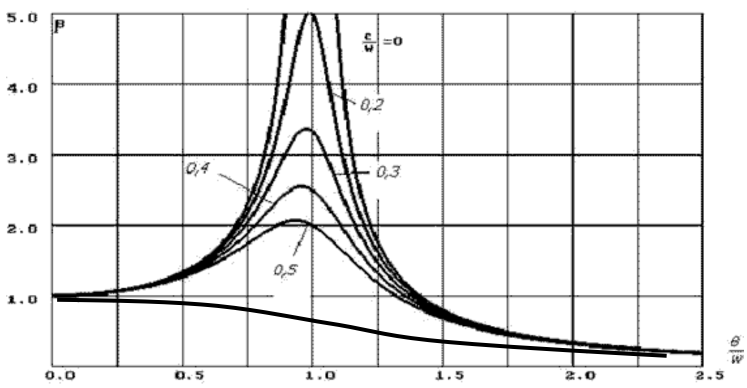
This root decreases with increasing resistance and disappears with resistance

$$\nu > \nu^* = \frac{1}{\sqrt{2}}$$

Let's find out the type of extreme point at zero.

$$y'' = -4(1 - z^2) + 8z^2 + 8\nu^2|_{z=0} = -4(1 - 2\nu^2)$$

When $\nu < \nu^*$ the derivative in zero is negative, y has a max and λ −minimum in zero.



It is at $\nu < \nu^*$, that there is a second root z_2, in which λ has a maximum, since the minimum is followed by a maximum.

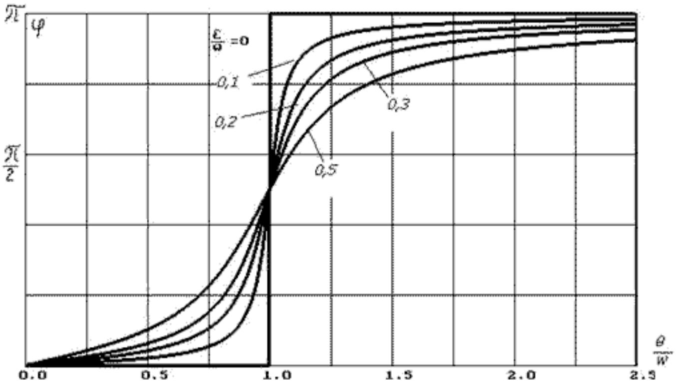
So, the graph of the function $\lambda(z)$ depends on the value ν of resistance: at $\nu < \nu^*$ $\lambda(z)$ has a minimum at zero and a maximum (***resonance***) at z_2. The value of z_2 and the value of the resonant amplitude decrease with increasing resistance ν. With high resistance $\nu > \nu^*$ $\lambda(z)$ has only a maximum at zero.

We see that at $\nu < \nu^*$, the coefficient of dynamism (amplitude) of forced oscillations reaches its maximum value at z_2. As is known, an increase in amplitude λ at some values of the forcing frequency (z) is called ***resonance***. Thus, in the presence of resistance, resonance occurs at z_2.

As the resistance increases, the value z_2 decreases, and resonance is reached earlier. It can be shown that in this case the resonant amplitude will decrease. When $\nu \geq \nu^*$the resonance disappears. As is known, in the absence of resistance, resonance occurs at $z = 1$

Graph *ε (z)*

When $z = 1$ all schedules go through $\pi/2$.



Findings:

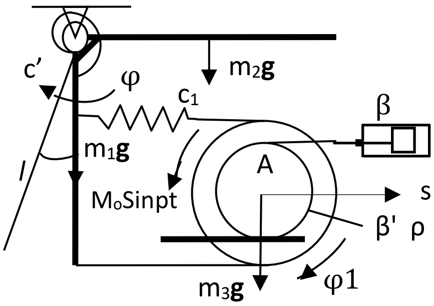
1) A conservative system (all forces are potential) makes undamped oscillations near the position of stable equilibrium ($c > 0$).
2) The medium (the force of viscous resistance) takes away the full mechanical energy from the system, so even with low resistance, the oscillations will be dampened, and with high resistance, there will be no vibrations at all.
3) If energy enters the system without resistance in the form of a periodic disturbing force, then forced oscillations with the frequency of the disturbing force appear. Their amplitude reaches a high value at $p = k$ (the phenomenon of resonance) if the system does not collapse first.
4) The most common model is the model of forced oscillations with resistance, in which an increase in resistance decreases the resonance amplitude and reduces the resonance phenomenon to zero when the resistance $\nu^*$is reached (the resonance disappears).
5) The dangerous phenomenon of resonance can be avoided if you:
   a) work away from the resonance zone
   b) eliminate resonance with the help of dampers.

There are mechanisms in which oscillations are useful, for example, a tamper, a jackhammer, a conveyor (oscillates).

Set of control oscillation problems for system with one degree of freedom is at https://disk.yandex.ru/d/sE-Djnx26knu9w

Example of solving the problem
on oscillations of a system with one degree of freedom.

The three-body system moves under the influence of alternating torque and experiences the action of two springs, the viscous resistance to the rotation of the roller moving without slippage, and a linear damper. The rods have different lengths and weights.



To find:
1. Ratio of static deformations of springs
2. The condition of stability of the depicted equilibrium position.
3. Differential equation of small motions of a system

Solution:
Let's denote the masses, stiffnesses and resistance coefficients. The system has one degree of freedom, since the thread is non-stretch and tensioned by a spring, and the roller rolls without slipping.

1. Let us make a quadratic form of kinetic energy. As is known, T acquires the form at the moment when the system passes the equilibrium position depicted in the figure.

$$T = \frac{1}{2}(J_1 + J_2)\dot{\varphi}^2 + \frac{1}{2}m_3\dot{s}^2 + \frac{1}{2}J_3\dot{\varphi_1}^2$$

$$J_1 = \frac{1}{3}m_1{l_1}^2, \qquad J_2 = \frac{1}{3}m_2{l_2}^2 \qquad J_3 = m_3\rho^2$$

Kinematic ratios:

$$\dot{\varphi}_1 = \frac{l_1}{R - r}\dot{\varphi}, \qquad \dot{s} = \dot{\varphi}_1 r = \frac{l_1 r}{R - r}\dot{\varphi}$$

Getting a quadratic form

$$T = \frac{1}{2}\left[\frac{1}{3}\left(m_1{l_1}^2 + m_2{l_2}^2\right) + m_3{l_1}^2\frac{r^2 + \rho^2}{(R - r)^2}\right]\dot{\varphi}^2 = \frac{1}{2}a\dot{\varphi}^2$$

Here, a is the coefficient of inertia of the system

2. Let us make the quadratic form of the Relay function Φ. As is known, the Φ acquires the form at the moment when the system passes the equilibrium position shown in the figure.

$$\Phi = \frac{1}{2}\beta{v_A}^2 + \frac{1}{2}\beta'\dot{\varphi_1}^2$$

Kinematic ratios

$$v_A = \frac{2l_1 r}{R - r}\dot{\varphi}$$

Quadratic form Φ

$$\Phi = \frac{1}{2}\left[\beta\frac{4{l_1}^2 r^2}{(R - r)^2} + \beta'\frac{{l_1}^2}{(R - r)^2}\right]\dot{\varphi}^2 = \frac{1}{2}b\dot{\varphi}^2$$

Here b is the coefficient of resistance of the system

3. Let's find the quadratic form of potential energy. As is known, the potential energy is equal to the work of potential forces when the system returns to the equilibrium position. The gravity of $m_3\mathbf{g}$ does not do the work as it is perpendicular to the movement of the center of the roller. The deformation of a linear spring in the deflected position consists of the static deformation and the sum of the displacements of the spring ends at rotation φ (the ends of the spring move in opposite directions).

$$\Pi = m_1 g\frac{l_1}{2}(1 - Cos\varphi) - m_2 g\frac{l_2}{2}Sin\varphi +$$
$$+\frac{c_1}{2}\left[\left(\Delta_{\text{ст}} + \frac{R + r}{R - r}l_1\varphi + (l_1 - 2R)\varphi\right)^2 - {\Delta_{\text{ст}}}^2\right] + \frac{c'}{2}\left[(\Delta'_{\text{ст}} + \varphi)^2 - {\Delta'_{\text{ст}}}^2\right]$$

The system is nonlinear, since trigonometric functions are series of φ

$$Cos\varphi = 1 - \frac{\varphi^2}{2} + \cdots \qquad Sin\varphi = \varphi - \cdots$$

It is necessary to consider small oscillations: $\varphi,\ \dot{\varphi} \ll 1$ and to discard in the expansions the terms of higher orders.

Let us show that the potential energy is the quadratic form of the generalized coordinate φ

$$\Pi = \frac{1}{2}c\varphi^2$$

The terms with zero degree $\Delta_{\text{ст}}$ are reduced. This is as it should be, since in the equilibrium position the potential energy is zero.

The terms of the first degree φ must also be absent due to the equilibrium condition.

$$\Pi'_0 = 0$$

Let us equate the coefficient with zero to the first power φ. It can be computed as the value of the first derivative Π'_0 in the equilibrium position. But it is easier to collect coefficients at the first degree φ

$$-m_2 g\frac{l_2}{2}+c_1\left(\frac{R+r}{R-r}l_1+l_1-2R\right)\Delta_{\text{ст}}+c'\Delta'_{\text{ст}}=0$$

This expression can be called the "ratio of static deformations". It shows that only one static deformation can be defined in the equilibrium position. The second must be determined from the ratio.

Thus, the potential energy is indeed a quadratic form of φ, from which we can find the stiffness coefficient of the system c. It is equal to .

$$\text{c}=\Pi''_0$$

But it is easier to find it as a coefficient by $\frac{\varphi^2}{2}$ in the expression of potential energy

$$\text{c}=m_1 g\frac{l_1}{2}+c_1\left(\frac{R+r}{R-r}l_1+l_1-2R\right)^2+c'$$

The condition for the stability of the equilibrium position is

$$\text{c}>0$$

We can see that the condition is met at any parameter values.

4. Let us find the generalized forcing force arising from the variable torque applied to the roller

$$M=M_0 Sinpt$$

by calculating the power of the moment at the positive possible generalized velocity $\dot{\varphi}>0$. The directions of momentum and angular velocity are opposite, so

$$N=-\dot{\varphi}_1 M_0 Sinpt=\left(-\frac{M_0 l_1}{R-r}Sinpt\right)\dot{\varphi}=Q_{\text{в}}\dot{\varphi}$$

$$Q_{\text{в}}=-\frac{M_0 l_1}{R-r}Sinpt=\text{H}Sinpt$$

Let us make a differential equation of small oscillations of the system. By substituting the quadratic forms T, Π, and Φ into the Lagrange equation

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\varphi}}-\frac{\partial T}{\partial\varphi}=-\frac{\partial\Pi}{\partial\varphi}-\frac{\partial\Phi}{\partial\dot{\varphi}}+Q_{\text{в}}$$

we get

$$a\ddot{\varphi}=-c\varphi-b\dot{\varphi}+\text{H}Sinpt$$

Dividing by a

$$\ddot{\varphi}+2n\dot{\varphi}+k^2\varphi=hSinpt$$

In here

$$2n=\frac{b}{a};\quad k^2=\frac{b}{a};\quad h=\frac{H}{a}$$

SYSTEM WITH TWO DEGREES OF FREEDOM

Quadratic form of potential energy. Condition for the stability of the equilibrium position.

We consider a system with 2 degrees of freedom and generalized coordinates q_1, q_2. All forces are potential, which means that there is a function $\Pi\ (q_1, q_2)$. The system has an equilibrium position in which we choose the origin and the zero level of potential energy $\Pi\ (0,0)\ =\ 0$. By equilibrium conditions:

$$\frac{\partial\Pi}{\partial q_1}(0,0)=0\qquad\qquad \frac{\partial\Pi}{\partial q_2}(0,0)=0$$

Let's decompose Π into a McLaren series at zero:

$$\Pi(q_1q_2) = \Pi(0,0) + \frac{\partial\Pi}{\partial q_1}(0,0)q_1 + \frac{\partial\Pi}{\partial q_2}(0,0)q_2 + \frac{1}{2}\left(\frac{\partial^2\Pi}{\partial {q_1}^2}(0,0){q_1}^2 + 2\frac{\partial^2\Pi}{\partial q_1\ \partial q_2}(0,0)q_1q_2 + \frac{\partial^2\Pi}{\partial {q_2}^2}(0,0){q_2}^2\right) + \cdots$$

In view of the choice of the zero level Π and the equilibrium conditions, the first non-zero term will be the quadratic form

$$\Pi(q_1q_2) = \frac{1}{2}(c_{11}{q_1}^2 + 2c_{12}q_1q_2 + c_{22}{q_2}^2)$$

Here are the stiffness coefficients of the system:

$$c_{11} = \frac{\partial^2\Pi}{\partial {q_1}^2}(0,0) \qquad c_{12} = \frac{\partial^2\Pi}{\partial q_1\partial q_2}(0,0) \qquad c_{22} = \frac{\partial^2\Pi}{\partial {q_2}^2}(0,0)$$

The system is called ***linear according to*** Π if there are no expansion terms following the quadratic form. If the system is not linearized, then it is "linearized" by considering small motions of the system near the equilibrium position. After linearization, the potential energy is practically a quadratic form.

The stiffness coefficients form a symmetrical stiffness matrix:

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$$

At the same time, since the order of taking the mixed derivative does not matter.$c_{12} = c_{21}$

Oscillations occur only near the position of stable equilibrium. The condition for the stability of the equilibrium position according to Lyapunov is the presence of min Π in the equilibrium position (at zero). Since Π (0,0) = 0, this means that in the vicinity of zero, Π must be a positively defined function.

It is known from mathematics that the condition for the positive definiteness of a quadratic form in the vicinity of zero is the Sylvester criterion: ***the main diagonal minors of the stiffness matrix must be positive***:

$$c_{11} > 0 |C| = c_{11}c_{22} - c_{12}^2 > 0$$

Quadratic form of kinetic energy.

$$T = \frac{1}{2}\sum m_k{V_k}^2$$

$$\boldsymbol{V}_k = \dot{\boldsymbol{r}}_k = \frac{\partial r_k}{\partial q_1}\dot{q}_1 + \frac{\partial r_k}{\partial q_2}\dot{q}_2; \qquad \frac{\partial r_k}{\partial q_1}(q_1q_2); \quad \frac{\partial r_k}{\partial q_2}(q_1q_2)$$

$$T = \frac{1}{2}\left({\dot{q}_1}^2\sum m_k\left(\frac{\partial r_k}{\partial q_1}\right)^2 + 2\dot{q}_1\dot{q}_2\sum m_k\left(\frac{\partial r_k}{\partial q_1}\right)\left(\frac{\partial r_k}{\partial q_2}\right) + {\dot{q}_2}^2\sum m_k\left(\frac{\partial r_k}{\partial q_2}\right)^2\right)$$

Thus, T is a quadratic form of generalized velocities with coefficients – in the general case, functions of coordinates:

$$a_{11} = \sum m_k\left(\frac{\partial r_k}{\partial q_1}\right)^2; \qquad a_{12} = \sum m_k\left(\frac{\partial r_k}{\partial q_1}\right)\left(\frac{\partial r_k}{\partial q_2}\right) \qquad a_{22} = \sum m_k\left(\frac{\partial r_k}{\partial q_2}\right)^2$$

$$T = \frac{1}{2}(a_{11}{\dot{q}_1}^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}{\dot{q}_2}^2)$$

The system is called ***linear with respect to T*** if the coefficients at the generalized velocities are constant. If the system is not linearized, then it is linearized, considering the small motions of the system

$a_{11}=a_{11}(0,0) \quad a_{12}=a_{12}(0,0) \quad a_{22}=a_{22}(0,0)$

This means that it is possible to obtain the desired form T by calculating T at zero.
Since kinetic energy is positive, the Sylvester criterion is always fulfilled for its coefficients:

$$a_{11}>0 \quad a_{11}a_{22}-a_{12}^2>0$$

Differential equations of motion of a system. Main forms.

By substituting into the Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{\partial T}{\partial q_1}=-\frac{\partial \Pi}{\partial q_1} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right)-\frac{\partial T}{\partial q_2}=-\frac{\partial \Pi}{\partial q_2}$$

forms T and Π, we obtain the ***differential equations of the oscillations of the system***:

$$a_{11}\ddot{q}_1+a_{12}\ddot{q}_2+c_{11}q_1+c_{12}q_2=0$$

$$a_{21}\ddot{q}_1+a_{22}\ddot{q}_2+c_{21}q_1+c_{22}q_2=0$$

We look for the solution of equations in the form of periodic in-phase functions with different amplitudes:

$$q_1=ASin(kt+\propto) \qquad q_2=BSin(kt+\propto)$$

By substituting these solutions into differential equations, after reducing by $Sin(kt+\propto)$, we obtain homogeneous algebraic equations with respect to the amplitudes A and B, with an unknown parameter – natural frequency k.

$$A(c_{11}-a_{11}k^2)+B(c_{12}-a_{12}k^2)=0$$

$$A(c_{12}-a_{12}k^2)+B(c_{22}-a_{22}k^2)=0$$

As is known, a non-trivial (non-zero) solution of such equations exists if the determinant of the matrix of the system is equal to zero:

$$(c_{11}-a_{11}k^2)(c_{22}-a_{22}k^2)-(c_{12}-a_{12}k^2)^2=0$$

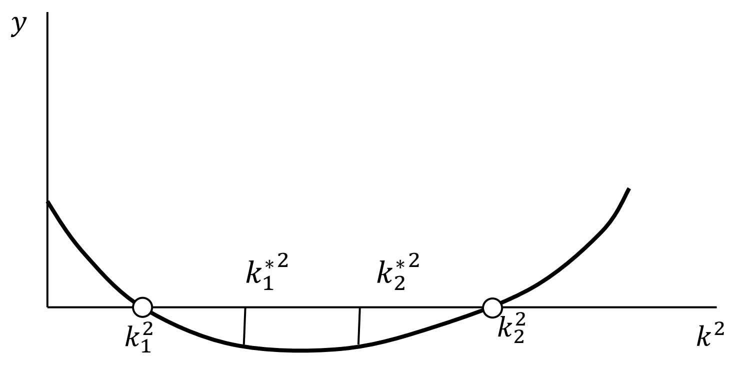
This gives a biquadratic "frequency equation" with respect to natural frequency k

$$y(k^2)=0$$

the left part of which looks like:

$$y(k^2)=k^4(a_{11}a_{22}-{a_{22}}^2)+k^2(2c_{12}a_{12}-a_{11}c_{22}-a_{22}c_{11})+(\,c_{11}c_{22}-{c_{12}}^2)$$

It has two roots. We accept only positive material decision, otherwise the decision will not be oscillating. Let us show that in a stable position of equilibrium they are just that.



Let's build a graph $y(k^2)$

$y(0) > 0$ due to the fulfillment of the condition of stability of the equilibrium position $c_{11}c_{22} - {c_{12}}^2 > 0$. that $y(\infty) > 0$ due to the fact that $a_{11}a_{22} - {a_{22}}^2 > 0$

At the same time, at frequency values called ***partial frequencies***

$$k_1^{*2} = \frac{c_{11}}{a_{11}} \qquad k_2^{*2} = \frac{c_{22}}{a_{22}}$$

$$y(k_1^{*2}) < 0 \qquad y(k_2^{*2}) < 0$$

which follows directly from the frequency equation, since

$$c_{11} - a_{11}k_1^{*2} = 0 \qquad c_{22} - a_{22}k_2^{*2} = 0$$

Thus, the frequency equation has two real positive roots k_1^2 and k_1^2 if the equilibrium position is stable.

Frequencies $k_1$and are called k_2***the natural frequencies of the system.***

Let's return to the amplitude equations. They become dependent at natural frequencies, so it is impossible to find amplitudes A and B from them. You can only find their ratios – ***the form coefficients*** for each frequency from any of the equations.

For example, from the first equation

$$A(c_{11} - a_{11}k^2) + B(c_{12} - a_{12}k^2) = 0$$

for each of the natural frequencies we find

$$\mu_1 = \frac{B_1}{A_1} = -\frac{c_{11} - a_{11}{k_1}^2}{c_{12} - a_{12}{k_1}^2} \qquad \mu_2 = \frac{B_2}{A_2} = -\frac{c_{11} - a_{11}{k_2}^2}{c_{12} - a_{12}{k_2}^2}$$

Now the law of motion of the system takes the form:

$$q_1 = A_1 Sin(k_1 t + \propto_1) + A_2 Sin(k_2 t + \propto_2)$$

$$q_2 = \mu_1 A_1 Sin(k_2 t + \propto_1) + \mu_2 A_2 Sin(k_2 t + \propto_2)$$

We see that the system performs 2 ***main oscillations*** with frequencies k_1 and k_2. There are four arbitrary constants in the solution

$$A_1; \ ; \ A_2 \propto_1; \ \propto_2$$

to be found from the initial conditions

$$t = 0: \ q_1 = q_{10}; \quad q_2 = q_{20} \quad \dot{q}_1 = \dot{q}_{10} \quad \dot{q}_2 = \dot{q}_{20}$$

A note about normal coordinates

It can be shown that for any system there are generalized coordinates, called ***normal,*** in which these coefficients of quadratic forms are absent

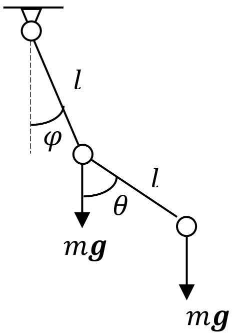
$$a_{12} = 0 \qquad c_{12} = 0$$

In normal coordinates, the equations are "separated":

$$a_{11}\ddot{q}_1 + c_{11}q_1 = 0$$

$$a_{22}\ddot{q}_2 + c_{22}q_2 = 0$$

Oscillations of a double mathematical pendulum



Let us consider the motion of a double mathematical pendulum. For simplicity, let's assume that their masses m and lengths l are the same. Lagrange equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = -\frac{\partial \Pi}{\partial \varphi} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial \Pi}{\partial \theta}$$

We find the quadratic form of kinetic energy by calculating T at the moment when the system passes the equilibrium position $\varphi, \theta = 0$. In the equilibrium position (the velocity of the lower mass is equal to) $l\dot{\varphi} + l\dot{\vartheta}$

$$T = \frac{m}{2}l^2\dot{\varphi}^2 + \frac{m}{2}(l\dot{\varphi} + l\dot{\vartheta})^2 = ml^2\dot{\varphi}^2 + ml^2\dot{\varphi}\dot{\vartheta} + \frac{m}{2}l^2\dot{\vartheta}^2 = \frac{1}{2}(a_{11}\dot{q}_1{}^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2{}^2)$$

Thus, the system is linear with respect to T

$$a_{11} = 2ml^2 \qquad a_{12} = a_{22} = ml^2$$

Potential energy of the system

$$\Pi = mgl(1 - Cos\varphi) + mgl[(1 - Cos\varphi) + (1 - Cos\theta)]$$

The system is not linear in Π, so it is necessary to consider small motions near the equilibrium position. For now

$$\Pi = mgl\varphi^2 + mgl\frac{\theta^2}{2} = \frac{1}{2}(\mathrm{c}_{11}\varphi^2 + 2\mathrm{c}_{12}\varphi\theta + \mathrm{c}_{22}\theta^2)$$

Hence

$$\mathrm{c}_{11} = 2mgl \qquad \mathrm{c}_{12} = 0 \qquad \mathrm{c}_{22} = mgl$$

Frequency equation:

$$k^4(2m^2l^4 - m^2l^4) + k^2(-2m^2l^3g - 2m^2l^3g) + 2(mgl)^2$$
$$= m^2l^4k^4 - 4m^2l^3gk^2 + 2(mgl)^2 = 0$$

By reducing by m^2l^4 we get

$$k^4 - 4\frac{g}{l}k^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

The solutions to this equation are natural frequencies

$$k^2{}_{1,2} = 2\frac{g}{l} \pm \sqrt{\left(2\frac{g}{l}\right)^2 - 2\left(\frac{g}{l}\right)^2} = \frac{g}{l}(2 \pm \sqrt{2})$$

Let us find the coefficients of form for $k^2{}_1 = \frac{g}{l}(2 + \sqrt{2})$

$$\mu_1 = -\frac{c_{11} - a_{11}k_1{}^2}{c_{12} - a_{12}k_1{}^2} = -\frac{2mgl - 2ml^2\frac{g}{l}(2 + \sqrt{2})}{-ml^2\frac{g}{l}(2 + \sqrt{2})} = -\frac{2\sqrt{2} + 2}{2 + \sqrt{2}} = -\sqrt{2}$$

For the second frequency, we get

$$\mu_2 = \sqrt{2}$$

A positive shape coefficient means that pendulums will oscillate in phase (a). A negative shape coefficient means that the pendulums will oscillate in the opposite phase (b). Characteristically, a higher frequency corresponds to a negative shape coefficient.

The system will oscillate along one of the shapes if the pendulums are deflected in proportion or as in the figure and released without initial speed. Under arbitrary initial conditions, both forms of oscillations $\mu_1 \mu_2$ will take place.,

Forced oscillations without resistance

Suppose there are disturbing forces applied to the conservative system, which are reduced to two generalized disturbing forces $Q_1(t)$and $Q_2(t)$. Then the differential equations of motion of the system will become inhomogeneous

$$a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + c_{11}q_1 + c_{12}q_2 = Q_1(t)$$

$$a_{21}\ddot{q}_1 + a_{22}\ddot{q}_2 + c_{21}q_1 + c_{22}q_2 = Q_2(t)$$

The solution of these equations consists of the general solution of the homogeneous equation (continuous oscillations with natural frequencies $k_1$and k_2) and forced oscillations.

As mentioned, from coordinates $q_1 \quad q_2$ it is possible to proceed to normal coordinates $\theta_1 \quad \theta_2$, in which differential equations are separated. Let the disturbing forces be harmonious, then

$$a_{11}\ddot{\theta}_1 + c_{11}\theta_1 = H_1 Sin(pt + \delta)$$

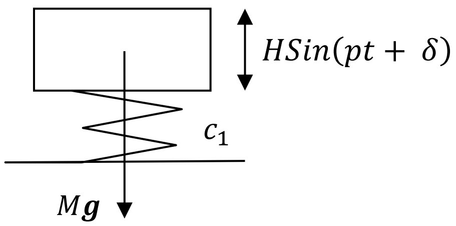
$$a_{22}\ddot{\theta}_2 + c_{22}\theta_2 = H_2 Sin(pt + \delta)$$

From these equations, it can be seen that the system has two resonances when each of the natural frequencies coincides with the forcing frequency p.

Dynamic vibration damper

The figure shows the diagram of the machine mass M on an elastic base of stiffness c_1.

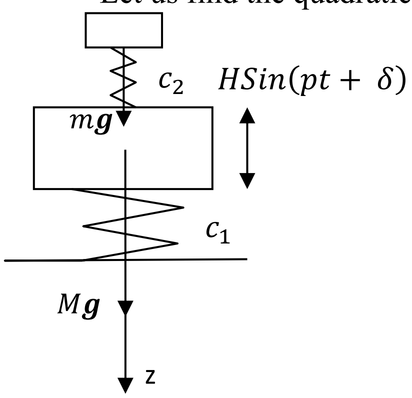
A periodic disturbing force $HSin(pt + \delta)$is applied to the machine, which can arise, for example, from the imbalance of the engine of the machine rotating at angular velocity $\omega = p$.



Obviously, the machine will make unwanted forced oscillations, especially dangerous near the resonance $\omega \rightarrow k$.

Let's show how to use a dynamic vibration damper to rid the machine of forced oscillations. A dynamic vibration damper is a body of mass m mounted on a spring of stiffness c_2 on the machine.

Let us find the quadratic forms of the kinetic and potential energies of such a construction. As the generalized coordinates, we choose absolute coordinates z_1 z_2, the origin of which is chosen in the position of equilibrium.



$$\Pi = -mz_2 - Mz_1 + \frac{c_1}{2}\left[(\Delta_{ст1}+z_1)^2 - {\Delta_{ст1}}^2\right] + \frac{c_2}{2}\left[(\Delta_{ст2}+z_2-z_1)^2 - {\Delta_{ст2}}^2\right] = \frac{1}{2}(c_{11}{z_1}^2 + 2c_{12}z_1z_2 + c_{22}{z_2}^2)$$

Hence

$$c_{11} = c_1 + c_2 \qquad c_{12} = -c_2 \qquad c_{22} = c_2$$

$$T = \frac{1}{2}M{\dot z_1}^2 + \frac{1}{2}m{\dot z_2}^2 == \frac{1}{2}(a_{11}{\dot z_1}^2 + 2a_{12}\dot z_1\dot z_2 + a_{22}{\dot z_2}^2)$$

Hence

$$a_{11} = M \quad a_{12} = 0 \quad a_{22} = m$$

By substituting the forms T *and* Π into the Lagrange equations, we obtain the differential equations of motion

$$M\ddot z_1 + (c_1+c_2)z_1 - c_2z_2 = \mathrm{H\,Sin}(\omega t + \delta)$$

$$m\ddot z_2 - c_2z_1 + c_2z_2 = 0$$

We are looking for a solution in the form of the right part.

$$z_1 = A\mathrm{Sin}(\omega t + \delta) \qquad z_2 = B\mathrm{Sin}(\omega t + \delta)$$

By substituting the solutions into the equations, after the reduction by $\mathrm{Sin}(\omega t + \delta)$ we obtain an algebraic system for determining the amplitudes of the stimulated oscillations A and B.

$$(c_1 + c_2 - \omega^2 M)A - c_2B = H$$

$$-c_2A + (c_2 - \omega^2 m)B = 0$$

System matrix determinant

$$\Delta = (c_1 + c_2 - \omega^2 M)(c_2 - \omega^2 m) - {c_2}^2$$

System solutions

$$A = \frac{H(c_2 - \omega^2 m)}{\Delta} \qquad B = \frac{Hc_2}{\Delta} \qquad \Delta = -{c_2}^2$$

From this it follows that it is possible to select its mass m and spring stiffness c_2 in such a way that

$$c_2 = \omega^2 m$$

the amplitude of forced oscillations of machine A will be equal to zero.

$$A = 0 \qquad B = -\frac{H}{c_2}$$

It can be seen that the damper acts on the machine with a force that balances the disturbing force at every moment, all the energy of which is used to swing the damper.

$$Bc_2\text{Sin}(\omega t + \delta) = -H\text{Sin}(\omega t + \delta)$$

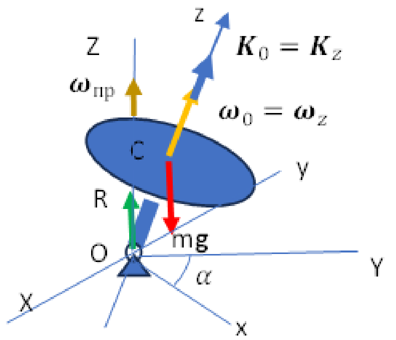
It is natural to choose a small damper mass $m \ll M$, but then the stiffness of its spring should be small. This, however, will lead to a large amplitude of vibrations of the damper itself. Therefore, the choice of specific damper parameters is the result of a compromise between the weight and amplitude of the damper.

ELEMENTARY GYROSCOPE THEORY

Gyroscope on a hinge

Gyroscope is an axisymmetric body m that performs spherical motion under the influence of gravity and the support reaction, which is given a large initial angular velocity ω_z around the axis of symmetry z.

The term gyroscope was proposed by Foucault in 1852.A gyroscope is, for example, an ordinary spinning top – a disk on an axis perpendicular to the disk (Fig) and resting on a hinge O.



At the initial moment, let the axis of the gyroscope be directed in the plane of the fixed axes YZ at an angle of α to the axis Z. Let us direct the y-axis in the plane XY, which it does not leave.

The moving axes x, y, z are the main axes of inertia of the gyroscope at O. Therefore, the gyroscope inertia tensor matrix at point O will be diagonal

$$J_o = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix}$$

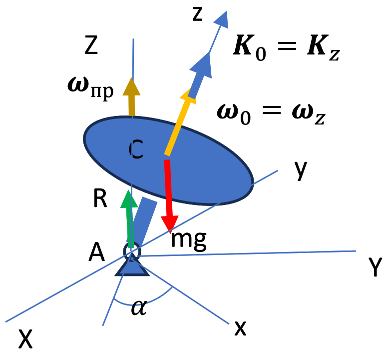
Column of projections of the angular momentum of the body with respect to the support O on the moving axes

$$K_o = J_o\omega = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} J_x\omega_x \\ J_y\omega_y \\ J_z\omega_z \end{pmatrix} \quad (1)$$

By spinning the gyroscope around its z-axis to angular velocity ω_0, we give it the initial angular momentum $\boldsymbol{K}_0$directed along the z-axis

The behavior of the vector $\boldsymbol{K}_0$ is described by the angular momentum change theorem

$$\frac{dK_o}{dt} = \boldsymbol{M}_o^e \qquad (2)$$



The main moment of external forces $\boldsymbol{M}_o^e$is created by the force of gravity m**g.** It is directed against the X-axis. According to the theorem, the velocity of the end of the vector $\boldsymbol{K}_0$is also directed there. Its modulus is equal to the value of the moment

$$mg\ OC\ Sin\alpha$$

The z-axis begins to rotate around the Z-axis with an angular velocity$\boldsymbol{\omega}_{пр}$ called the angular ***velocity of precession***. The y-axis rotates at this speed in the XY plane.

Vectors $\boldsymbol{K}_{\text{o}}\boldsymbol{M}_{O}^{e}$, and $\boldsymbol{\omega}_{\text{пр}}$ turn out to be related by the relation:

$$\boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{K}_{\text{o}} = \boldsymbol{M}_{O}^{e}$$

Absolute angular velocity becomes a vector sum

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{z} + \boldsymbol{\omega}_{\text{пр}}$$

It follows from formula (1) that $\boldsymbol{K}_{\text{o}}$ will no longer be directed along the z-axis of the gyroscope, as well as the vector of absolute angular velocity $\boldsymbol{\omega}$.

However, at a high velocity of proper rotation (ω_z up to 30000 rpm), this difference can be neglected, and it can be considered that $\boldsymbol{K}_{\text{o}}$ is directed along the gyroscope's proper z axis (***approximate gyroscope theory***).

Let us find the angular velocity of precession. It is equal to the velocity modulus of the end of the vector $\boldsymbol{K}_{\text{o}}$ (the modulus of the moment of gravity) divided by its distance to the Z axis

$$\omega_{\text{пр}} = \frac{mgOCSin\,\alpha}{J_z\omega_0 Sin\alpha} = \frac{OC}{\rho^2\omega_0}g$$

We can see that the angular velocity of precession $\omega_{\text{пр}}$ is constant, does not depend on the angle of inclination α of the gyroscope axis. It is inversely proportional to the square of the gyroscope's radius of inertia ρ and its angular velocity ω_0, and is proportional to the distance of OC.

To reduce the rate of precession, gyroscopes are made in the form of massive rings, and give them a large angular velocity of their own. Then

$$\omega_z \gg \omega_{\text{пр}}$$

The z-axis of the gyroscope moves along a conical surface with a central angle of α with a constant angular velocity $\omega_{\text{пр}}$of precession.

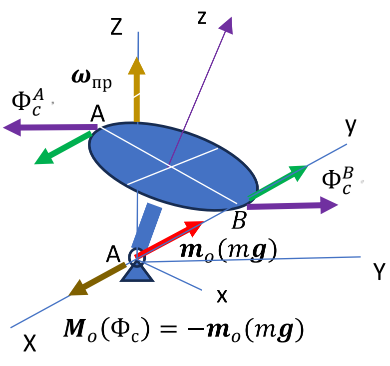
Role of Coriolis forces of inertia

We see that the moment of gravity $\boldsymbol{m}_{o}(m\boldsymbol{g})$ does not create the expected rotation. Let's find out what forces balance it.

Let's do this in the xyz reference frame associated with the gyroscope axis. Since it rotates at angular velocity, it is non-inertial. In a non-inertial frame of reference, Coriolis and portable inertia forces act on the points of the body. The latter can be neglected due to their small size.$\boldsymbol{\omega}_{\text{пр}}$

Let us consider the points A and B of a gyroscope on its diameter parallel to the x-axis.

$$\omega_z \gg \omega_{\text{пр}}$$



Coriolis inertia forces acting on the masses dm of these points

$$\Phi_c^A = -\Phi_c^B = -2dm(\boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{V}_A)$$

are directed opposite and at the moment parallel to the Y-axis.

We see that the Coriolis inertia forces of the points above and below the white diameter parallel to the y-axis create a moment of inertial forces directed against the y-axis and the moment of gravity.

Let us show that the total moment of Coriolis forces of inertia is modulo equal to the moment of gravity, and opposite in direction. Thus, both actions are balanced.

The gyroscope is at relative rest with respect to the moving coordinates x, y, z, uniformly rotating around the vertical axis with the angular velocity of precession $\boldsymbol{\omega}_{\text{пр}}$.

This means that the force of gravity and the forces of inertia are in equilibrium. We are interested in the condition of the invariability of the angle ∝, that is, the equality of the moments of these forces with respect to O.

Radius vector of point dm $\boldsymbol{\rho} = \boldsymbol{OC} + \boldsymbol{r}$ has its attached matrix

$$\rho = \begin{pmatrix} 0 & -\rho_z & \rho_y \\ \rho_z & 0 & -\rho_x \\ -\rho_y & \rho_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -OC & -rSin\varphi \\ OC & 0 & rCos\varphi \\ rSin\varphi & -rCos\varphi & 0 \end{pmatrix}$$

Coriolis force of inertia of point dm

$$d\,\Phi_c = -2dm(\boldsymbol{\omega}_{пр} \times \boldsymbol{\omega} \times \boldsymbol{r}), \qquad dm = \gamma hrdrd\varphi$$

The moment of this force relative to the support O

$$\boldsymbol{m}_O(d\,\Phi_c) = -2dm(\boldsymbol{\rho} \times \boldsymbol{\omega}_{пр} \times \boldsymbol{\omega} \times \boldsymbol{r})$$

In matrix form

$$\begin{pmatrix} m_x(d\,\Phi_c) \\ m_y(d\,\Phi_c) \\ m_z(d\,\Phi_c) \end{pmatrix} = -2dm \begin{pmatrix} 0 & -OC & -rSin\varphi \\ OC & 0 & rCos\varphi \\ rSin\varphi & rCos\varphi & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_{пр}Cos\alpha & -\omega_{пр}Sin\alpha \\ \omega_{пр}Cos\alpha & 0 & 0 \\ \omega_{пр}Sin\alpha & 0 & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -rCos\varphi \\ -rSin\varphi \\ 0 \end{pmatrix} = -2\gamma hrdrd\varphi\omega_{пр}r\omega \begin{pmatrix} -OCSin\varphi Cos\alpha - rSin^2\varphi Sin\alpha \\ OCCos\varphi Cos\alpha + rCos\varphi Sin\varphi Sin\alpha \\ 2rSin\varphi Cos\varphi Cos\alpha) \end{pmatrix}$$

$$\omega_{пр} = \frac{2OC}{R^2\omega} g$$

$$M_x(\,\Phi_c) = \frac{4OC\gamma h}{R^2} g \left[OCCos\alpha \int_0^R (r^2)\,dr \int_0^{2\pi} Sin\varphi\,d\varphi + Sin\alpha \int_0^R (r^3)\,dr \int_0^{2\pi} Sin^2\varphi\,d\varphi \right] =$$

$$= \frac{4OC\gamma h}{R^2} g \left[\left(-OCCos\alpha \frac{R^3}{3} Cos\varphi \right)_0^{2\pi} + Sin\alpha \frac{R^4}{4} \left[\frac{\varphi}{2} - \frac{1}{4} Sin2\varphi \right]_0^{2\pi} \right] =$$

$$= \gamma h\pi R^2 gOCSin\alpha = mgOCSin\alpha \qquad M_y(\,\Phi_c) = 0, \quad M_z(\,\Phi_c) = 0$$

We see that the moment of Coriolis inertia forces in modulo equal to the moment of gravity, and opposite in direction. This means that the force of gravity is balanced by the Coriolis forces of inertia, so the angle ∝ is unchanged.

Role of transport inertial forces

Find the portable acceleration of the gyroscope particle.

Radius vector of the gyroscope particle of mass

$$dm = \gamma hrdrd\varphi$$

(h is the thickness of the disk, γ is the density of the disk material) is

$$\boldsymbol{\rho} = \boldsymbol{OC} + \boldsymbol{r}$$

Absolute particle velocity

$$\boldsymbol{V}_a = \dot{\boldsymbol{\rho}} = \dot{\boldsymbol{OC}} + \dot{\boldsymbol{r}} = \boldsymbol{\omega}_{пр} \times \boldsymbol{OC} + (\boldsymbol{\omega}_{пр} + \boldsymbol{\omega}) \times \boldsymbol{r} = \boldsymbol{\omega}_{пр}(\boldsymbol{OC} + \boldsymbol{r}) + \boldsymbol{\omega} \times \boldsymbol{r}$$

Absolute point acceleration

$$\boldsymbol{W_a} = \ddot{\boldsymbol{\rho}} = \boldsymbol{\omega}_{\text{пр}} \times (\dot{\boldsymbol{OC}} + \dot{\boldsymbol{r}}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}} =$$
$$= \boldsymbol{\omega}_{\text{пр}} \times \left[\boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{OC} + \left(\boldsymbol{\omega}_{\text{пр}} + \boldsymbol{\omega}\right) \times \boldsymbol{r}\right] + \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega}_{\text{пр}} + \boldsymbol{\omega}\right) \times \boldsymbol{r} =$$
$$= \boldsymbol{W_e} + \boldsymbol{W_r} + \boldsymbol{W_c}$$

In here

$$\boldsymbol{W_e} = \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{r}$$
$$\boldsymbol{W_r} = \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{r}$$
$$\boldsymbol{W_c} = 2\boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\omega} \times \boldsymbol{r}$$

Transport force of inertia of particle dm

$$d\,\Phi_e = -dm\left(\boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{r}\right), \qquad dm = \gamma h r dr d\varphi$$

The moment of this force relative to the support O

$$\boldsymbol{m}_O(d\,\Phi_e) = -dm\boldsymbol{\rho} \times \left(\boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{r}\right)$$

Let's write this ratio in matrix form in the coordinates xyz

$$\begin{pmatrix} m_x(d\,\Phi_e) \\ m_y(d\,\Phi_e) \\ m_z(d\,\Phi_e) \end{pmatrix} = -dm \begin{pmatrix} 0 & -OC & -rSin\varphi \\ OC & 0 & rCos\varphi \\ rSin\varphi & rCos\varphi & 0 \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} 0 & -\omega_{\text{пр}}Cos\alpha & -\omega_{\text{пр}}Sin\alpha \\ \omega_{\text{пр}}Cos\alpha & 0 & 0 \\ \omega_{\text{пр}}Sin\alpha & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_{\text{пр}}Cos\alpha & -\omega_{\text{пр}}Sin\alpha \\ \omega_{\text{пр}}Cos\alpha & 0 & 0 \\ \omega_{\text{пр}}Sin\alpha & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -rCos\varphi \\ -rSin\varphi \\ OC \end{pmatrix} +$$
$$+ \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega_{\text{пр}}Cos\alpha & -\omega_{\text{пр}}Sin\alpha \\ \omega_{\text{пр}}Cos\alpha & 0 & 0 \\ \omega_{\text{пр}}Sin\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} -rCos\varphi \\ -rSin\varphi \\ 0 \end{pmatrix} =$$
$$= dm\left[r\omega^2{}_{\text{пр}} \begin{pmatrix} -OCCos^2\alpha\, Sin\varphi - rSin\alpha\, Cos\alpha Sin^2\varphi \\ OC(Cos^2\alpha\, Cos\varphi + Sin\alpha Cos\varphi) + rSin\alpha\, Cos\alpha Sin\varphi Cos\varphi \\ rSin\varphi(Cos^2\alpha\, Cos\varphi + Sin\alpha Cos\varphi) + rSin\alpha\, Cos\alpha Sin\varphi Cos\varphi \end{pmatrix} \right.$$
$$\left. + \omega\omega_{\text{пр}} \begin{pmatrix} -OCCos\alpha\, Sin\varphi \\ OCCos\alpha\, Cos\varphi \\ rCos\alpha\, Cos\varphi Sin\varphi + rCos\alpha\, Sin\varphi Cos\varphi \end{pmatrix} \right]$$

We leave it only $Sin^2\varphi$ because

$$\int_0^{2\pi} Sin\varphi\, d\varphi = 0, \quad \int_0^{2\pi} Cos\varphi\, d\varphi = 0, \quad \int_0^{2\pi} Sin\varphi Cos\varphi\, d\varphi = 0$$

$$\begin{pmatrix} m_x(d\,\Phi_e) \\ m_y(d\,\Phi_e) \\ m_z(d\,\Phi_e) \end{pmatrix} = \begin{pmatrix} -\omega^2{}_{\text{пр}}Sin\alpha\, Cos\alpha\, r^2 Sin^2\varphi \\ 0 \\ 0 \end{pmatrix} \gamma h r dr d\varphi$$

The main moment of the transport forces of inertia with respect to the x-axis

$$M_x(\,\Phi_e) = -\omega^2{}_{\text{пр}}\gamma h Sin\alpha\, Cos\alpha \int_0^R r^3 dr \int_0^{2\pi} Sin^2\varphi d\varphi = -\frac{4OC^2}{R^4\omega^2} g^2 \gamma h \frac{R^4}{4}\left[\frac{\varphi}{2} - \frac{1}{4}Sin2\varphi\right]_0^{2\pi}$$
$$= -\frac{\gamma h \pi R^2 OC^2}{R^2\omega^2} g^2 = -m\left(\frac{OCg}{R\omega}\right)^2$$

We see that this moment tends to increase the angle α , but it is small due to the high speed ω of the gyroscope's own rotation, and cannot be taken into account by an approximate theory

Angular acceleration of the gyroscope

The change in the direction of the angular velocity vector $\boldsymbol{\omega}$ during regular precession is acceleration. Since the angular velocity modulus is constant, $\omega = Const$

$$\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \boldsymbol{\omega}_{\text{пр}} \times \boldsymbol{\omega}$$

The vector $\boldsymbol{\varepsilon}$ is directed towards precession and the moment of gravity that generates it, and its modulus within the framework of the approximate theory is equal to

$$\varepsilon = \omega_{\text{пр}}\omega Sin \propto = \frac{OC}{\rho^2\omega} g\omega Sin \propto = \frac{OCSin \propto}{\rho^2} g$$

Note that

$$J_z\varepsilon = m\rho^2 \frac{OCSin \propto}{\rho^2} g = mgOCSin \propto = m_{\text{o}}(mg)$$

Interestingly, $\boldsymbol{\varepsilon}$ is not directed along z, but as well as the moment $\boldsymbol{m}_{\text{o}}(mg)$.

A gyroscope supported in the center of mass ($OC = 0$) will not perform regular precession, since there is no moment of external forces. This is the gyroscope in the Cardan suspension.

The axis of such a gyroscope retains its direction in an inertial frame of reference (the geocentric frame of reference is not inertial), which is widely used in navigational instruments such as the gyrocompass, the "artificial horizon" in airplanes, the gyroscopic stabilizers.

Like Foucault's pendulum, such a gyroscope allows you to measure the speed of the Earth's rotation.

Gyroscopic effects

The gyroscopic effect is an unexpected rotation of the gyroscope axis, the cause of which was shown above.

The gyroscopic effect is easy to observe by holding the axis of a rapidly rotating bicycle wheel with both hands. If you try to rotate the axle of the wheel in the vertical plane, the axle will rotate in the horizontal plane.

The gyroscopic effect helps to ride a bicycle "without hands". The kinetic moment of the bicycle wheels is directed to the left. When you lose your balance, you lean along with the axle of the front wheel, for example, to the left. At the same time, you apply a backward torque to the wheel axis. The end of the kinetic momentum vector of the front wheel, directed to the left, receives a velocity also directed backwards. The wheel turns to the left, which helps you regain your balance.

If, during normal riding, you sharply turn the handlebar to the right without tilting the bike, then the gyroscopic effect will tilt the bike to the left, which is dangerous.

However, this maneuver is useful if you want to make a sharp left turn because it instantly creates the necessary tilt of you and the bike to the left.

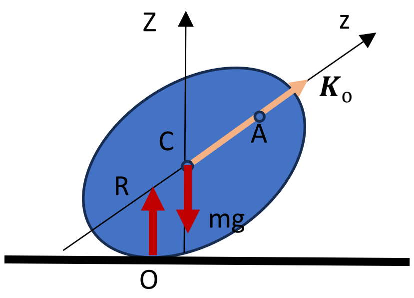
It is also useful for avoiding a trapdoor-sized obstacle. Your body continues to move straight by inertia, and the bike below you goes around the hatch.

Free gyroscope on the plane

Suppose that the support plane is absolutely smooth (there is no friction). The gyroscope is affected by the m**g** and **R**, equal in modulus and opposite in direction.

Let us assume that the speed of proper rotation allows us to apply the simplified theory of the gyroscope and the kinetic moment of the gyroscope is directed along the z axis

The moment of reaction **R** relative to the center of mass C creates a regular precession of the gyroscope axis z around the axis Z. Looking from the end of the Z axis, we can see the direction of precession of the gyroscope axis counterclockwise.



Let us try to accelerate the precession by applying a force at point A of the gyroscope axis, directed behind the drawing - in the direction of the precession velocity of point A

The moment of such a force relative to the center of mass C is directed in the plane of the drawing perpendicular to the z-axis to the Z-axis.

According to theorem (2), the z-axis of the gyroscope will rotate to the vertical axis of precession Z, and will raise the center of mass C as much as possible.

Inference: Any force that seeks to accelerate precession raises the center of mass of the gyroscope. The opposite is also true.

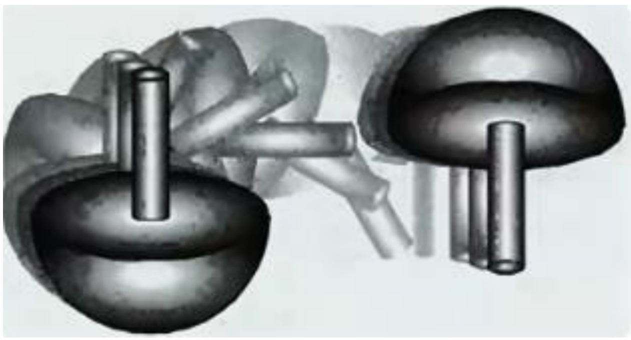
Columbus Egg

Now suppose that there is friction at the contact point O. The frictional force is directed against the sliding velocity of the point of contact O. This velocity is composed of the velocity from its own rotation and from the velocity of precession. The first one is directed behind the drawing, and it is much larger than the second one directed towards us. The total speed of sliding is directed beyond the drawing, and the frictional force is directed towards us, in the direction of precession.

It has been shown that such a force raises the center of gravity of the gyroscope, aligning the z-axis of the gyroscope with the vertical axis of precession Z. The center of mass of the gyroscope will take the highest position on the Z-axis.

The force of gravity, as well as the force of friction, will do a negative job, reducing the kinetic energy (angular velocity) of the gyroscope.

This effect explains the behavior of the so-called Columbus egg and the Chinese spinning top.



A spun Chinese spinning top rises from the sphere to the leg.

The name Columbus Egg comes from the legend of how Columbus allegedly won a bet that he would put the egg "on his butt" while sailing. He simply broke the egg at one end and set the egg down.

If the egg-shaped body in the position (Fig a) is strongly twisted, the frictional force will lift the egg to an upright position (Fig b).